

CS-184: Computer Graphics

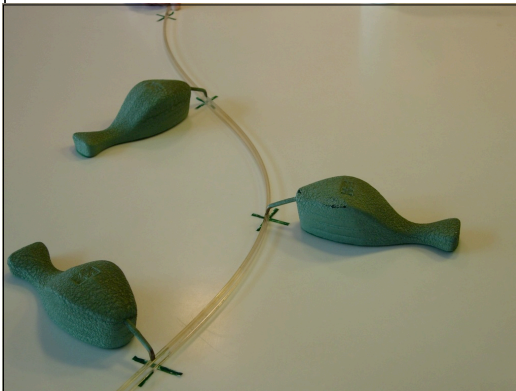
Lecture #13: Natural Splines, B-Splines, and NURBS

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V2009-F-13-1.0

Natural Splines

- Draw a “smooth” line through several points



A real draftsman's spline.

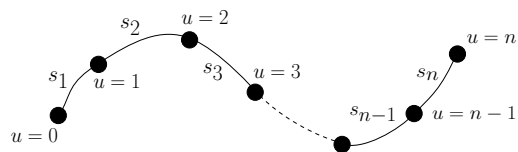
Image from Carl de Boor's webpage.

Natural Cubic Splines

- Given $n + 1$ points
 - Generate a curve with n segments
 - Curves passes through points
 - Curve is C^2 continuous
- Use cubics because lower order is better...

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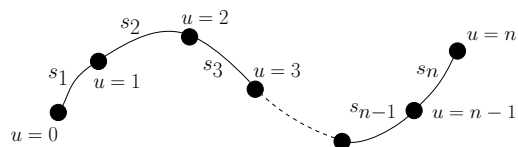
Natural Cubic Splines



$$\mathbf{x}(u) = \begin{cases} \mathbf{s}_1(u) & \text{if } 0 \leq u < 1 \\ \mathbf{s}_2(u-1) & \text{if } 1 \leq u < 2 \\ \mathbf{s}_3(u-2) & \text{if } 2 \leq u < 3 \\ \vdots & \\ \mathbf{s}_n(u-(n-1)) & \text{if } n-1 \leq u \leq n \end{cases}$$

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Natural Cubic Splines



$$s_i(0) = p_{i-1} \quad i = 1 \dots n$$

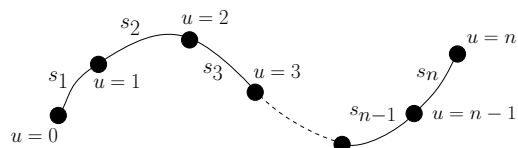
$$s_i(1) = p_i \quad i = 1 \dots n$$

$$s'_i(1) = s'_{i+1}(0) \quad i = 1 \dots n - 1$$

$$s''_i(1) = s''_{i+1}(0) \quad i = 1 \dots n - 1$$

$$s''_1(0) = s''_n(1) = 0$$

Natural Cubic Splines



$$s_i(0) = p_{i-1} \quad i = 1 \dots n \quad \leftarrow n \text{ constraints}$$

$$s_i(1) = p_i \quad i = 1 \dots n \quad \leftarrow n \text{ constraints}$$

$$s'_i(1) = s'_{i+1}(0) \quad i = 1 \dots n - 1 \quad \leftarrow n-1 \text{ constraints}$$

$$s''_i(1) = s''_{i+1}(0) \quad i = 1 \dots n - 1 \quad \leftarrow n-1 \text{ constraints}$$

$$s''_1(0) = s''_n(1) = 0 \quad \leftarrow 2 \text{ constraints}$$

Total 4n constraints

Natural Cubic Splines

- Interpolate data points
- No convex hull property
- Non-local support
 - Consider matrix structure...
- C^2 using cubic polynomials

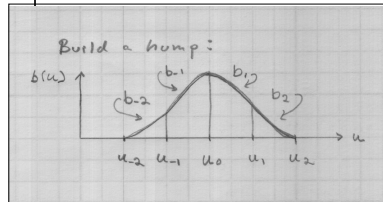
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B-Splines

- Goal: C^2 cubic curves with local support
 - Give up interpolation
 - Get convex hull property
- Build basis by designing “hump” functions

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B-Splines



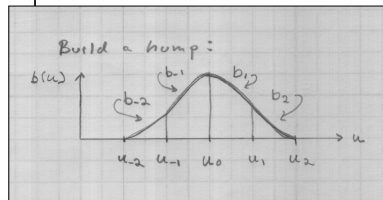
$$\mathbf{b}(u) = \begin{cases} \mathbf{b}_{-2}(u) & \text{if } u_{-2} \leq u < u_{-1} \\ \mathbf{b}_{-1}(u) & \text{if } u_{-1} \leq u < u_0 \\ \mathbf{b}_{+1}(u) & \text{if } u_0 \leq u < u_{+1} \\ \mathbf{b}_{+2}(u) & \text{if } u_{+1} \leq u \leq u_{+2} \end{cases}$$

$$\begin{aligned} b''_{-2}(u_{-2}) = b'_{-2}(u_{-2}) = b_{-2}(u_{-2}) &= 0 \\ b''_{+2}(u_{+2}) = b'_{+2}(u_{+2}) = b_{+2}(u_{+2}) &= 0 \end{aligned}$$

$$\begin{aligned} b_{-2}(u_{-1}) &= b_{-1}(u_{-1}) \\ b_{-1}(u_0) &= b_{+1}(u_0) \\ b_{+1}(u_{+1}) &= b_{+2}(u_{+1}) \end{aligned}$$

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B-Splines



$$\mathbf{b}(u) = \begin{cases} \mathbf{b}_{-2}(u) & \text{if } u_{-2} \leq u < u_{-1} \\ \mathbf{b}_{-1}(u) & \text{if } u_{-1} \leq u < u_0 \\ \mathbf{b}_{+1}(u) & \text{if } u_0 \leq u < u_{+1} \\ \mathbf{b}_{+2}(u) & \text{if } u_{+1} \leq u \leq u_{+2} \end{cases}$$

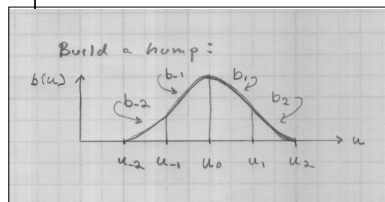
$$\begin{aligned} b''_{-2}(u_{-2}) = b'_{-2}(u_{-2}) = b_{-2}(u_{-2}) &= 0 && \leftarrow 3 \text{ constraints} \\ b''_{+2}(u_{+2}) = b'_{+2}(u_{+2}) = b_{+2}(u_{+2}) &= 0 && \leftarrow 3 \text{ constraints} \end{aligned}$$

$$\begin{aligned} b_{-2}(u_{-1}) &= b_{-1}(u_{-1}) \\ b_{-1}(u_0) &= b_{+1}(u_0) \\ b_{+1}(u_{+1}) &= b_{+2}(u_{+1}) \end{aligned} \quad \leftarrow \begin{cases} \text{Repeat for } b' \text{ and } b'' \\ 3 \times 3 = 9 \text{ constraints} \end{cases}$$

Total 15 constraints need one more

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B-Splines



$$\mathbf{b}(u) = \begin{cases} \mathbf{b}_{-2}(u) & \text{if } u_{-2} \leq u < u_{-1} \\ \mathbf{b}_{-1}(u) & \text{if } u_{-1} \leq u < u_0 \\ \mathbf{b}_{+1}(u) & \text{if } u_0 \leq u < u_{+1} \\ \mathbf{b}_{+2}(u) & \text{if } u_{+1} \leq u \leq u_{+2} \end{cases}$$

$$b''_{-2}(u_{-2}) = b'_{-2}(u_{-2}) = b_{-2}(u_{-2}) = 0 \quad \leftarrow 3 \text{ constraints}$$

$$b''_{+2}(u_{+2}) = b'_{+2}(u_{+2}) = b_{+2}(u_{+2}) = 0 \quad \leftarrow 3 \text{ constraints}$$

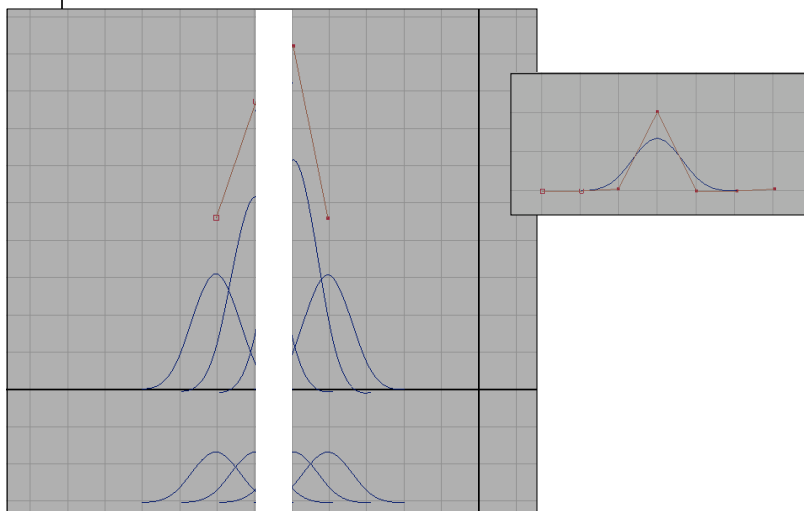
$$\begin{aligned} b_{-2}(u_{-1}) &= b_{-1}(u_{-1}) \\ b_{-1}(u_0) &= b_{+1}(u_0) \\ b_{+1}(u_{+1}) &= b_{+2}(u_{+1}) \end{aligned} \quad \leftarrow \begin{cases} \text{Repeat for } b' \text{ and } b'' \\ 3 \times 3 = 9 \text{ constraints} \end{cases}$$

$$b_{-2}(u_{-2}) + b_{-1}(u_{-1}) + b_{+1}(u_0) + b_{+2}(u_{+1}) = 1 \quad \leftarrow 1 \text{ constraint (convex hull)}$$

Total 16 constraints

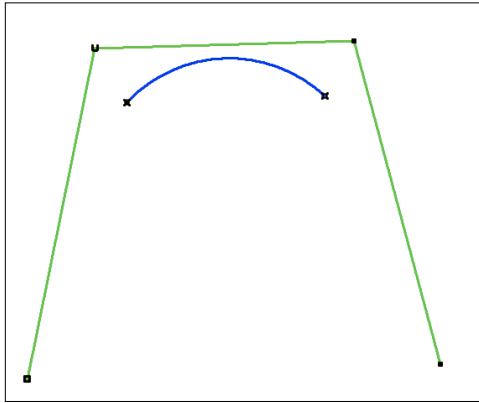
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B-Splines



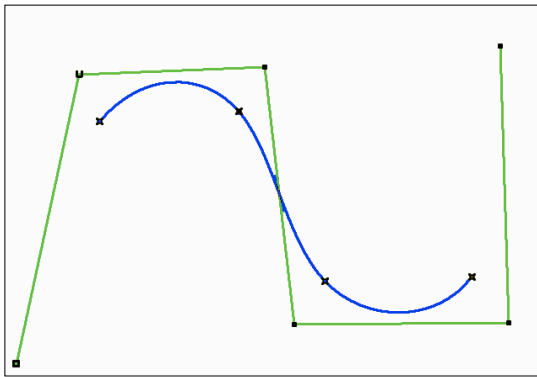
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B-Splines



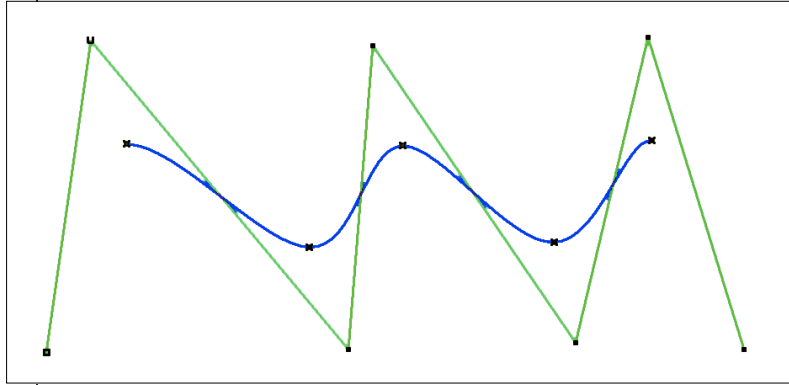
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B-Splines



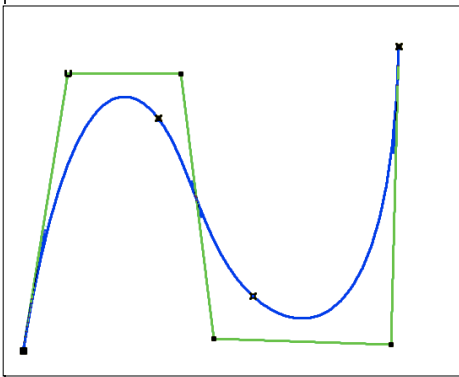
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B-Splines



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B-Splines



Example with end knots repeated

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B-Splines

- Build a curve w/ overlapping bumps
- Continuity
 - Inside bumps C^2
 - Bumps “fade out” with C^2 continuity
- Boundaries
 - Circular
 - Repeat end points
 - Extra end points

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B-Splines

- Notation
 - The basis functions are the $b_i(u)$
 - “Hump” functions are the concatenated function
 - Sometimes the humps are called basis... can be confusing
 - The u_i are the knot locations
 - The weights on the hump/basis functions are control points

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B-Splines

- Similar construction method can give higher continuity with higher degree polynomials
- Repeating knots drops continuity
 - Limit as knots approach each other
- Still cubics, so conversion to other cubic basis is just a matrix multiplication

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B-Splines

- Geometric construction
 - Due to Cox and de Boor
 - My own notation, beware if you compare w/ text
- Let hump centered on u_i be $N_{i,4}(u)$

Cubic is order 4

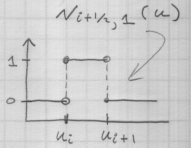
$N_{i,k}(u)$ Is order k hump, centered at u_i

Note: i is integer if k is even
else $(i + 1/2)$ is integer

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B-Splines

$$N_{i,1}(u) = \begin{cases} 1 & \text{if } u_{i-1/2} \leq u < u_{i+1/2} \\ 0 & \text{else} \end{cases}$$

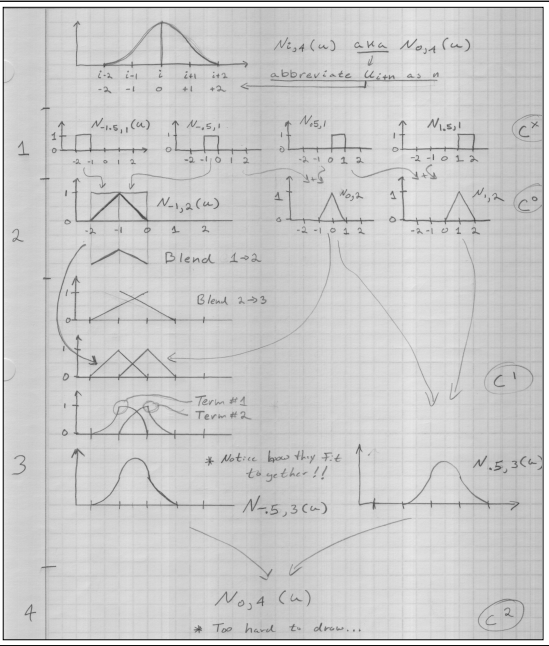


$$N_{i,2}(u) = \frac{(u - u_{i-1/2}) N_{i-1/2,1}(u)}{u_{i+1/2} - u_{i-1/2}} + \frac{(u_{i+1/2} - u) N_{i+1/2,1}(u)}{u_{i+1/2} - u_{i-1/2} + 1}$$

(Note: $k=2$ is indicated near the first term)

(Note: "Term #1" and "Term #2" are labeled with arrows pointing to the two fractions)

Recursive defn.



$N_{i,1}(u)$ aka $N_{0,1}(u)$ abbreviate u as n

$N_{i,2}(u)$

Blend 1 to 2

Blend 2 to 3

Term #1
 Term #2

$N_{i,3}(u)$

$N_{i,4}(u)$

* Notice how they fit together!!

* Too hard to draw...

NURBS

- **Non**uniform **R**ational **B**-Splines
 - Basically B-Splines using homogeneous coordinates
 - Transform under perspective projection
 - A bit of extra control

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NURBS

$$\mathbf{p}_i = \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \\ p_{iw} \end{bmatrix} \quad \mathbf{x}(u) = \frac{\sum_i \begin{bmatrix} p_{ix} \\ p_{iy} \\ p_{iz} \end{bmatrix} N_i(u)}{\sum_i p_{iw} N_i(u)}$$

- Non-linear in the control points
- The p_{iw} are sometimes called "weights"

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