CS–172 Computability & Complexity, Spring 2009
Feb. 26, 2009

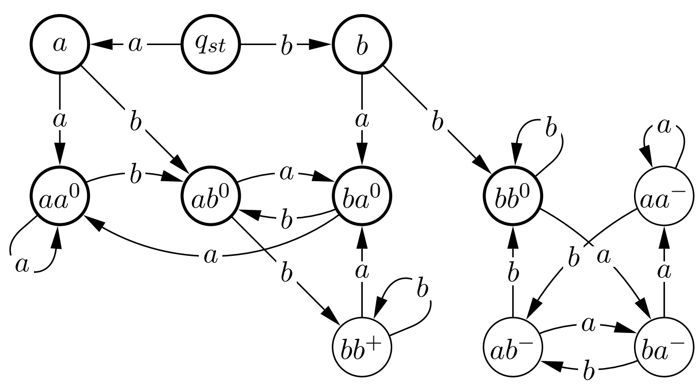
Some Homework 3 Solutions

Note: These solutions are not necessarily model answers. Rather, they are designed to be tutorial in nature, and sometimes contain a little more explanation than an ideal solution. Also, bear in mind that there may be more than one correct solution.

1. (a) Not regular. Proof by contradiction: assume that $L = \{w : w$ has balanced parentheses$\}$ is regular. Let n be the constant guaranteed to exist by the pumping lemma. Consider the string $w = (^n)^n$ — i.e., n ('s followed by n)'s. Clearly w has balanced parentheses, so $w \in L$. Thus, since $|w| \geq n$, by the pumping lemma we must be able to write $w = xyz$ with $|xy| \leq n$, $|y| \geq 1$, and such that $xy^iz \in L$ for all $i \geq 0$. However, since w starts with n ('s, y must consist entirely of one or more ('s. Therefore, for any $i > 1$, $xy^iz \notin L$ since it has more ('s than)'s. This is a contradiction, so L is not regular.

(b) Regular. The key observation here is that successive occurrences of abb and of bba in any string over $\{a, b\}$ must *alternate* along the string. To see this, one can show that in any string w, between any two occurrences of abb there is an occurrence of bba and vice versa. Consider an arbitrary substring of w delimited by two occurrences of abb. This string has the form $abbuabb$, where u is a possibly empty string. If u contains no a symbols, then the string $bbua$ ends in bba. Otherwise, suppose that the first a in u occurs at position i; then the string $bbu_1 \dots u_i$ ends in bba. For the other direction, again consider an arbitrary substring of w delimited by two occurrences of bba. Then the *reversal* w^R of w has the form $abbuabb$ for some string u. By the above argument, w^R must contain bba as a substring, so w itself contains an occurrence of abb.

For a string w, let $D(w)$ denote the difference between the number of occurrences of abb and of bba in w. By the above argument, for any w, $|D(w)| \leq 1$. At this point it is not difficult to see what a DFA for our language should look like. The states should keep track of the last two symbols seen, as well as the sign of the quantity $D(w)$. (See diagram; the start state is q_{st}, and the accept states are marked in thicker lines.)



[*The key point in this question is to observe that occurrences of abb and bba must alternate along the string.*]

(e) Not regular. Assume that this language is regular, and let n be any number bigger than both 2 and the pumping length. Consider the string $w = a^n b^n$; this contains $n-2$ copies of aaa and $n-2$ copies of bbb, so it is in the language. By the pumping lemma there exists a split $w = xyz$ such that $|xy| \leq n$, $y \neq \epsilon$ and xy^2z is also in the language. However, whenever x and y satisfy the first two conditions, the string xy^2z will be of the form $a^m b^n$, for some $m > n$. This string has more copies of aaa than bbb, so it cannot be in the language. Contradiction.

(f) Not regular. We do a proof by contradiction using closure properties. (Note that it apparently isn't possible to use the pumping lemma directly here, because we'd have to show that any possible pumping of a substring y leads to a string that's *not* in L, which is hard as L is not very tightly constrained.)

So assume $L = \{0^i 1^j : i, j \geq 0 \text{ and } i \neq j\}$ is regular. Since regular languages are closed under complementation, the complement $\overline{L}$ is also regular. Now consider the language $L' = \{0^i 1^j : i, j \geq 0\}$, which is certainly regular (it is denoted by the regular expression $\mathtt{0}^*\mathtt{1}^*$). Since regular languages are also closed under intersection, $\overline{L} \cap L'$ must also be regular. However, $\overline{L} \cap L' = \{0^i 1^i : i \geq 0\}$, which we know is *not* regular (as we saw in class, by the same argument we used to show that the set of 0-1 strings with equal numbers of 0's and 1's is not regular). Therefore we have a contradiction, so we deduce that L itself must not be regular.

An alternative argument for this part, using the Myhill-Nerode Theorem, goes as follows. We show that the relation $\sim_L$ splits $\{0,1\}^*$ into infinitely many equivalence classes, which implies that L is not regular. Indeed, consider the collection of strings $C = \{0^n : n \geq 0\}$. We claim that all strings in C are in distinct equivalence classes. For suppose that there exists $m \neq n$ such that $0^m \sim_L 0^n$. Then, by the definition of $\sim_L$, $0^m 1^n \sim_L 0^n 1^n$. But this is impossible since $0^m 1^n \in L$ while $0^n 1^n \notin L$.

2. (a) Since L is regular, there is a DFA M that accepts it. Modify the DFA in the following way: take all outgoing edges from all the accepting states (including self-loops) and reroute them to point to a dead state. We claim that the resulting DFA M' decides $\min(L)$. To see this, note that M' certainly cannot accept any string that is not accepted by M. And a string w is accepted by M' iff the accepting computation of M on w does not pass through any intermediate accepting states. But this latter condition corresponds precisely to saying that no proper prefix of w is accepted by M, as required.

(b) (not *) This language is finite and therefore regular, since every finite language is regular. (To see this, just write down a regular expression that takes the union of the singleton strings.) However, note that given a FA for L we do *not* in general know how to construct a FA for L_{10}: we know only that such an FA exists. Thus, unlike parts (a) and (c) of this problem, this proof is *not* constructive.

(d) (coming)

4 (b) False. E.g., let $\Sigma = \{0,1\}$, $L = \{0^n 1^n : n \geq 0\}$, and $L' = \{0^m 1^n : m \neq n\}$. Then L and L' are both non-regular. However $L_1 \cap L_2 = \varnothing$, which is regular.

(c) False. For a counterexample, let L be any non-regular language (e.g., $L = \{0^i 1^i : i \geq 0\}$). Then we can write $L = \bigcup_{i=1}^{\infty} L_i$, where each L_i consists just of the ith string in L in lexicographic order. Clearly each L_i is finite and hence regular. However, the union of all of the L_i is L, which is not regular.