2-3. (a) $\mathcal{F}\mathcal{F}\{g(x,y)\} = \iint\limits_{-\infty}^{\infty} df_X df_Y \left\{ \iint\limits_{-\infty}^{\infty} d\xi d\eta\, g(\xi,\eta)\, e^{-j2\pi(\xi f_X + \eta f_Y)} \right\} e^{-j2\pi(f_X x + f_Y y)}.$

Interchange the orders of integration, yielding

$$\iint\limits_{-\infty}^{\infty} d\xi d\eta\, g(\xi,\eta) \quad \iint\limits_{-\infty}^{\infty} df_X df_Y \exp\left\{-j2\pi\left[(\xi + x)f_X + (\eta + y)f_Y\right]\right\}.$$

But the right-hand double integral is identically the same as $\delta(\xi+x, \eta+y)$, and the sifting property can be applied to the remaining double integral,

$$\iint\limits_{-\infty}^{\infty} d\xi d\eta\, g(\xi,\eta)\, \delta(\xi + x, \eta + y)\, d\xi d\eta = g(-x, -y).$$

The result for $\mathcal{F}^{-1}\mathcal{F}^{-1}\{g(x,y)\}$ is derived the same way with a change of sign in both exponentials.

(b) The simplest method of proof is to show that

$$\mathcal{F}^{-1}\left\{G(f_X, f_Y) \otimes H(f_X, f_Y)\right\} = g(x,y)\, h(x,y).$$

Remembering that the $\mathcal{F}^{-1}$ operator operates on the variables (f_X, f_Y),

$$\begin{aligned}
&\mathcal{F}^{-1}\left\{ \iint\limits_{-\infty}^{\infty} G(\xi,\eta) H(f_X - \xi, f_Y - \eta) d\xi d\eta \right\} \\
&= \iint\limits_{-\infty}^{\infty} G(\xi,\eta) \mathcal{F}^{-1}\left\{H(f_X - \xi, f_Y - \eta)\right\} d\xi d\eta \\
&= \iint\limits_{-\infty}^{\infty} G(\xi,\eta)\ \exp[j2\pi(\xi x + \eta y)]\, d\xi d\eta\ h(x,y) \\
&= g(x,y)\, h(x,y)
\end{aligned}$$

where the shift theorem for inverse transforms has been used.

(c) $\mathcal{F}\left\{\nabla^2 g(x,y)\right\} = \mathcal{F}\left\{\frac{\partial^2}{\partial x^2} g(x,y) + \frac{\partial^2}{\partial y^2} g(x,y)\right\}$. Now

$$\mathcal{F}\left\{\frac{\partial^2}{\partial x^2} g(x,y)\right\} = \frac{\partial^2}{\partial x^2} \iint\limits_{-\infty}^{\infty} G(f_X, f_Y)\, e^{j2\pi(f_X x + f_Y y)}\, df_X df_Y$$

$$= \int\int_{-\infty}^{\infty} G(f_X, f_Y)\, \frac{\partial^2}{\partial x^2} e^{j2\pi(f_X x + f_Y y)}\, df_X df_Y$$

$$= \int\int_{-\infty}^{\infty} \left[-4\pi^2\left(f_X^2 + f_Y^2\right)\right] G(f_X, f_Y)\, e^{j2\pi(f_X x + f_Y y)}\, df_X df_Y.$$

We conclude that

$$\mathcal{F}\left\{\nabla^2 g(x,y)\right\} = -4\pi^2(f_X^2 + f_Y^2)\, G(f_X, f_Y) = -4\pi^2(f_X^2 + f_Y^2)\, \mathcal{F}\left\{g(x,y)\right\}.$$

2-5. Note that since $G(f_X, f_Y) = \int\int_{-\infty}^{\infty} g(x,y)\, e^{-j2\pi(f_X x + f_Y y)}\, dx dy$, we see that

$$G(0,0) = \int\int_{-\infty}^{\infty} g(x,y)\, dx dy.$$

Similarly, since $g(x,y) = \int\int_{-\infty}^{\infty} G(f_X, f_Y)\, e^{j2\pi(f_X x + f_Y y)}\, df_X df_Y$, we have

$$g(0,0) = \int\int_{-\infty}^{\infty} G(f_X, f_Y)\, df_X df_Y.$$

Thus

$$\Delta_{XY} = \left| \frac{\iint\limits_{-\infty}^{\infty} g(x,y)\,dxdy}{g(0,0)} \right| = \left| \frac{G(0,0)}{g(0,0)} \right| = \left| \frac{G(0,0)}{\iint\limits_{-\infty}^{\infty} G(f_X,f_Y)\,df_X df_Y} \right| = \frac{1}{\Delta_{f_X f_Y}}.$$

Hence $\Delta_{xy}\Delta_{f_X f_Y} = 1$.

2-8. To avoid confusion, let's call the frequencies of the applied cosinusoidal signal $(\bar{f}_X, \bar{f}_Y)$. Note that the input can be expanded into a sum of two complex exponentials,

$$g(x,y) = \cos[2\pi(\bar{f}_X x + \bar{f}_Y y)] = \frac{1}{2}\exp[j2\pi(\bar{f}_X x + \bar{f}_Y y)] + \frac{1}{2}\exp[-j2\pi(\bar{f}_X x + \bar{f}_Y y)].$$

Now to have any hope of producing a cosine at the output, we had better insist that the system be *invariant*, for only then can we expect the exponential nature of the two input components to be preserved. For an invariant system, each complex-exponential input produces a complex-exponential output of the same frequency, but with a possible change of amplitude and phase, as determined by the transfer function. Remembering that the complex exponentials are eigenfunctions of linear, invariant systems, we write the output $v(x,y)$ as

$$v(x,y) = \frac{1}{2}H(\bar{f}_X, \bar{f}_Y)\exp[j2\pi(\bar{f}_X x + \bar{f}_Y y)] + \frac{1}{2}H(-\bar{f}_X, -\bar{f}_Y)\exp[-j2\pi(\bar{f}_X x + \bar{f}_Y y)],$$

where $H(f_X, f_Y)$ is the transfer function of the system, given by the Fourier transform of the impulse response $h(x,y)$. The transfer function can be written as the product of an amplitude function and a phase function,

$$H(f_X, f_Y) = A(f_X, f_Y)\, e^{j\phi(f_X, f_Y)},$$

where $A(f_X, f_Y) \geq 0$. Thus the output can be written

$$\begin{aligned} v(x,y) &= \frac{1}{2} A(\bar{f}_X, \bar{f}_Y) \exp[j2\pi(\bar{f}_X x + \bar{f}_Y y) + \phi(\bar{f}_X, \bar{f}_Y)] \\ &+ \frac{1}{2} A(-\bar{f}_X, -\bar{f}_Y) \exp[-j2\pi(\bar{f}_X x + \bar{f}_Y y) + \phi(-\bar{f}_X, -\bar{f}_Y)] \end{aligned}$$

Now we ask under what conditions can the above two exponential terms be combined to form a cosinusoidal output of frequency $(\bar{f}_X, \bar{f}_Y)$? The answer is that the following two conditions must be satisfied:

$$\begin{aligned} A(-\bar{f}_X, -\bar{f}_Y) &= A(\bar{f}_X, \bar{f}_Y) \\ \phi(-\bar{f}_X, -\bar{f}_Y) &= -\phi(\bar{f}_X, \bar{f}_Y), \end{aligned}$$

i.e. the magnitude of the transfer function must be even and the phase must be odd. These symmetry relations will be satisfied if and only if the impulse response of the system, $h(x,y)$, is *real-valued*. Thus, to summarize, the required conditions are that the system be linear and invariant, and that its impulse response be real-valued.

2-11. (a) By the convolution theorem,

$$P(f_X, f_Y) = G(f_X, f_Y)\, XY \operatorname{comb}(X f_X) \operatorname{comb}(Y f_Y),$$

where we have used the similarity theorem and the fact that the Fourier transform of a comb function is another comb function. Further simplification results from the following relation:

$$\begin{aligned} XY \operatorname{comb}(X f_X) \operatorname{comb}(Y f_Y) &= XY \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta(X f_X - n, Y f_Y - m) \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \delta\left(f_X - \frac{n}{X}, f_Y - \frac{m}{Y}\right), \end{aligned}$$

where we have used the fact that $\delta(ax, by) = \frac{1}{|a,b|}\delta(x, y)$. We have assumed in the above that $X \geq 0, Y \geq 0$.

(b) The Fourier transform of the given $g(x, y)$ is found as follows:

$$\mathcal{F}\{g(x,y)\} = \frac{XY}{4} \operatorname{sinc}\left(\frac{X}{2} f_X\right) \operatorname{sinc}\left(\frac{Y}{2} f_Y\right),$$

where the similarity theorem has been used. The figure below shows sketches of $g(x, 0)$ and $p(x, 0)$ in this case.

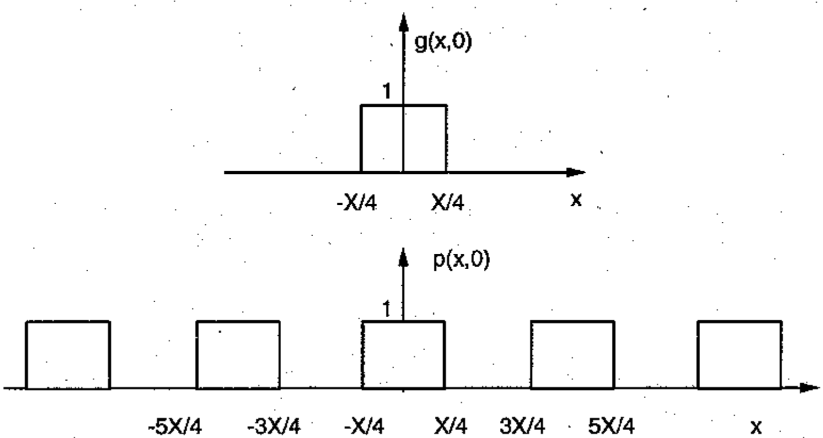

Figure 2-11:

2-13. The object $U_o(x,y)$ has a band-unlimited spectrum, while the transfer function $H(f_X,f_Y)$ of the system is bandlimited to the region $|f_X| \le B_X, |f_Y| \le B_Y$. Because of the bandlimitation on H, it is possible to write

$$H(f_X,f_Y) = H(f_X,f_Y)\ \mathrm{rect}\left(\frac{f_X}{2B_X}\right)\mathrm{rect}\left(\frac{f_Y}{2B_Y}\right).$$

Since the imaging system is both linear and invariant, the image and object spectra, G_i and G_o, respectively, can be related by

$$G_i(f_Y,f_Y) = H(f_X,f_Y)\,G_o(f_X,f_Y) = H(f_X,f_Y)\left[\mathrm{rect}\left(\frac{f_X}{2B_X}\right)\mathrm{rect}\left(\frac{f_Y}{2B_Y}\right)G_o(f_X,f_Y)\right].$$

From this equation we can see directly that the output spectrum can be viewed as resulting from the application of a new fictitious object with spectrum

$$G_o'(f_X,f_Y) = \mathrm{rect}\left(\frac{f_X}{2B_X}\right)\mathrm{rect}\left(\frac{f_Y}{2B_Y}\right)G_o(f_X,f_Y).$$

In the space domain, the relation between the fictitious object and the actual object is

$$\begin{aligned} U_o'(x,y) &= U_o(x,y)\otimes 4B_XB_Y\mathrm{sinc}(2B_Xx)\mathrm{sinc}(2B_Yy) \\ &= 4B_XB_Y \iint\limits_{-\infty}^{\infty} U_o(\xi,\eta)\,\mathrm{sinc}\left[2B_X(x-\xi)\right]\mathrm{sinc}\left[2B_Y(y-\eta)\right]d\xi d\eta. \end{aligned}$$

Since U_o' is bandlimited, it can be reconstructed from samples taken at the Nyquist rate, i.e. samples taken at coordinates $x_n = \frac{n}{2B_X}, y_m = \frac{m}{2B_Y}$. The sampled object which will yield U_o'

after low pass filtering is given by

$$
\begin{aligned}
\hat{U}_o'(x,y) &= \operatorname{comb}(2B_X x)\operatorname{comb}(2B_Y y)\,U_o'(x,y) \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} U_o'\left(\frac{n}{2B_X x},\frac{m}{2B_Y y}\right)\delta\left(x-\frac{n}{2B_X},y-\frac{m}{2B_Y}\right).
\end{aligned}
$$

Substituting the expression derived above for U_o',

$$
\begin{aligned}
\hat{U}_o'(x,y) &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[\iint_{-\infty}^{\infty} U_o(x,y)\operatorname{sinc}(n-2B_X\xi)\operatorname{sinc}(m-2B_Y\eta)\,d\xi d\eta\right] \\
&\times\ \delta\left(x-\frac{n}{2B_X},m-\frac{m}{2B_Y}\right).
\end{aligned}
$$

This array of point sources will yield the same image as the original object $U_o(x,y)$.

3-5. Using Eq. (3-63) we have the following:

(a) Circular aperture of diameter d:

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)=\mathcal{B}\left\{\operatorname{circ}\left(\frac{2r}{d}\right)\right\}\Big|_{\substack{f_X=\alpha/\lambda\\ f_Y=\beta/\lambda}}$$

using the similarity theorem for Fourier-Bessel transforms (Eq. (2-34)) and the Fourier-Bessel transform pair of Eq. (2-35),

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)=\frac{d^2}{4}\frac{J_1\left(\frac{2\pi\rho d}{2}\right)}{\frac{d\rho}{2}}=\frac{d}{2}\frac{J_1(\pi\rho d)}{\rho}.$$

Finally, note that $\rho=\sqrt{f_X^2+f_Y^2}=\sqrt{\left(\frac{\alpha}{\lambda}\right)^2+\left(\frac{\beta}{\lambda}\right)^2}$ yielding

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)=\frac{d^2}{4}\frac{J_1\left(\frac{2\pi\rho d}{2}\right)}{\frac{d\rho}{2}}=\frac{d}{2}\frac{J_1\left(\pi\sqrt{\left(\frac{\alpha}{\lambda}\right)^2+\left(\frac{\beta}{\lambda}\right)^2}\,d\right)}{\sqrt{\left(\frac{\alpha}{\lambda}\right)^2+\left(\frac{\beta}{\lambda}\right)^2}}.$$

(b) A circular opaque disk of diameter d can be modeled by the following amplitude transmittance function:

$$t_A(x,y)=1-\operatorname{circ}\left(\frac{2r}{d}\right).$$

From the linearity theorem of Fourier analysis it follows that the angular spectrum of this structure is

$$A\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda};0\right)=\delta\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)-\frac{d}{2}\frac{J_1\left(\pi\sqrt{\left(\frac{\alpha}{\lambda}\right)^2+\left(\frac{\beta}{\lambda}\right)^2}\,d\right)}{\sqrt{\left(\frac{\alpha}{\lambda}\right)^2+\left(\frac{\beta}{\lambda}\right)^2}}.$$

3-7. (a) Substituting $U(x,y,z) \approx A(x,y,z)e^{jkz}$ into the Helmholtz equation $(\nabla^2 + k^2)U = 0$,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right] A(x,y,z)e^{jkz} = 0.$$

Then,

$$\begin{aligned}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] Ae^{jkz} + \frac{\partial}{\partial z}\left[\frac{\partial A}{\partial z}e^{jkz} + jkAe^{jkz}\right] + k^2 Ae^{jkz} &= 0 \\ \nabla_t^2 Ae^{jkz} + \frac{\partial^2 A}{\partial z^2}e^{jkz} + 2jk\frac{\partial A}{\partial z}e^{jkz} + (jk)^2 Ae^{jkz} + k^2 Ae^{jkz} &= 0.\end{aligned}$$

Dividing by e^{jkz} and simplifying,

$$\nabla_t^2 A + j2k\frac{\partial}{\partial z}A + \frac{\partial^2}{\partial z^2}A = 0.$$

The “slowly varying” approximation for A implies that:

$$\frac{\partial^2}{\partial z^2}A \ll j2k\frac{\partial A}{\partial z}$$

leaving,

$$\nabla_t^2 A + j2k\frac{\partial A}{\partial z} = 0.$$

(b) We first evaluate a number of different derivatives:

$$\begin{aligned}
A(x,y,z) &= \frac{A_1}{q} e^{jk\frac{x^2+y^2}{2q(z)}} \\
\frac{\partial}{\partial z} A(x,y,z) &= -\frac{A_1}{q^2}\frac{dq}{dz} e^{jk\frac{x^2+y^2}{2q}} - \frac{A_1}{q}\left(jk\frac{x^2+y^2}{2q^2}\right)\frac{dq}{dz} e^{jk\frac{x^2+y^2}{2q}} \\
&= -\left(\frac{1}{q} + jk\frac{x^2+y^2}{2q^2}\right)\frac{dq}{dz} A(x,y,z) \\
\frac{\partial}{\partial x} A(x,y,z) &= jk\frac{xA_1}{q^2} e^{jk\frac{x^2+y^2}{2q}} \\
\frac{\partial^2}{\partial x^2} A(x,y,z) &= jk\frac{A_1}{q^2} e^{jk\frac{x^2+y^2}{2q}} + \left(jk\frac{x}{q}\right)^2 \frac{A_1}{q} e^{jk\frac{x^2+y^2}{2q}} \\
&= \left(jk\frac{1}{q} - k^2\frac{x^2}{q^2}\right) A(x,y,z)
\end{aligned}$$

Similarly,

$$\frac{\partial^2}{\partial y^2} A(x,y,z) = \left(jk\frac{1}{q} - k^2\frac{y^2}{q^2}\right) A(x,y,z).$$

Now substitute the partial derivatives of A into the paraxial Helmholtz equation. Noting that dq/dz is equal to 1,

$$\begin{aligned}
\nabla_t^2 A + j2k\frac{\partial A}{\partial z} &= \left(2jk\frac{1}{q} - k^2\frac{x^2+y^2}{q^2} - 2jk\frac{1}{q}\frac{dq}{dz} + k^2\frac{x^2+y^2}{q^2}\frac{dq}{dz}\right) A \\
&= 0.
\end{aligned}$$

(c) Substituting the given expression into the result from part (b),

$$\begin{aligned}
A &= A_1\left(\frac{1}{R} + j\frac{\lambda}{\pi W^2}\right)\exp\left[\frac{jk}{2}(x^2+y^2)\left(\frac{1}{R} + j\frac{\lambda}{\pi W^2}\right)\right] \\
&= A_1\left(\frac{1}{R} + j\frac{\lambda}{\pi W^2}\right)\exp\left[-\frac{\rho^2}{W^2}\right]\exp\left[jk\frac{\rho^2}{2R}\right] \\
U &= A\exp[jkz] \\
&= A_1\left(\frac{1}{R} + j\frac{\lambda}{\pi W^2}\right)\exp\left[-\frac{\rho^2}{W^2}\right]\exp\left[jkz + jk\frac{\rho^2}{2R}\right] \\
&= A_0\frac{W_0}{W(z)}\exp\left[-\frac{\rho^2}{W^2}\right]\exp\left[jkz + jk\frac{\rho^2}{2R} + j\theta(z)\right]
\end{aligned}$$

where:

$$\begin{aligned}
\frac{A_0 W_0}{W} &= A_1\left[\left(\frac{1}{R}\right)^2 + \left(\frac{\lambda}{\pi W^2}\right)^2\right]^{1/2} \\
\theta(z) &= \tan^{-1}\left(\frac{\lambda R}{\pi W^2}\right).
\end{aligned}$$

To show that W_0 is independent of z, we differentiate W_0^2 with respect to z and show that it equals zero:

$$\begin{aligned} W_0^2 &= \left(\frac{A_1}{A_0}\right)^2 \left[\left(\frac{W}{R}\right)^2 + \left(\frac{\lambda}{\pi W}\right)^2\right] \\ \frac{d(W_0^2)}{dz} &= \left(\frac{A_1}{A_0}\right)^2 \left(\frac{2WR^2W' - 2RW^2R'}{R^4} - \frac{2\lambda^2 W'}{\pi^2 W^3}\right) \\ &= 2\left(\frac{A_1}{A_0}\right)^2 \frac{\pi^2 W^4 RW' - \pi^2 W^5 R' - \lambda^2 R^3 W'}{\pi^2 W^3 R^3}, \end{aligned}$$

where the prime sign designates a derivative with respect to z. Now, using the condition $dq/dz = 1$, we can express R' and W' in terms of R and W:

$$\begin{aligned} \frac{d(1/q)}{dz} &= \frac{-1}{q^2}\frac{dq}{dz} = \frac{-1}{q^2} \\ \frac{d}{dz}\left(\frac{1}{R} + j\frac{\lambda}{\pi W^2}\right) &= -\left(\frac{1}{R} + j\frac{\lambda}{\pi W^2}\right)^2 \\ \frac{-R'}{R^2} - 2j\frac{\lambda W'}{\pi W^3} &= \frac{-1}{R^2} + \frac{\lambda^2}{\pi^2 W^4} - 2j\frac{\lambda}{\pi R W^2}. \end{aligned}$$

Solving for both the real and the complex parts of the equation, we get

$$\begin{aligned} R' &= 1 - \frac{\lambda^2 R^2}{\pi^2 W^4} \\ W' &= \frac{W}{R}. \end{aligned}$$

Substituting,

$$\begin{aligned} \frac{d(W_0^2)}{dz} &= 2\left(\frac{A_1}{A_0}\right)^2 \frac{1}{\pi^2 W^3 R^3}\left[\pi^2 W^4 R\frac{W}{R} - \pi^2 W^5\left(1 - \frac{\lambda^2 R^2}{\pi^2 W^4}\right) - \lambda^2 R^3 \frac{W}{R}\right] \\ &= 2\left(\frac{A_1}{A_0}\right)^2 \frac{1}{\pi^2 W^3 R^3}\left(\pi^2 W^5 - \pi^2 W^5 + \lambda^2 R^2 W - \lambda^2 R^2 W\right) \\ &= 0. \end{aligned}$$