Chapter 11
Generalized Van Cittert - Zernike theorem for partial coherence

We now extend the analysis to partially coherent sources. We assume:

- There is a small but nonzero coherence area on the source.
- The coherence factor of the source depends *only* on the coordinate differences.
- The source size and spatial structure in the source is large compared to the coherence area.

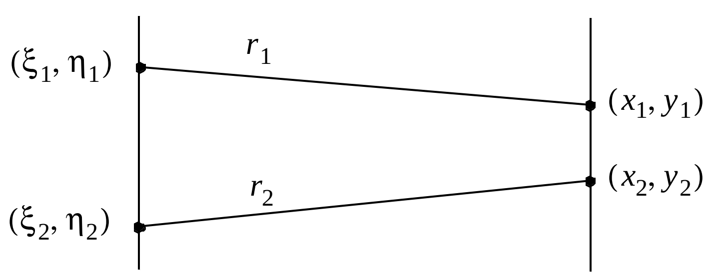
Under these assumptions, the source mutual intensity is:

(11.1)

where $\bar{\xi} = \dfrac{\xi_1 + \xi_2}{2}$ $\quad \bar{\eta} = \dfrac{\eta_1 + \eta_2}{2}$

We now use the general propagation law for mutual intensity, with paraxial conditions

$$\tilde{J}(x_1, y_1; x_2, y_2) = \frac{1}{(\bar{\lambda} z)^2} \int\int_{-\infty}^{\infty}\int\int J(\xi_1, \eta_1; \xi_2, \eta_2) \exp\left[-j\frac{2\pi}{\bar{\lambda}}(r_2 - r_1)\right] d\xi_1 d\eta_1 d\xi_2 d\eta_2 \qquad (11.2)$$



previously, $(\xi_1, \eta_1)(\xi_2, \eta_2)$ points merged due to an idealized incoherent δ-function approximation.

Now, with a little algebra, we can write

$$r_2 - r_1 \cong \frac{1}{z}[\bar{x}\Delta x + \bar{y}\Delta y + \bar{\xi}\Delta\xi + \bar{\eta}\Delta\eta - \Delta x\bar{\xi} - \bar{x}\Delta\xi - \Delta y\bar{\eta} - \bar{y}\Delta\eta] \qquad (11.3)$$

A small coherence area means $\tilde{\mu}$ is only appreciable for small $\Delta\xi, \Delta\eta$. We can drop the $\Delta\eta, \Delta\xi$ terms for

$$z > 4\frac{\bar{\xi}\Delta\xi}{\bar{x}} \qquad z > 4\frac{\bar{\eta}\Delta\eta}{\bar{\lambda}}$$

We can then write

$$J(x_1, y_1; x_2, y_2) = \frac{e^{-j\psi}}{(\bar{\lambda}z)^2}\int_{-\infty}^{\infty}\int I(\bar{\xi}, \bar{\eta})\exp\left[j\frac{2\pi}{\bar{\lambda}z}(\Delta x\bar{\xi} + \Delta y\bar{\eta})\right]d\bar{\xi}d\bar{\eta} \tag{11.4}$$

$$\times\int_{-\infty}^{\infty}\int \mu(\Delta\xi, \Delta\eta)\exp\left[j\frac{2\pi}{\bar{\lambda}z}(\bar{x}\Delta\xi + \bar{y}\Delta\eta)\right]d\Delta\xi\, d\Delta\eta$$

(11.5)

same as before. The second integral depends on $\bar{x}, \bar{y}$

$$\kappa(\bar{x}, \bar{y}) = \int_{-\infty}^{\infty}\int \mu(\Delta\xi, \Delta\eta)\exp\left[j\frac{2\pi}{\bar{\lambda}z}(\bar{x}\Delta\xi + \bar{y}\Delta\eta)\right]d\Delta\xi\, d\Delta\eta \tag{11.6}$$

This is just the Fourier Transform of μ. It gives the coarse variation of the average intensity.

$$J(x_1, y_1; x_2, y_2) = \frac{\kappa(\bar{x}, \bar{y})}{(\bar{\lambda}z)^2}e^{-j\psi}\int_{-\infty}^{\infty}\int I(\bar{\xi}, \bar{\eta})\exp\left[j\frac{2\pi}{\bar{\lambda}z}(\Delta x\bar{\xi} + \Delta y\bar{\eta})\right]d\bar{\xi}d\bar{\eta} \tag{11.7}$$

This looks just like the incoherent case, except that κ varies in the *(x, y)* plane.

Since μ is narrow, κ is broad. The integral is narrow in $\Delta x, \Delta y$. When $\Delta x = \Delta y = 0$ we recover the diffraction pattern described by κ.

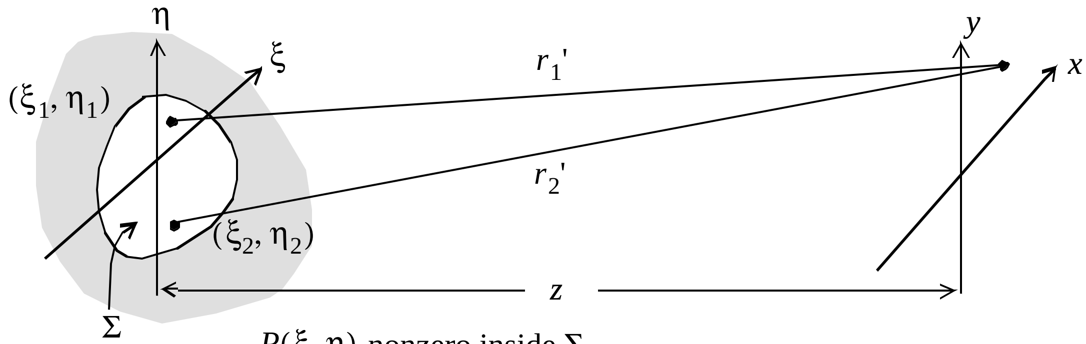
In the far-field:

- The source size in the (ξ, η) plane determines the coherence area.
- The source coherence area in $(\Delta\xi, \Delta\eta)$ determines the intensity distribution.

Here, our small coherence area approximation is valid under the condition: $z > 2\frac{Dd_c}{\bar{\lambda}}$, where D is the dimension of source and d_c is the dimension of the coherence area of the source. This is just the geometric mean of the far field distances for D, d_c. (Recall that $z_{\text{far-field}} > \pi\frac{D^2}{\bar{\lambda}}$.)

<u>Diffraction revisited for a partially coherent light</u>

What is the diffraction pattern from an aperture when the illumination is partially coherent?



Usual geometry $P(\xi, \eta)$ nonzero inside Σ.

Aperture amplitude transmittance $P(\xi, \eta)$ (later this will be our pupil function).

<u>Effect of an aperture on mutual intensity:</u>

The effect of an aperture on the field is:

(11.8)

τ_o : any possible time delay through diffracting structure

Then

$$J_t(\xi_1, \eta_1; \xi_2, \eta_2) \equiv \langle U_t(\xi_1, \eta_1; t) U^*_t(\xi_2, \eta_2; t) \rangle \quad (11.9)$$
$$= P(\xi_1, \eta_1) P^*(\xi_2, \eta_2) \langle U_i(\xi_1, \eta_1; t - \tau_o) U^*_i(\xi_2, \eta_2; t - \tau_o) \rangle$$

$$J_t(\xi_1, \eta_1; \xi_2, \eta_2) = P(\xi_1, \eta_1) P^*(\xi_2, \eta_2) J_i(\xi_1, \eta_1; \xi_2, \eta_2) \quad (11.10)$$

Under similar approximations to those used to obtain the general Van Cittert - Zernike theorem:

(11.11)

Thus

$$J_i(\xi_1, \eta_1; \xi_2, \eta_2) = P(\xi_1, \eta_1) P^*(\xi_2, \eta_2) I(\bar{\xi}, \bar{\eta}) \tilde{\mu}_i(\Delta\xi, \Delta\eta) \quad (11.12)$$

$$= P\left(\bar{\xi} - \frac{\Delta\xi}{2}, \bar{\eta} - \frac{\Delta\eta}{2}\right) P^*\left(\bar{\xi} + \frac{\Delta\xi}{2}, \bar{\eta} + \frac{\Delta\eta}{2}\right) I(\bar{\xi}, \bar{\eta}) \tilde{\mu}_i(\Delta\xi, \Delta\eta) \quad (11.13)$$

This depends on both $(\bar{\xi}, \bar{\eta})$ and $(\Delta\xi, \Delta\eta)$ variables. It is not a <u>simple</u> <u>product</u> separation.

Using the general Van Cittert - Zernike in the paraxial approximation:

$$J(x_1, y_1; x_2, y_2) = \frac{e^{-j\psi}}{(\bar{\lambda} z)^2} \int\int_{-\infty}^{\infty} \int\int d\Delta\xi\, d\Delta\eta\, d\bar{\xi}\, d\bar{\eta}\, P\left(\bar{\xi} - \frac{\Delta\bar{\xi}}{2}, \bar{\eta} - \frac{\Delta\bar{\eta}}{2}\right) P^*\left(\bar{\xi} + \frac{\Delta\bar{\xi}}{2}, \bar{\eta} + \frac{\Delta\bar{\eta}}{2}\right)$$
$$\times I(\bar{\xi}, \bar{\eta}) \tilde{\mu}(\Delta\xi, \Delta\eta) \exp\left[j\frac{2\pi}{\bar{\lambda} z}(\Delta x \bar{\xi} + \Delta y \bar{\eta} + \bar{x}\Delta\xi + \bar{y}\Delta\eta)\right] \quad (11.14)$$

(11.15)

To get the intensity of the diffraction pattern, we set $x_1 = x_2$, $y_1 = y_2$.

$$\Delta x \to 0 \qquad \Delta y \to 0 \qquad \psi \to 0$$

$$I(x, y) = \frac{1}{(\bar{\lambda} z)^2} \int\int_{-\infty}^{\infty} \mathcal{P}(\Delta\xi, \Delta\eta) \tilde{\mu}_i(\Delta\xi, \Delta\eta) \exp\left[j\frac{2\pi}{\bar{\lambda} z}(x\Delta\xi + y\Delta\eta)\right] d\Delta\xi\, d\Delta\eta \quad (11.16)$$

where

$$\mathcal{P}(\Delta\xi, \Delta\eta) = \int\int_{-\infty}^{\infty} I(\bar{\xi}, \bar{\eta}) P\left(\bar{\xi} - \frac{\Delta\xi}{2}, \bar{\eta} - \frac{\Delta\eta}{2}\right) P^*\left(\bar{\xi} + \frac{\Delta\xi}{2}, \bar{\eta} + \frac{\Delta\eta}{2}\right) d\bar{\xi}\, d\bar{\eta} \quad (11.17)$$

If $I(\bar{\xi}, \bar{\eta})$ is constant over the aperture, then $\mathcal{P}$ is just the autocorrelation function of the complex pupil function.

(11.18)

The intensity distribution is a Fourier Transform of $\mathcal{P}(\Delta\xi, \Delta\eta)\tilde{\mu}_i(\Delta\xi, \Delta\eta)$. Note the valid range on z is $z > 2\frac{Dd_c}{\bar{\lambda}}$, assuming that $d_c < D$. However, if $d_c > D$ (as with coherent illumination), then

$$z > 2\frac{D^2}{\bar{\lambda}} \qquad \text{(same as Fraunhofer)}$$

<u>Interpretation</u>

Check the coherent limit. We should get the Fraunhofer formula. Full coherence: $\tilde{\mu} = 1$.

Then

(11.19)

<u>checks</u>

Now let's check for nearly incoherent illumination, where the coherence area << aperture area $(d_c \ll D)$. Then $\tilde{\mu}(\Delta\xi, \Delta\eta)$ is sharply peaked near the origin.

Near $(\Delta\xi, \Delta\eta) = (0, 0)$, P has its maximum value of $I_o A$.

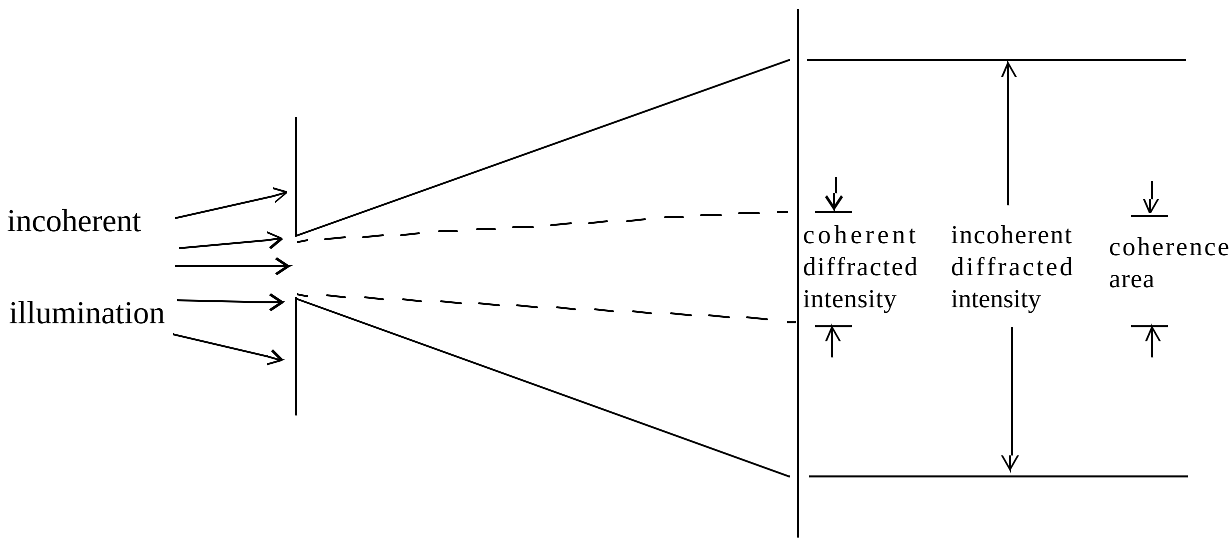
$$I(x, y) \cong \frac{I_o A}{(\bar{\lambda} z)^2} \int_{-\infty}^{\infty}\int \mu_i(\Delta\xi, \Delta\eta) \exp\left[j\frac{2\pi}{\bar{\lambda} z}(\Delta\xi x + \Delta\eta y)\right] d\Delta\xi d\Delta\eta \tag{11.20}$$

<u>The intensity of the diffraction pattern is a Fourier Transform of the mutual coherence function.</u>

- Since $d_c \ll D$, the light diffracts more rapidly, with a larger diffraction angle than for coherent illumination

- In fact, the divergence is controlled only by d_c in this limit. The aperture in this case has a negligible effect!

- Recall the result from the Van Cittert - Zernike theorem. The coherence area of diffracted light in a far-field is related to the Fourier Transform of the <u>aperture</u>.



Now we consider the intermediate case: <u>partially coherent illumination</u>

Both P and $\tilde{\mu}$ play a role in determining the shape of $I(x, y)$. $I(x, y)$ is determined by the convolution of the transforms of P and μ.

$$I(x, y) = \frac{I_o}{(\bar{\lambda} z)^2} F[P \cdot \tilde{\mu}] = \frac{I_o}{(\bar{\lambda} z)^2} F[P] \otimes F[\tilde{\mu}] \tag{11.21}$$

(11.22)

Thus, the Fraunhofer diffraction pattern is convolved with $F[\tilde{\mu}]$.

As the coherence area is reduced, the diffraction pattern gets smoothed and broadened out.