CS172 Computability & Complexity (Spring'09)

Instructor: Mihai Pătraşcu   GSI: Omid Etesami

$\boxed{\text{Sample Exam}}$ *1.5 hours*

All electronics must be off. You may bring one sheet of paper (double-sided, letter size) containing any notes you care to use. No other materials allowed. You may answer questions in any order.

To prove that a language is **NP**-hard, you may only reduce from **NP**-complete languages we discussed in class (SAT, 3SAT, circuitSAT, Set Cover, Subset Sum, Directed Hamiltonian Path).

1. Write your name, email, and student ID on top of all sheets of paper. Make sure your cell phone (or other device) will not produce sound during the exam.

2. You are the manager of a company. Your n employees are on strike. Each one is making a set of demands, out of m possible demands ("free coffee!"; "2-hour lunch breaks!"; etc). For any demand, you know the dollar cost C_i of meeting it. Each employee will agree to end the strike if at least one of his own demands are met (he feels he has won something). You goal is to stop the strike with minimal cost.

(a) Formulate a language L that "asks" whether the strike can be stopped with cost at most x. Show that $L \in$ **NP**.

(b) Show that L is **NP**-hard.

3. The Socialist Travelling Salesman (STS) is considerably less efficient than the capitalist version: he does not need to complete all work, just pretend to work until the day is over; and he can visit a customer multiple times.

Formally, the salesman has a complete directed graph G with n customers, with weights on the edges representing the cost of a bus ticket between the two customers. The salesman must visit k customers, after which the work day is over. After seeing one customer, he must travel to a different customer; however, he may return to any customer as many times as he likes. The salesman wishes to visit k customers paying as little money on bus fares as possible.

(a) Give a language L to formalize the question "can the salesman spend at most d dollars on tickets?" Show that $L \in$ **NP**.

(b) Show that $L \in$ **P** by dynamic programming (note: L is always in **P**, regardless of the size of the dollar figures).

(c) Assume the salesman is not allowed to visit the same customer twice. Show that the problem is **NP**-complete. Your reduction should only use costs in $\{1, 2\}$.

In this problem, we saw that sometimes the size of the numbers is irrelevant: with repeated visits allowed, the problem is poly-time for any costs; without repeated visits, it is **NP**-complete even if the costs are $O(1)$.

4. Consider the function $H : \mathbb{N} \to \mathbb{N}$ defined as:

$H(n) =$ the number of Turing Machines with exactly n states which halt when started with an empty tape

Show that $H(\cdot)$ is uncomputable. (Hint: First show that it is undecidable whether a given machine halts on the empty tape.)