Assignment #2

Point Value: 15 points

Due Date: Sept. 8th, 5:00 pm

CS 184: Foundations of Computer Graphics
Fall 2009
Prof. James O'Brien

The purpose of this assignment is to test your prerequisite math background. In general, you should find this assignment to be fairly easy. This assignment must be done solo. To submit, please slide under Prof. O'Brien's door.

Introduction: Some of these questions may look a little difficult. If they do, consider reviewing how dot products and cross products are formed, how matrices are multiplied with vectors and with each other, how determinants are formed and what eigenvalues are. This exercise could take a very long time if you don't think carefully about each problem, but good solutions will use one or two lines per question. Many textbooks supply expressions for the area of a triangle and for the volume of a pyramid, which you may find very useful.

You must write neatly. If the assignment cannot be read, it cannot be graded. If you find that you make a mess as you work out the solution, then you should use scrap paper for working it out and copy it down neatly for turning in. If you are feeling particularly intrepid, consider formatting your solutions with LaTeX. If you are feeling particularly perverse, consider formatting your solutions using Microsoft' Word's equation editor.

I know that many of you have access to solutions to this, or a very similar assignment from previous years. To be honest, if you need to cheat on this assignment then you should just go drop the class now.

Question 1: Two vectors in the plane, $\mathbf{i}$ and $\mathbf{j}$, have the following properties ($\mathbf{i} \cdot \mathbf{i}$ means the dot product between $\mathbf{i}$ and $\mathbf{i}$): $\mathbf{i} \cdot \mathbf{i} = 1$, $\mathbf{i} \cdot \mathbf{j} = 0$, $\mathbf{j} \cdot \mathbf{j} = 1$.

1. Is there a vector $\mathbf{k}$, that is not equal to $\mathbf{i}$, such that: $\mathbf{k} \cdot \mathbf{k} = 1$, $\mathbf{k} \cdot \mathbf{j} = 0$? What is it? Are there many vectors with these properties?
2. Is there a vector $\mathbf{k}$ such that: $\mathbf{k} \cdot \mathbf{k} = 1$, $\mathbf{k} \cdot \mathbf{j} = 0$, $\mathbf{k} \cdot \mathbf{i} = 0$? Why not?
3. If $\mathbf{i}$ and $\mathbf{j}$ were vectors in 3D, how would the answers to the above questions change?

Question 2: For three points on the plane (x_1, y_1), (x_2, y_2) and (x_3, y_3) show that the determinant of

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

is proportional to the area of the triangle whose corners are the three points. If these points lie on a straight line, what is the value of the determinant? Does this give a useful test to tell whether three points lie on a line? Why do you think so?

Question 3: The equation of a line in the plane is $ax + by + c = 0$. Given two points on the plane, show how to find the values of a, b, c for the line that passes through those two points. You may find the answer to question 2 useful here.

Question 4: Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$. Show that if $\{i, j, k\}$ is $\{1, 2, 3\}$, $\{2, 3, 1\}$, or $\{3, 1, 2\}$, then $\mathbf{e}_i \times \mathbf{e}_j = \mathbf{e}_k$, where $\times$ is the cross product. Now show that if $\{i, j, k\}$ is $\{1, 3, 2\}$, $\{3, 2, 1\}$ or $\{2, 1, 3\}$, then $\mathbf{e}_i \times \mathbf{e}_j = -\mathbf{e}_k$.

Question 5: For four points in space (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) show that the determinant of

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is proportional to the volume of the tetrahedron whose corners are the four points. If these points lie on a plane, what is the value of the determinant? Does this give a useful test to tell whether four given points lie on a plane? Why do you think so?

Question 6: The equation of a plane in space is $ax + by + cz + d = 0$. Given three points in space, show how to find the values of a, b, c, d for the plane that passes through those three points. You may find the answer to question 4 useful here.

Question 7: Let $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$ be three distinct points in space. Now consider the cross product $\mathbf{n} = (\mathbf{p}_0 - \mathbf{p}_1) \times (\mathbf{p}_0 - \mathbf{p}_2)$. What does this vector mean geometrically? Let $\mathbf{p}$ be any point on the plane formed by $\mathbf{p}_0$, $\mathbf{p}_1$ and $\mathbf{p}_2$; what is the geometric relationship between $\mathbf{n}$ and $\mathbf{p} - \mathbf{p}_0$ (look at the dot product)? Show that, if $\mathbf{p} = (x, y, z)$ the equation of the plane must be $\mathbf{n} \cdot (\mathbf{p} - \mathbf{p}_0) = 0$.

Question 8: Suppose that $\mathbf{A}$ is a square matrix, and its inverse and transpose exist, and are equal; a matrix with these properties is called an *orthonormal* matrix. Show that, for any angle θ, the matrix $\mathbf{M}(\theta)$, defined by the array below, is orthonormal.

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Show also that $\mathbf{M}(\theta_1 + \theta_2) = \mathbf{M}(\theta_1)\mathbf{M}(\theta_2)$; use this to argue that the inverse of $\mathbf{M}(\theta)$ is $\mathbf{M}(-\theta)$. Finally, for the point $\mathbf{p} = (0, 1)$, what geometrical figure is given by taking the points given by $\mathbf{M}(\theta)\mathbf{p}$ for every possible value of θ?

Question 9: You are given four vectors in the plane, $\mathbf{x}_1$ and $\mathbf{x}_2$, $\mathbf{b}_1$ and $\mathbf{b}_2$, and you are told that there is a matrix $\mathbf{M}$ such that $\mathbf{M}\mathbf{x}_1 = \mathbf{b}_1$ and $\mathbf{M}\mathbf{x}_2 = \mathbf{b}_2$. Now if you are given a vector $\mathbf{x}_3$, how do you determine $\mathbf{M}\mathbf{x}_3$? (hint: show how to find out what $\mathbf{M}$ is from the data, and then apply it to $\mathbf{x}_3$).

Question 10: Write down the parametric equation of the points on a line that passes through two points in space, $\mathbf{p}_1$ and $\mathbf{p}_2$; now write down the parametric equation of the points on a plane that contains these two points and a third point, $\mathbf{p}_3$. Show how to use these parametric equations and some inequalities on the parameters to specify (a) all the points on the line that lie between $\mathbf{p}_1$ and $\mathbf{p}_2$ and (b) all the points on the triangle whose vertices are $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$.

Question 11: $\mathbf{M}$ is a symmetric real 2x2 matrix. It has two eigenvalues, λ_0 and λ_1.

- If both eigenvalues are positive, show that, for any 2D vector $\mathbf{x}$, $0 \leq \mathbf{x}^T\mathbf{M}\mathbf{x} \leq (\max(\lambda_0, \lambda_1))\mathbf{x}^T\mathbf{x}$.
- If one eigenvalue is zero, show that there is some vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{M}\mathbf{x} = \mathbf{0}$.
- If one eigenvalue is positive and one is negative, show that there is some vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x}^T\mathbf{M}\mathbf{x} = 0$
- A matrix $\mathbf{M}$ is **positive definite** if, for any vector $\mathbf{x}$, $\mathbf{x}^T\mathbf{M}\mathbf{x} > 0$. Assume that $\mathbf{M}$ is symmetric, real, and positive definite. What can you say about its eigenvalues?

Question 12: $\mathbf{M}$ is a 2x2 matrix.

- Show that the eigenvalues of $\mathbf{M}$ are the roots of the equation $\lambda^2 - \mathrm{trace}(\mathbf{M})\lambda + \det(\mathbf{M}) = 0$.