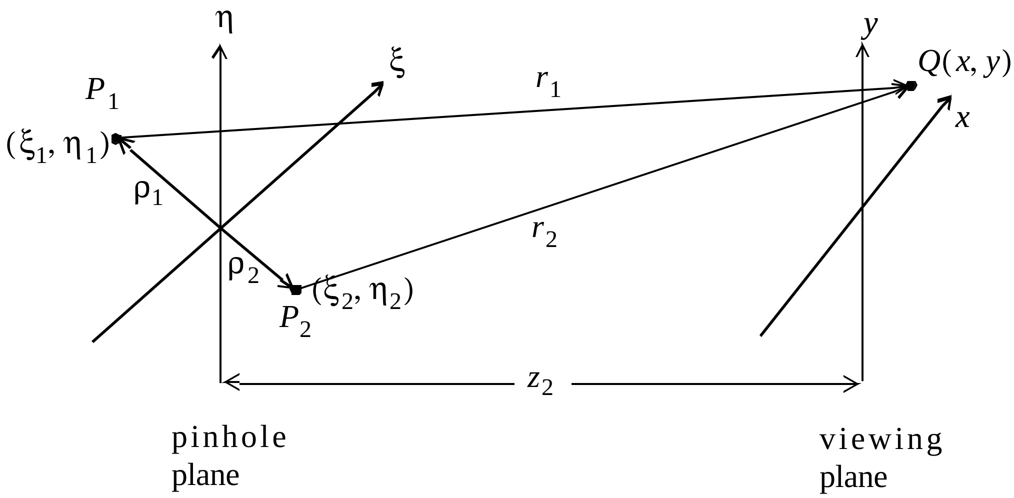
Chapter 10
Spatial Coherence – Part 2

Geometry of the fringe pattern:



$$r_1 = \sqrt{z_2^2 + (\xi_1 - x)^2 + (\eta_1 - y)^2} \tag{10.1}$$

$$r_2 = \sqrt{z_2^2 + (\xi_2 - x)^2 + (\eta_2 - y)^2} \tag{10.2}$$

Define:

$$\rho_1 \equiv \sqrt{\xi_1^2 + \eta_1^2} \qquad \rho_2 \equiv \sqrt{\xi_2^2 + \eta_2^2} \tag{10.3}$$

as the radial distances of the pinholes from the optic axis. The pinhole spacings are given by:

(10.4)

We define the pinhole spacing vector as:

(10.5)

and $\vec{Q} \equiv (x, y)$ is the observation point in x, y plane

Under the paraxial approximation $z_2 \gg |\vec{Q}|, \rho_1, \rho_2$

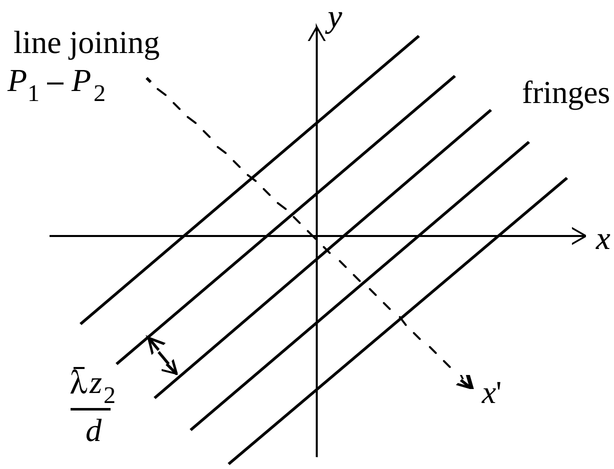
$$r_2 - r_1 \cong \frac{1}{2z_2}[\rho_2^2 - \rho_1^2 - 2\Delta\xi x - 2\Delta\eta y]$$
$$= \frac{1}{2z_2}[\rho_2^2 - \rho_1^2 - 2\overrightarrow{\Delta P} \cdot \vec{Q}] \tag{10.6}$$

Assuming that α_{12} is constant, the fringe modulation has the form

(10.7)

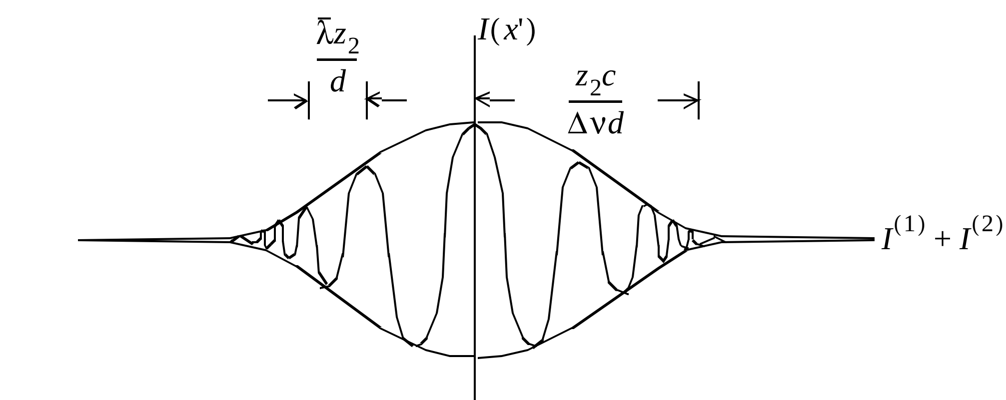
This has the form of a plane wave with a wavevector in the *x*-*y* plane: $\frac{2\pi}{\bar{\lambda} z_2}\overrightarrow{\Delta P}$

Defining $d = |\overrightarrow{\Delta P}|$, as the pinhole spacing, then the fringe period is $\frac{\bar{\lambda} z_2}{d}$. The fringes are straight, and run perpendicular to $\overrightarrow{\Delta P}$.



$$I(Q) = I^{(1)}(Q) + I^{(2)}(Q) + 2\sqrt{I^{(1)}(Q) I^{(2)}(Q)}\gamma_{12}\left(\frac{r_2 - r_1}{c}\right) \cos\left(\frac{2\pi}{\bar{\lambda}}\left[\frac{\rho_2^2 - \rho_1^2}{2z_2} - \frac{\Delta\vec{P}\cdot\vec{Q}}{z_2}\right]\right)$$

Let's look at a cut through the fringe pattern along the x' axis. Assume that $I^{(1)}(Q)$, $I^{(2)}(Q)$ are nearly constant (tiny pinholes) and $\rho_2^2 - \rho_1^2 = 0$ (where the pinholes are equidistant from the axis).



half-width of fringe packet, (similar to Michelson) = $\frac{z_2 c}{\Delta\nu d}$

The total number of fringes observable is roughly:

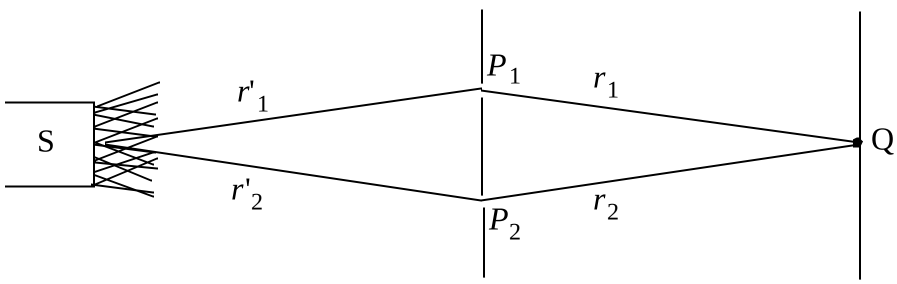
(10.8)

Each fringe represents a path difference of $\bar{\lambda}$]

The coherence length is $c\tau_c \cong \frac{c}{\Delta\nu} = \frac{c}{\bar{\nu}}\frac{\bar{\nu}}{\Delta\nu} = \bar{\lambda}\frac{\bar{\nu}}{\Delta\nu} = \frac{1}{[\text{fractional bandwidth}]} \cdot \bar{\lambda}$

Quasimonochromatic condition

We wish to concentrate on the spatial coherence effects and make the temporal coherence effects negligible. Assume that the light is narrowband. Not just $\Delta\nu \ll \bar{\nu}$, but for all source points and observation points, the path lengths are the same within τ_c.



Thus, for all source points and all Q. Also

$$(r_2 - r_1)/c,\ (r'_2 - r'_1)/c \ll \tau_c \tag{10.9}$$

If these conditions are satisfied, the fringe contrast is constant over the observing region. The coherence function simplifies

(10.10)

(10.11)

The mutual intensity is:

$$\tilde{J}_{12} \equiv \Gamma_{12}(0) = \langle u(P_1, t)u^*(P_2, t)\rangle \tag{10.12}$$

$$\tilde{\mu}_{12} \equiv \tilde{\gamma}_{12}(0) = \frac{J_{12}}{[I(P_1)I(P_2)]^{1/2}} \qquad 0 \le |\tilde{u}_{12}| \le 1 \tag{10.13}$$

Under these conditions, the intensity in the *x-y* plane becomes

$$I(x,y) = I^{(1)} + I^{(2)} + 2\kappa_1\kappa_2 J_{12}\cos\left[\frac{2\pi}{\lambda z_2}\Delta\vec{P}\cdot\vec{Q} + \phi_{12}\right] \tag{10.14}$$

$$= I^{(1)} + I^{(2)} + 2\sqrt{I^{(1)}I^{(2)}}\mu_{12}\cos\left[\frac{2\pi}{\lambda z_2}\Delta\vec{P}\cdot\vec{Q} + \phi_{12}\right]$$

(10.15)

(10.16)

(10.17)

The fringe visibility is given by:

$$V = \frac{2\sqrt{I^{(1)}I^{(2)}}}{I^{(1)} + I^{(2)}}\mu_{12} \tag{10.18}$$

and $V = \mu_{12}$ if $I^{(1)} = I^{(2)}$.

$\mu_{12} = 0$ tells us there are no fringes. The two waves are <u>mutually incoherent</u>.

$\mu_{12} = 1$ tells us that the waves are perfectly correlated, and are <u>mutually coherent</u>.

For $0 < \mu_{12} < 1$ the waves are <u>partially coherent</u>.

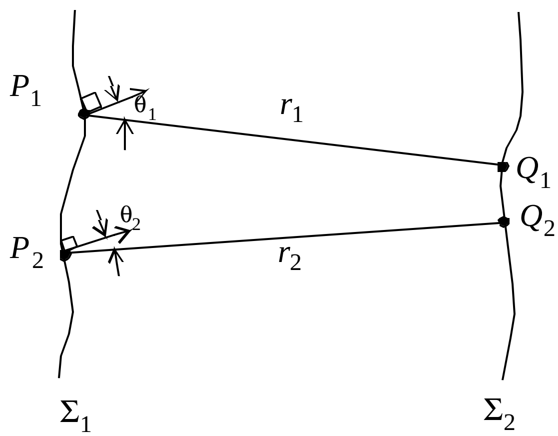
The above cases refer to two particular pinhole points P_1, P_2. This is a limiting case of illumination over the full object plane. Then we are concerned with all the pairs of P_1, P_2 over the full object plane.

Various measures of coherence

Symbol	Definition	Name	Temporal or Spatial
$\Gamma_{11}(\tau)$	$\langle u(P_1, t-\tau)u^*(P_1, t)\rangle$	Self coherence function	Temporal
$\gamma_{11}(\tau)$	$\Gamma_{11}(\tau)/\Gamma_{11}(0)$	Complex degree of self-coherence	Temporal
$\Gamma_{12}(\tau)$	$\langle u(P_1, t-\tau)u^*(P_2, t)\rangle$	Mutual coherence function	Spatial and Temporal

Symbol	Definition	Name	Temporal or Spatial
$\gamma_{12}(\tau)$	$\dfrac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\Gamma_{22}(0)}}$	Complex degree of coherence	Spatial and Temporal
J_{12}	$\Gamma_{12}(0)$	Mutual intensity	Spatial quasimono-chromatic
μ_{12}	$\gamma_{12}(0)$	Complex coherence factor	Spatial quasimono-chromatic

Propagation of Mutual coherence



As light propagates from Σ_1 to Σ_2, the mutual coherence function changes (propagates).

Given $\Gamma(P_1, P_2;\tau)$ for all P_1, P_2 pairs, how to find $\Gamma(Q_1, Q_2;\tau)$?

Start with narrowband light. Later we will specialize to quasi-monochromatic light.

$$\boxed{} \tag{10.19}$$

The generalized Huygens-Fresnel integral relates the fields on Σ_2 to fields on Σ_1.

$$u(Q_1, t+\tau) = \iint_{\Sigma_1} \frac{1}{j\bar{\lambda}r_1} u\left(P_1, t+\tau-\frac{r_1}{c}\right)\cos\theta_1 ds_1 \tag{10.20}$$

$$u^*(Q_2, t) = \iint_{\Sigma_1} \frac{-1}{j\bar{\lambda}r_2} u^*\left(P_2, t-\frac{r_2}{c}\right)\cos\theta_2 ds_2 \tag{10.21}$$

Then

$$\Gamma(Q_1, Q_2, \tau) = \iint_{\Sigma_1} ds_1 \iint_{\Sigma_1} ds_2 \frac{\left\langle u\left(P_1, t + \tau - \frac{r_1}{c}\right) u^*\left(P_2, t - \frac{r_2}{c}\right)\right\rangle}{(\bar{\lambda})^2 r_1 r_2} \cos\theta_1 \cos\theta_2 \qquad (10.22)$$

Thus

$$\Gamma(Q_1, Q_2; \tau) = \iint_{\Sigma_1} ds_1 \iint_{\Sigma_1} ds_2 \Gamma\left(P_1, P_2; \tau + \frac{r_2 - r_1}{c}\right) \frac{\cos\theta_1}{\bar{\lambda} r_1} \frac{\cos\theta_2}{\bar{\lambda} r_2} \qquad (10.23)$$

As a side note, using the generalized Huygens-Fresnel integral for <u>broadband</u> light given earlier, we obtain the propagation law for mutual coherence of broadband light:

$$\Gamma(Q_1, Q_2; \tau) = -\iint_{\Sigma_1} ds_1 \iint_{\Sigma_1} ds_2 \frac{\partial^2}{\partial \tau^2} \Gamma\left(P_1, P_2; \tau + \frac{r_2 - r_1}{c}\right) \frac{\cos\theta_1}{2\pi c r_1} \frac{\cos\theta_2}{2\pi c r_2} \qquad (10.24)$$

Now make a quasimonochromatic approximation; that is, the (maximum difference in path-lengths) << (the coherence of the length of light).

At Σ_2,

(10.25)

The integrand in equation (1) with $\tau = 0$,

(10.26)

$$\Gamma\left(P_1, P_2; \frac{r_2 - r_1}{c}\right) = \tilde{J}(P_1, P_2) \exp\left[-j\frac{2\pi}{\bar{\lambda}}(r_2 - r_1)\right] \qquad (10.27)$$

This leads us to the propagation law for mutual intensity

$$\tilde{J}(Q_1, Q_2) = \iint_{\Sigma_1} ds_1 \iint_{\Sigma_1} ds_2 \tilde{J}(P_1, P_2) \exp\left[-j\frac{2\pi}{\bar{\lambda}}(r_2 - r_1)\right] \frac{\cos\theta_1}{\bar{\lambda} r_1} \frac{\cos\theta_2}{\bar{\lambda} r_2} \qquad (10.28)$$

In general, the intensity distribution can be found by letting Q_1, Q_2 merge in the mutual intensity.

(10.29)

<u>Limiting forms of the coherence function</u>

<u>Fully coherent field in the quasi-monochromatic limit</u>

$\mu_{12} = 1$ for all pairs (P_1, P_2).

This means

$$\frac{|J_{12}|}{[I(P_1)I(P_2)]^{1/2}} = 1 \tag{10.30}$$

$$\frac{|\langle U(P_1,t)U^*(P_2,t)\rangle|}{[\langle|U(P_1,t)|^2\rangle\langle|U(P_2,t)|^2\rangle]^{1/2}} = 1 \tag{10.31}$$

Apply the Schwarz inequality

$$\left|\int f(t)g^*(t)dt\right| \le \left[\int |f(t)|^2 dt \int |g(t)|^2 dt\right]^{1/2} \tag{10.32}$$

There is equality if and only if $g(t) = kf(t)$ with k a complex constant.

Thus $U(P_2,t) = K_{12}U(P_1,t)$ K_{12} depends on P_1, P_2 but <u>not *t*.</u>

Choose some reference point P_o. We can then write

$$U(P_1,t) = U(P_1)\frac{U(P_o,t)}{[I(P_o)]^{1/2}};\quad U(P_2,t) = U(P_2)\frac{U(P_o,t)}{[I(P_o)]^{1/2}} \tag{10.33}$$

Thus

$$\tag{10.34}$$

correspondingly

$$\tilde{\mu}_{12} = \exp\{j[\phi(P_1) - \phi(P_2)]\} \tag{10.35}$$

where

$$\phi(P_1) = \arg[U(P_1)] \quad \phi(P_2) = \arg[U(P_2)] \tag{10.36}$$

<u>The incoherent limit:</u>

For two pinholes $\mu_{12} = 0$ represents incoherent waves. A fully coherent field has $\mu_{12} = 1$ for all pairs (P_1, P_2). A logical extension might be to say that for an <u>incoherent field</u>:

$$|\Gamma_{12}(\tau)| = 0 \quad \text{for all } P_1 \neq P_2 \quad \text{and for all } \tau$$

However, if we examine the consequence of this in equation (1), the first integration over $\Sigma_1 = 0$ except at $P_1 = P_2$ where the value is *finite* (with a finite singularity = $I(P_1)$). Hence, the result of integration is precisely zero.

Thus $\Gamma(Q_1, Q_2;\tau) = 0$ for an incoherent field at Σ_1, but $I(Q_1) = \Gamma(Q_1, Q_2;0) = 0$ also.

An incoherent wave field on Σ_1 does not propagate!

Why is this? An incoherent field in this sense has an infinitesimally fine spatial structure. Recall our discussion of the spatial frequency cutoff. The spatial frequencies $f_{x,y} > \frac{1}{\bar{\lambda}}$ don't propagate. A perfectly incoherent surface *does not* radiate.

For a propagating wavefield, coherence must exist over a linear dimension of at least $\bar{\lambda}$. For the quasimonochromatic case, the mutual intensity representing a propagating incoherent wave field can be represented by:

$$J(P_1, P_2) = \sqrt{I(P_1)I(P_2)}\left[2\frac{J_1(\bar{k}\sqrt{\Delta x^2 + \Delta y^2})}{\bar{k}\sqrt{\Delta x^2 + \Delta y^2}}\right], \tag{10.37}$$

where J_1 is the Bessel function.

This is somewhat cumbersome for calculations. If an optical system has a resolution coarser than $\bar{\lambda}$ at (x, y) plane, the exact shape of $J(P_1, P_2)$ is not important. This will become clearer as we go along.

Under this assumption, it is reasonable to approximate

(10.38)

(10.39)

If coherence extends over more than $\bar{\lambda}$, but is still not resolved, then the δ-function is still valid, but the value of κ changes.

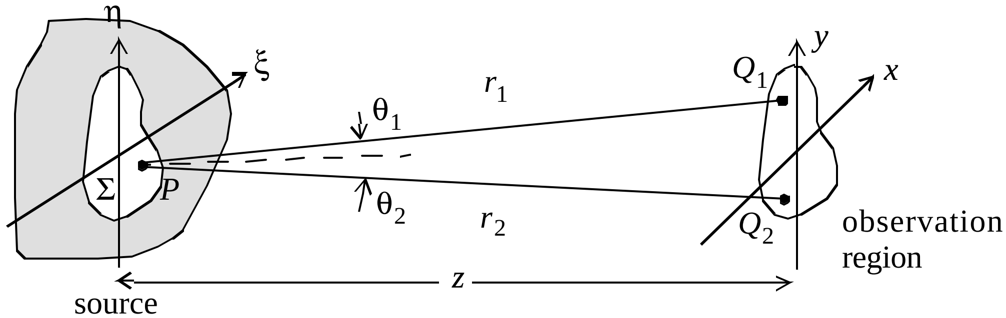
For a wavefield that is incoherent in this sense, the mutual intensity propagates as follows:

$$J(Q_1, Q_2) = \iint_{\Sigma_1}\iint_{\Sigma_1} J(P_1, P_2)\exp\left[-j\frac{2\pi}{\bar{\lambda}}(r_2 - r_1)\right]\frac{\cos\theta_1}{\bar{\lambda}r_1}\frac{\cos\theta_2}{\bar{\lambda}r_2}ds_1 ds_2 \tag{10.40}$$

Using equation (10.38):

$$= \frac{k}{(\bar{\lambda})^2}\iint_{\Sigma} I(P_1)\exp\left[-j\frac{2\pi}{\bar{\lambda}}(r_2 - r_1)\right]\frac{\cos\theta_1}{\bar{\lambda}r_1}\frac{\cos\theta_2}{\bar{\lambda}r_2}ds_1 \tag{10.41}$$

The geometry:



Under the conditions of the Fresnel approximation:

(10.42)

(10.43)

$$r_2 \cong z + \frac{(x_2-\xi)^2+(y_2-\eta)^2}{2z} \tag{10.44}$$

$$r_1 = z + \frac{(x_1-\xi)^2+(y_1-\eta)^2}{2z} \tag{10.45}$$

Then

$$r_2 - r_1 = \frac{1}{2z}[x_2^2+y_2^2-(x_1^2+y_1^2)+2\Delta x\xi+2\Delta y\eta] \tag{10.46}$$

(the pure ξ^2, η^2 terms cancel). Let:

(10.47)

$$J(\Delta x, \Delta y) = \frac{ke^{-j\psi}}{(\bar{\lambda}z)^2}\int\int_{-\infty}^{\infty} I(\xi,\eta)\exp\left[j\frac{2\pi}{\bar{\lambda}z}(\Delta x\xi+\Delta y\eta)\right]d\xi d\eta \tag{10.48}$$

$I(\xi, \eta)$ is defined so that it is zero outside Σ.

(10.49)

<u>Van Cittert - Zernike theorem</u>

The normalized form of the propagation law gives the complex coherence factor:

$$\tilde{\mu}(\Delta x, \Delta y) = \frac{e^{-j\Psi}\int\int_{-\infty}^{\infty} I(\xi, \eta)\exp\left[j\frac{2\pi}{\bar{\lambda}z}(\Delta x\xi + \Delta y\eta)\right]d\xi d\eta}{\int\int_{-\infty}^{\infty} I(\xi, \eta)\, d\xi\, d\eta} \tag{10.50}$$

Interpretation

Aside from the phase factor $e^{-j\Psi}$,

- $J(\Delta x, \Delta y)$ is found from a two-dimensional Fourier Transform of intensity $I(\xi, \eta)$ across the source.
- The mutual intensity is analogous to Fraunhofer diffraction pattern of source aperture.
- But it *is* valid in the *Fresnel* region.
- Modulus of the coherence factor $|\tilde{\mu}|$ depends only on coordinate differences. Width of $|\tilde{\mu}|$ defines a coherence area A_c. We define this by analogy to τ_c.

(10.51)

- The coherence area grows with the distance z from the source like divergence of a diffraction pattern.

(10.52)

where A_s is the source area

Example

A circular source, with radius a, which is uniformly bright, quasimonochromatic, and incoherent.

$$I(\xi, \eta) = I_o \operatorname{circ}\left(\frac{\sqrt{\xi^2 + \eta^2}}{a}\right) \tag{10.53}$$

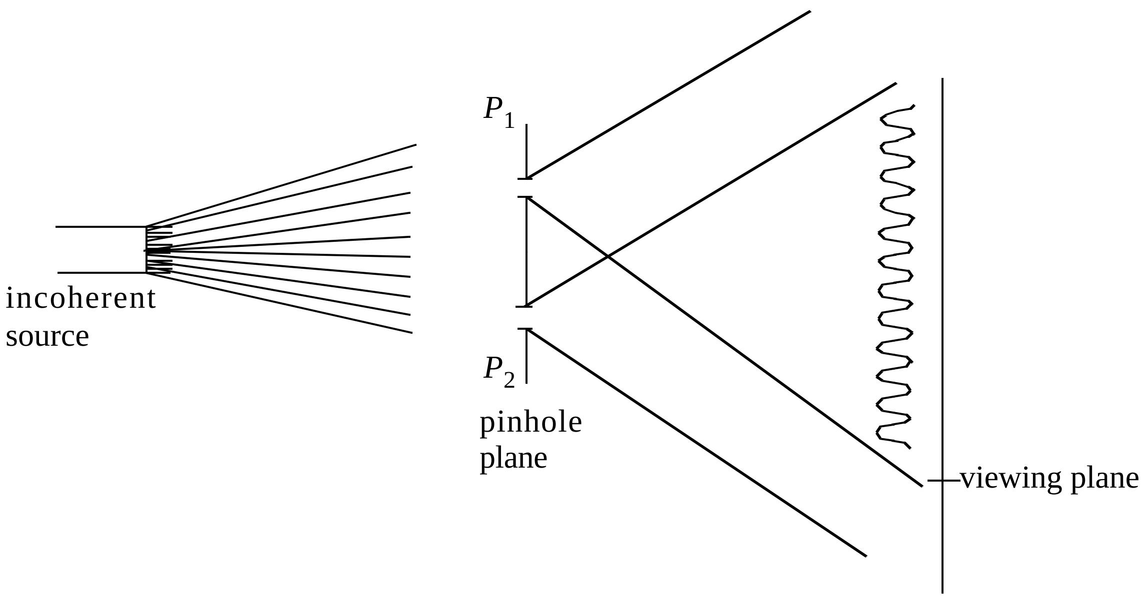
Then the mutual intensity has the form of an Airy pattern

$$J(x,y;x_2, y_x) = \frac{\pi a^2 I_o \kappa}{(\bar{\lambda}z)^2} e^{-j\Psi}\left[2\frac{J_1\left(\frac{2\pi a}{\bar{\lambda}z}\sqrt{\Delta x^2 + \Delta y^2}\right)}{\frac{2\pi a}{\bar{\lambda}z}\sqrt{\Delta x^2 + \Delta y^2}}\right] \tag{10.54}$$

$$\tilde{\mu}(x_1, y_1; x_2, y_2) = e^{-j\Psi} 2\left[\frac{J_1\left(\frac{2\pi a}{\bar{\lambda} z}\right)\sqrt{\Delta x^2 + \Delta y^2}}{\frac{2\pi a}{\bar{\lambda} z}\sqrt{\Delta x^2 + \Delta y^2}}\right] \tag{10.55}$$

Recall that $\psi = \frac{\pi}{\bar{\lambda} z}[(x_2^2 + y_2^2) - (x_1^2 + y_1^2)]$

Note: this is a coherence factor, not a field or intensity. It relates to coherence at 2 points. It can then be used to predict the result of the interference experiment.



The Van Cittert - Zernike theorem gives the coherence between P_1, P_2 as a function of the distance between P_1, P_2. In turn we can then predict the fringe visibility at the viewing plane. Also, a careful measurement of the fringe phase and visibility gives $\bar{\mu}(x_1, y_1; x_2, y_2)$ at the pinhole plane. Young's experiment can be used to measure the coherence.

For an incoherent source, what is the intensity of the pattern at the pinhole plane?

(10.56)

Ideally the incoherent source has a fine spatial structure of $\sim\bar{\lambda}$. This diffracts out to all angles.