1. a) Aberrations always decrease MTF

Schwarz inequality: $\left|\iint X Y \, dp\, dq\right|^2 \le \left[\iint |X|^2 dp\, dq\right]\left[\iint |Y|^2 dp\, dq\right]$

OTF: $$\mathcal{H}(f_x, f_y) = \frac{\iint_{a(f_x,f_y)} dx\, dy}{\iint_{a(0,0)} dx\, dy}$$ for diffraction limited system.

$a(f_x, f_y)$: overlap area of displaced pupil

1.

with aberrations

$$\mathcal{H}(f_x, f_y) = \frac{\iint_{a(f_x,f_y)} dx\, dy \exp\left\{jk\left[W\left(x+\frac{\lambda z_i f_x}{2}, y+\frac{\lambda z_i f_y}{2}\right) - W\left(x-\frac{\lambda z_i f_x}{2}, y-\frac{\lambda z_i f_y}{2}\right)\right]\right\}}{\iint_{a(0,0)} dx\, dy}$$

(2)

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let $X = \exp\left[jk W\left(x+\frac{\lambda z_i f_x}{2}, y+\frac{\lambda z_i f_y}{2}\right)\right]$

$Y = \exp\left[-jk W\left(x-\frac{\lambda z_i f_x}{2}, y-\frac{\lambda z_i f_y}{2}\right)\right]$

Comparing to Schwarz inequality, we see that

$$\left|\mathcal{H}(f_x, f_y)\right|^2_{\text{aberrated}} \le \left|\mathcal{H}(f_x, f_y)\right|^2_{\text{diffraction limited}}$$

b) Strehl ratio

$$S \equiv \frac{|h(0,0)|^2 \text{ aberrated}}{|h(0,0)|^2 \text{ aberration free}}$$ where $h(u,v)$ is amplitude of point spread function.

OTF is normalized Fourier transform of $|h(u,v)|^2$

$$\mathcal{H}(f_x,f_y) = \frac{\mathcal{F}\left[|h(u,v)|^2\right]}{\iint_{-\infty}^{\infty} |h(u,v)|^2 \, du\, dv}$$

so $$|h(u,v)|^2 = C\,\mathcal{F}^{-1}\left[\mathcal{H}(f_x,f_y)\right]$$

$$= C \iint_{-\infty}^{\infty} \mathcal{H}(f_x,f_y) \exp\left[j2\pi(f_x u + f_y v)\right] df_x\, df_y$$

thus

$$|h(0,0)|^2 = C \iint_{-\infty}^{\infty} \mathcal{H}(f_x,f_y)\, df_x\, df_y$$

as long as the aberrations <u>only</u> introduce a phase variation in the aperture, then $\iint_{-\infty}^{\infty} |h(u,v)|^2 du\,dv$ is unaffected by aberrations, so C is the same as well.

so

$$S = \frac{\iint_{-\infty}^{\infty} \mathcal{H}(f_x,f_y)\, df_x\, df_y \text{ with aberrations}}{\iint_{-\infty}^{\infty} \mathcal{H}(f_x,f_y)\, df_x\, df_y \text{ diffraction limited}}$$

2. <u>Coherence lengths</u>

He Ne laser. $\lambda = 633$ nm. Doppler-broadened → Gaussian lineshape

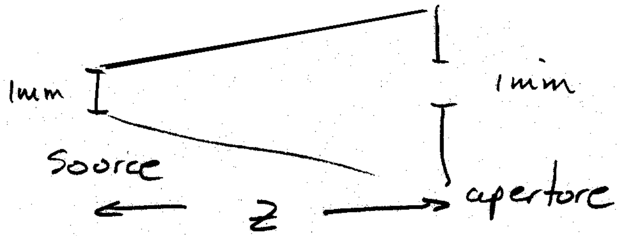
$\Delta\nu = 1.5 \times 10^9$ Hz. $\tau_c = \dfrac{0.664}{\Delta\nu} = \underline{0.44 \times 10^{-9} \text{ sec}}$

$l_c = c\tau_c = \underline{13.3 \text{ cm}}$

Ar laser $\lambda = 488$ nm $\Delta\nu = 7.5 \times 10^9$ Hz

$\tau_c = \underline{0.089 \times 10^{-9} \text{ sec}}$ $l_c = c\tau_c = \underline{2.66 \text{ cm}}$

3. <u>Coherent illumination</u>



at aperture, we want coherent illumination.

So we need coherence diameter ~ 1 mm.

Take $A_c = \pi\left(\frac{d_c}{2}\right)^2 \qquad A_s = \pi\left(\frac{d_s}{2}\right)^2$

since $A_c = \frac{(\lambda z)^2}{A_s} \qquad \frac{\pi d_c^2}{4} = \frac{(\lambda z)^2}{\pi d_s^2/4}$

$$z_{min} = \frac{\pi d_c d_s}{4\lambda} = 142 \text{ cm}$$

4. From class notes,

$$A_c = \frac{(\lambda z)^2}{A_s}$$

where A_c is the coherence area observed at distance z from an incoherent source with area A_s. For a circular source, we can rewrite this

$$d_c^2 = \frac{(\lambda z)^2}{d_s^2}$$

where d_c is the coherence diameter, and d_s is the source diameter. The angle subtended by the source, as seen at the observation region is just

$$\theta_s = \frac{d_s}{z}, \quad \text{so} \quad d_c = \frac{\lambda}{\theta_s}$$

For $\theta_s = 32 \text{ min} \cong 8.7 \times 10^{-3}$ rad, and $\lambda = 550$ nm,

$$d_c = 100 \mu m$$

5. a) General H-F integral for broadband light from class:

$$u(Q,t) = \iint_{\Sigma} \frac{\cos(\hat{n}, \vec{r}_{01})}{2\pi c r_{01}} \frac{d}{dt} u(P_1, t - \frac{r_{01}}{c})\, ds$$

for small pinholes, the field from each pinhole is just:

$$u(Q,t) = \frac{A_1 \cos(\hat{n}, \vec{r}_{01})}{2\pi c r_{01}} \frac{d}{dt} u(P_1, t - \frac{r_{01}}{c})$$

the field from 2 pinholes is then

$$u_{TOT}(Q,t) = K_1 \frac{d}{dt} u(P_1, t - \frac{r_{01}}{c}) + K_2 \frac{d}{dt} u(P_2, t - \frac{r_{02}}{c})$$

b) Taking $I = \langle |u_{TOT}(Q,t)|^2 \rangle$

$$I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + K_1 K_2 \left\langle \frac{d}{dt} u(P_1, t - \frac{r_{01}}{c}) \frac{d}{dt} u^*(P_2, t - \frac{r_{02}}{c}) \right\rangle$$
$$+ K_1 K_2 \left\langle \frac{d}{dt} u^*(P_1, t - \frac{r_{01}}{c}) \frac{d}{dt} u(P_2, t - \frac{r_{02}}{c}) \right\rangle$$

(K_1, K_2 are both real)

consider the expression:

$$\left\langle \frac{d}{dt} u(P_1, t+\tau) \frac{d}{dt} u^*(P_2, t) \right\rangle$$

The definition of the time average tells us this is written:

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{d}{dt} u(P_1, t+\tau) \frac{d}{dt} u^*(P_2, t)\, dt$$

integrate by parts

$$\lim_{T\to\infty} \frac{1}{T} \left\{ \frac{d}{dt} u(P_1, t+\tau)\, u^*(P_2, t) \Big|_{-\frac{T}{2}}^{\frac{T}{2}} - \int_{-T/2}^{T/2} \frac{d^2}{dt^2} u(P_1, t+\tau)\, u^*(P_2, t) \right\}$$

The first term goes to zero in the limit $T \to \infty$, so finally we have shown that

$$\left\langle \frac{d}{dt} u(P_1, t+\tau) \frac{d}{dt} u^*(P_2, t) \right\rangle = -\left\langle \frac{d^2}{dt^2} u(P_1, t+\tau) u^*(P_2, t) \right\rangle$$

but it is also true by definition of the derivative that

$$\frac{d^2}{dt^2} u(P_1, t+\tau) = \frac{\partial^2}{\partial \tau^2} u(P_1, t+\tau)$$

Thus $\left\langle \frac{d}{dt} u(P_1, t+\tau) \frac{d}{dt} u^*(P_2, t) \right\rangle = -\frac{\partial^2}{\partial \tau^2} \left\langle u(P_1, t+\tau) u^*(P_2, t) \right\rangle$

Therefore

$$\begin{aligned} I(Q) &= I^{(1)}(Q) + I^{(2)}(Q) - K_1 K_2 \frac{\partial^2}{\partial \tau^2} \left\langle u(P_1, t+\tau) u^*(P_2, t) \right\rangle \\ &\quad - K_1 K_2 \frac{\partial^2}{\partial \tau^2} \left\langle u^*(P_1, t+\tau) u(P_2, t) \right\rangle \\ &= I^{(1)}(Q) + I^{(2)}(Q) - K_1 K_2 \left[\frac{\partial^2}{\partial \tau^2} \Gamma_{12}(\tau) + \frac{\partial^2}{\partial \tau^2} \Gamma_{12}^*(\tau) \right] \\ &= I^{(1)}(Q) + I^{(2)}(Q) - 2 K_1 K_2 \, \mathrm{Re} \left[\frac{\partial^2}{\partial \tau^2} \Gamma_{12}(\tau) \right] \end{aligned}$$

c) we can write, in general, the inverse transform:

$$u(P_1, t+\tau) = \int_{-\infty}^{\infty} U(P_1, \nu) \exp(j 2\pi\nu[t+\tau]) d\nu$$

substitute this into:

$$\frac{\partial^2}{\partial\tau^2} \langle u(P_1, t+\tau) u^*(P_2, t) \rangle = \frac{\partial^2}{\partial\tau^2} \langle \int_{-\infty}^{\infty} U(P_1, \nu) e^{j2\pi\nu(t+\tau)} d\nu \; u^*(P_2, t) \rangle$$

$$= -4\pi^2 \langle \int_{-\infty}^{\infty} \nu^2 U(P_1, \nu) e^{j2\pi\nu(t+\tau)} d\nu \; u^*(P_2, t) \rangle$$

if u_1 is narrowband, this reduces to:

$$-4\pi^2\nu^2 \langle u(P_1, t+\tau) u^*(P_2, t) \rangle$$

$$= -4\pi^2\nu^2 \Gamma_{12}(\tau)$$

therefore, for narrowband light

$$I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + 2(4\pi^2\nu^2) K_1 K_2 \operatorname{Re} \Gamma_{12}(\tau)$$

our previous narrowband case result was

$$I^{(1)}(Q) + I^{(2)}(Q) + 2\mathcal{K}_1\mathcal{K}_2 \operatorname{Re} \Gamma_{12}(\tau)$$

but our $K_1 = \dfrac{A_1 \cos(\hat{n}, \vec{r}_1)}{2\pi c r_1}$

and $\mathcal{K}_1 = \dfrac{A_1 \cos(\hat{n}, \vec{r}_1)}{\lambda r'}$ so $4\pi^2\nu^2 K_1 K_2 = \mathcal{K}_1 \mathcal{K}_2$.

by similar reasoning $I^{(1)}(Q)$ and $I^{(2)}(Q)$ for broadband light reduce to the narrowband expressions.

6. <u>Image Intensity</u> : $I_i(u,v)$

mutual intensity: $J_i(u_1, v_1; u_2, v_2)$

$$I_i(u,v) = J_i(u,v;u,v) \quad (A)$$

Fourier transform of mutual intensity: $\mathcal{J}_i(\nu_1, \nu_2, \nu_3, \nu_4)$

$$J_i(u_1; v_1; u_2, v_2) = \mathcal{F}^{-1}\left[\mathcal{J}_i(\nu_1, \nu_2, \nu_3, \nu_4)\right]$$

$$= \iiiint_{-\infty}^{\infty} \mathcal{J}_i(\nu_1,\nu_2,\nu_3,\nu_4)\exp\left[-j2\pi(\nu_1 u_1 + \nu_2 v_1 + \nu_3 u_2 + \nu_4 v_2)\right] d\nu_1 d\nu_2 d\nu_3 d\nu_4$$

take $u_1 = u_2 = u \quad v_1 = v_2 = v$ to get $I(u,v)$, using (A)

$$I(u,v) = \iiiint_{-\infty}^{\infty} \mathcal{J}_i(\nu_1,\nu_2,\nu_3,\nu_4)\exp\left\{-j2\pi\left[(\nu_1+\nu_3)u + (\nu_2+\nu_4)v\right]\right\} d\nu_1 d\nu_2 d\nu_3 d\nu_4$$

7. <u>Image Spectrum</u>

by definition: $\mathcal{I}_i(\nu_u, \nu_v) = \iint_{-\infty}^{\infty} I(u,v)\exp\left[j2\pi(u\nu_u + v\nu_v)\right] du\,dv$

using previous problem 8, we have:

$$\mathcal{I}_i(\nu_u,\nu_v) = \iiiint\!\!\iint_{-\infty}^{\infty} \mathcal{J}_i(\nu_1,\nu_2,\nu_3,\nu_4)\exp\left\{j2\pi\left[(\nu_u-\nu_1-\nu_3)u + (\nu_v-\nu_2-\nu_4)v\right]\right\} d\nu_1 d\nu_2 d\nu_3 d\nu_4\,du\,dv$$

changing order of integration

$$= \iiiint_{-\infty}^{\infty} d\nu_1 d\nu_2 d\nu_3 d\nu_4\, \mathcal{J}_i(\nu_1,\nu_2,\nu_3,\nu_4) \underbrace{\iint_{-\infty}^{\infty} du\,dv \exp\left\{j2\pi\left[(\nu_u-\nu_1-\nu_3)u + (\nu_v-\nu_2-\nu_4)v\right]\right\}}_{\delta(\nu_u-\nu_1-\nu_3)\,\delta(\nu_v-\nu_2-\nu_4)}$$

perform the integrals over ν_3, ν_4. The δ functions sift out giving:

$$\mathcal{I}_i(\nu_u,\nu_v) = \iint_{-\infty}^{\infty} d\nu_1 d\nu_2\, \mathcal{J}_i(\nu_1,\nu_2,\nu_u-\nu_1,\nu_v-\nu_2) \quad ✓$$