CS 184: Foundations of Computer Graphics

Kinematics of Articulated Bodies

Rahul Narain

Kinematics vs. dynamics

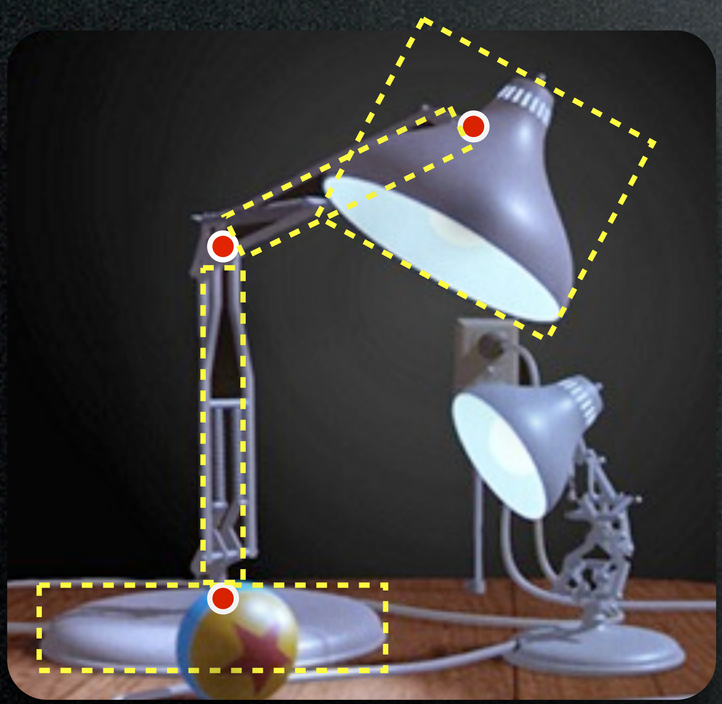
- Kinematics = motion but no forces
  - Keyframing, motion capture, etc.
- Dynamics = motion from forces
  - Simulation

If you want to know more about dynamics and simulation...

- Particles and rigid bodies:
  - Baraff and Witkin, "Physically Based Modeling", 2001
- Deformable bodies:
  - James O'Brien's CS 283 slides and readings on elastic simulation
- Fluids:
  - Bridson and Müller-Fischer, "Fluid Simulation for Computer Animation", 2007

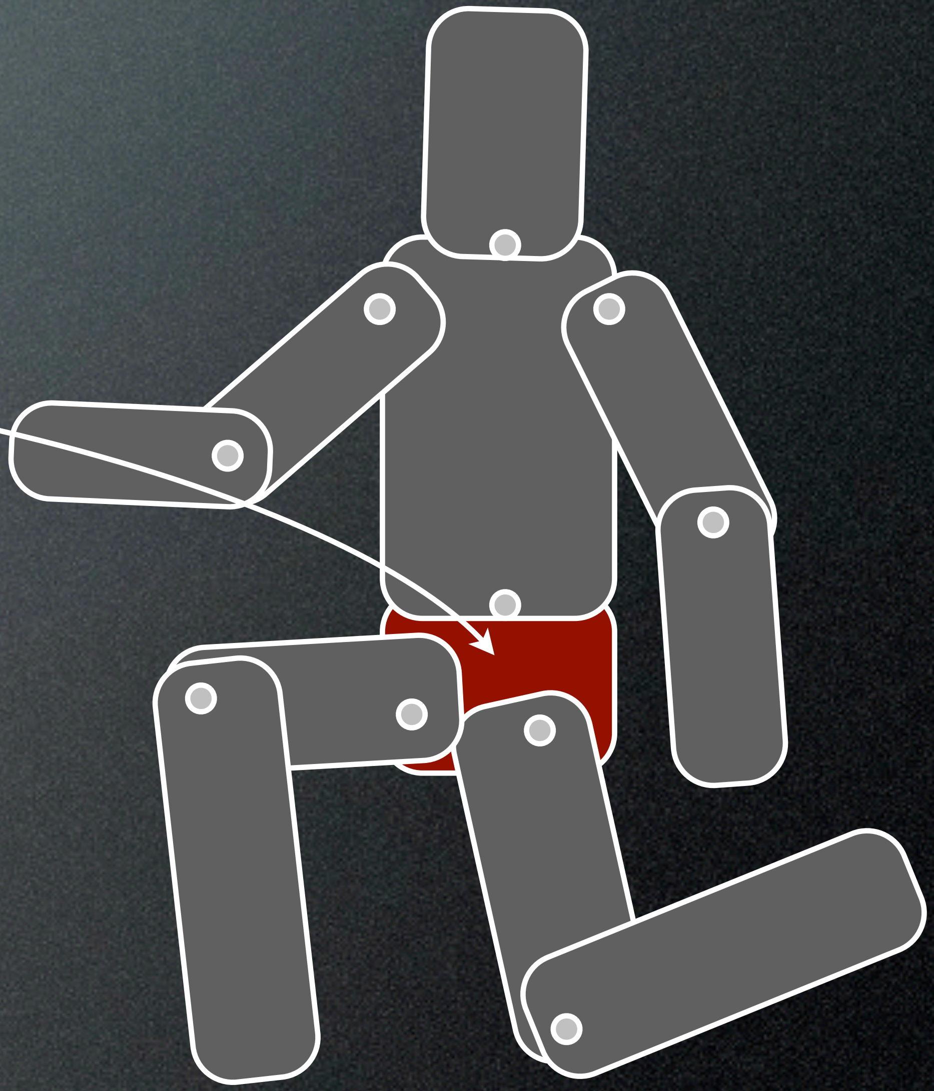
Articulated bodies

- Rigid bodies connected with joints
  - Topology (what's connected to what)
  - Geometric relations from joints
- Not necessarily what's displayed in the end



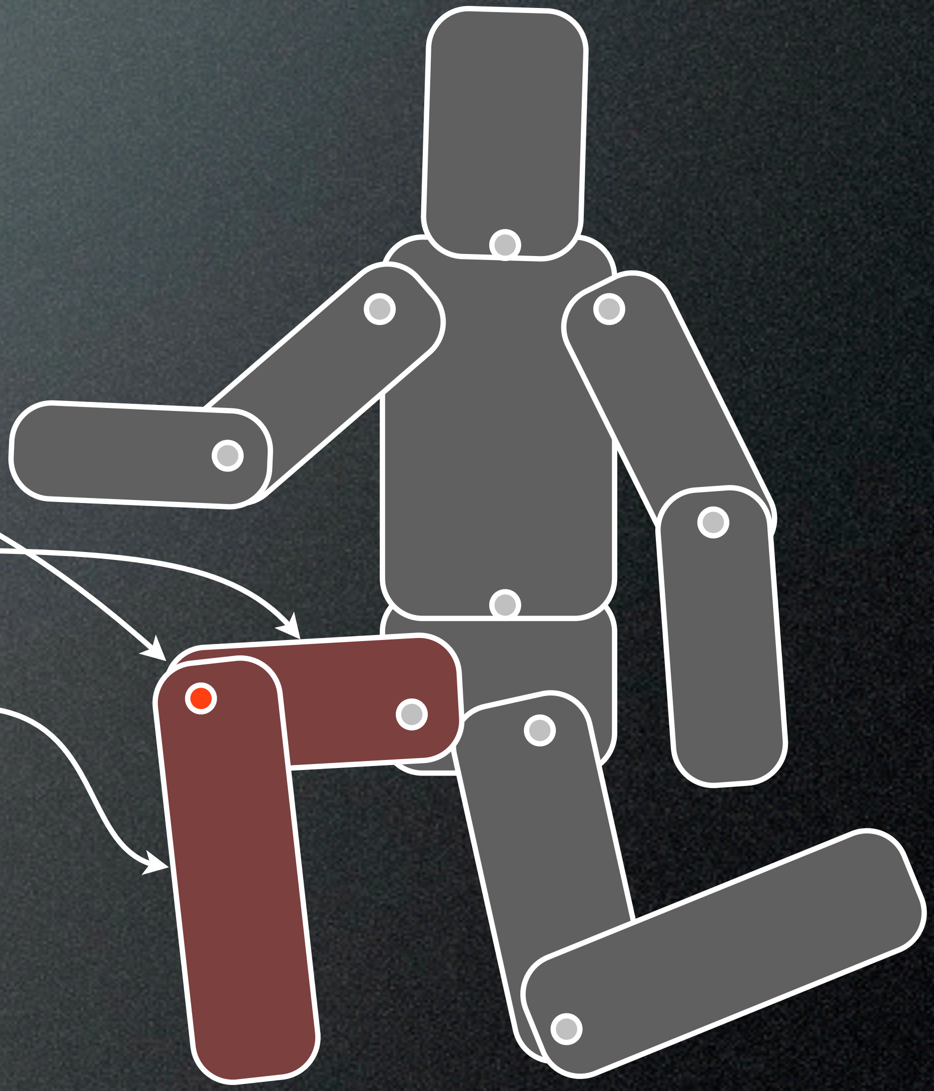
Articulated bodies

- Root body
  - Position and orientation set by "global" transformation
  - Other bodies move relative to root



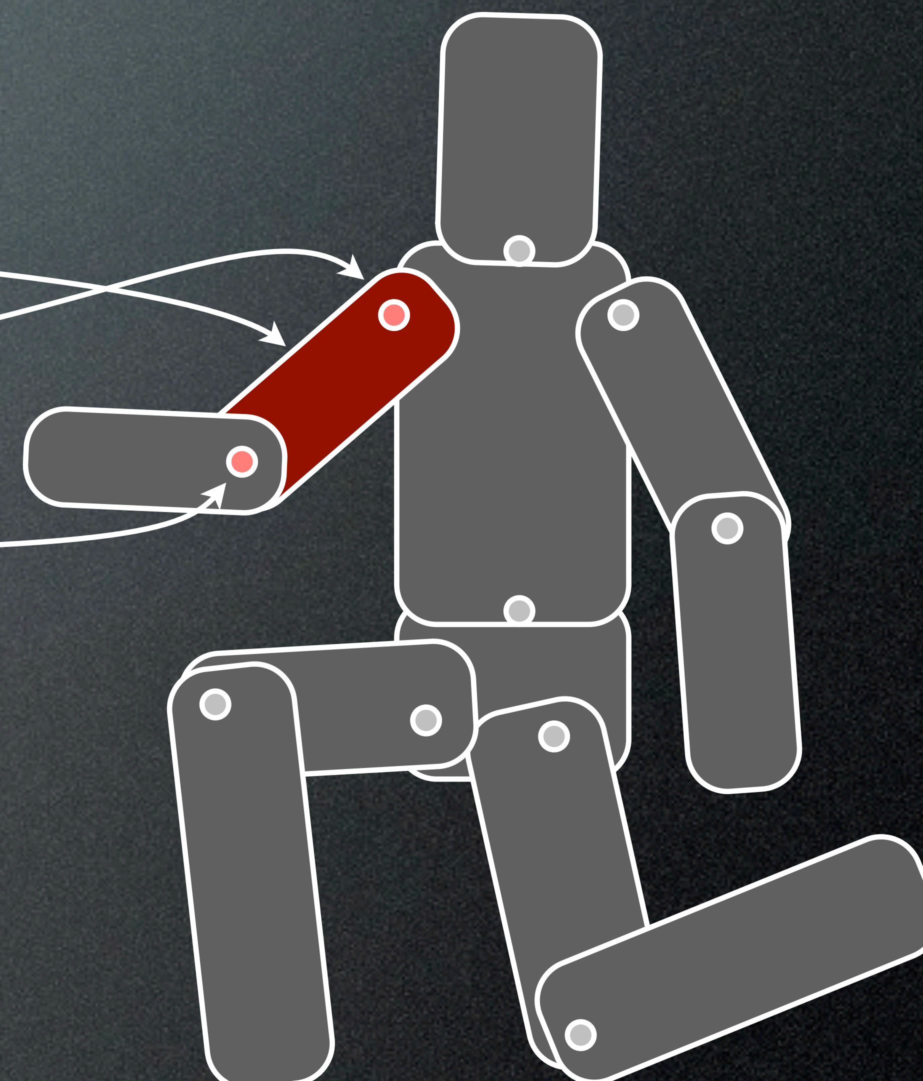
Articulated bodies

- A joint
  - Inboard body (towards root)
  - Outboard body (away from root)



Articulated bodies

- A body
  - Inboard joint
  - Outboard joint(s)
  - Parent
  - Child(ren)



Articulated bodies

- A body
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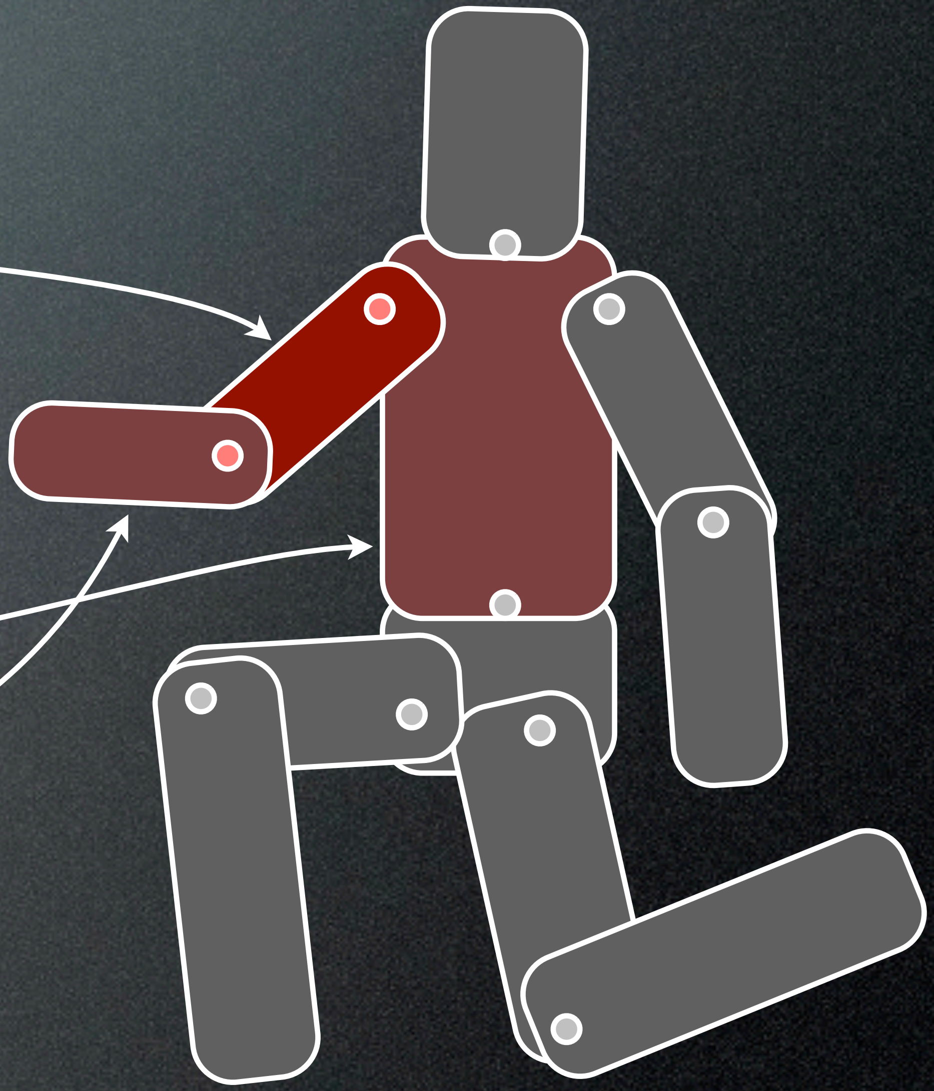
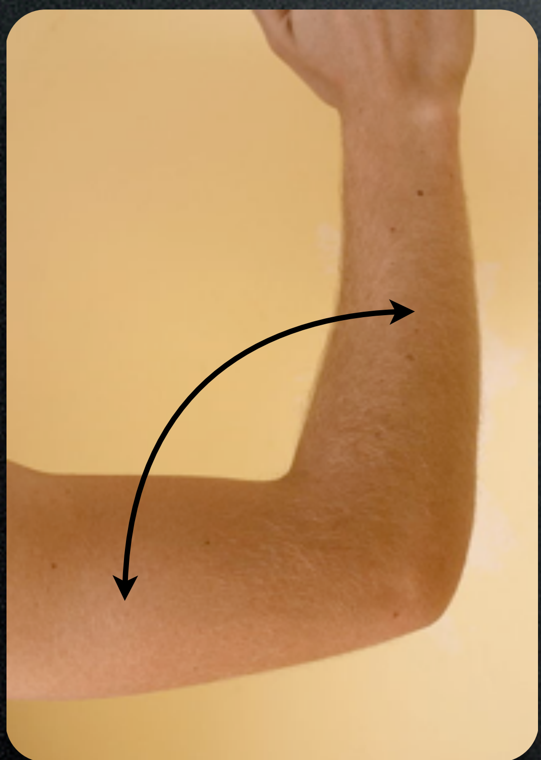


Articulated bodies

- Interior joints are typically not 6 DOF

Pin joint:
rotation about one axis



Ball:
arbitrary rotation

Prism joint:
translation along one axis

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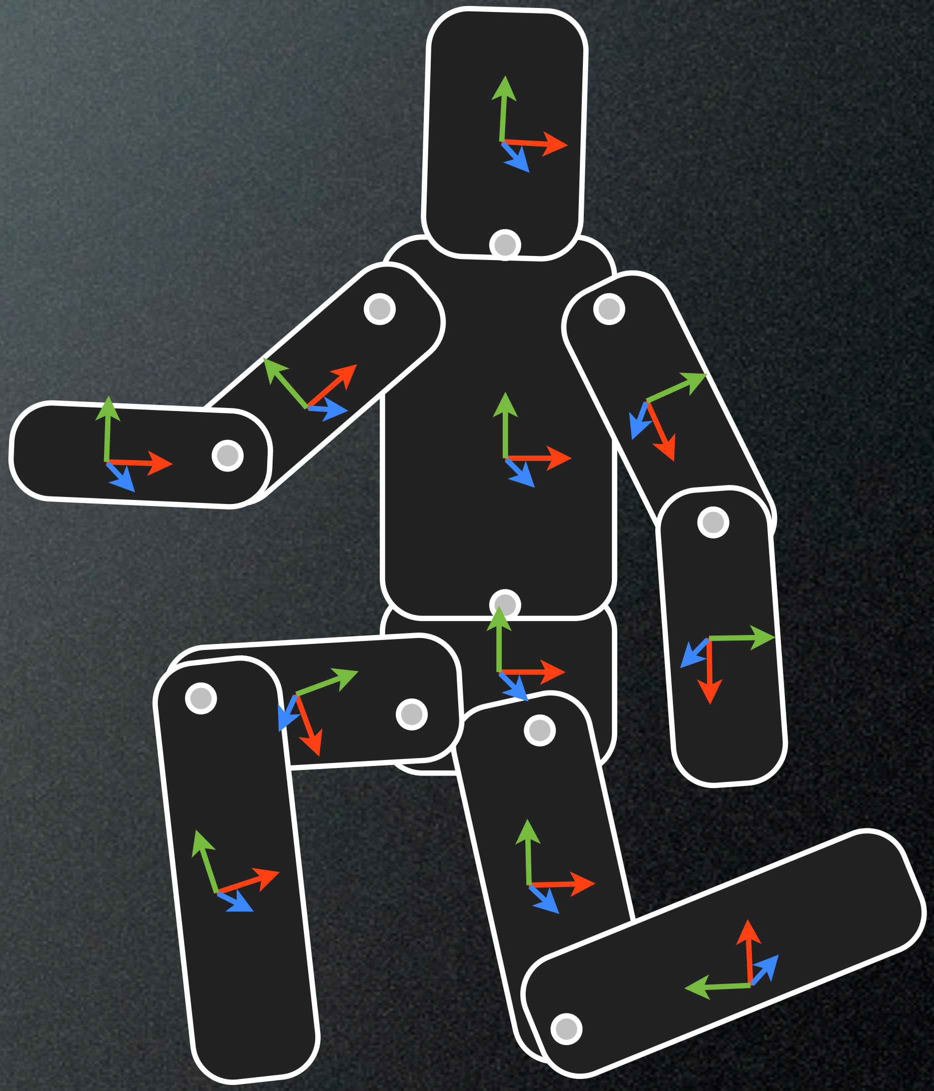
Forward and inverse kinematics

- Forward kinematics:
  - Given all the joint parameters, where are the bodies?
- Inverse kinematics:
  - Given where I want some body to be, what joint parameters do I need to set?

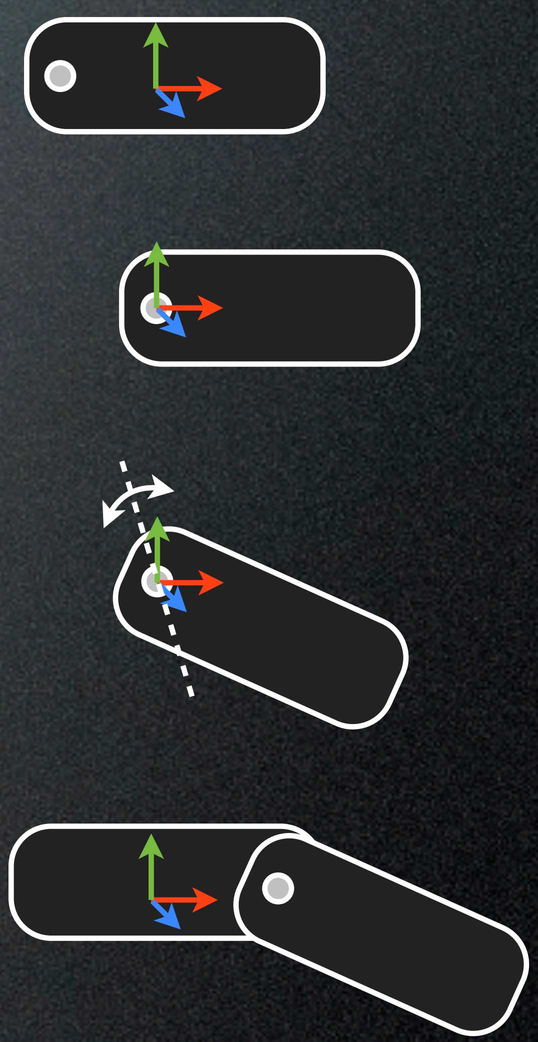
Forward kinematics

- Each body gets its own local coordinate system
  - Position of a vertex is fixed relative to local coordinate system
  - What is its position relative to world coordinates?



Forward kinematics

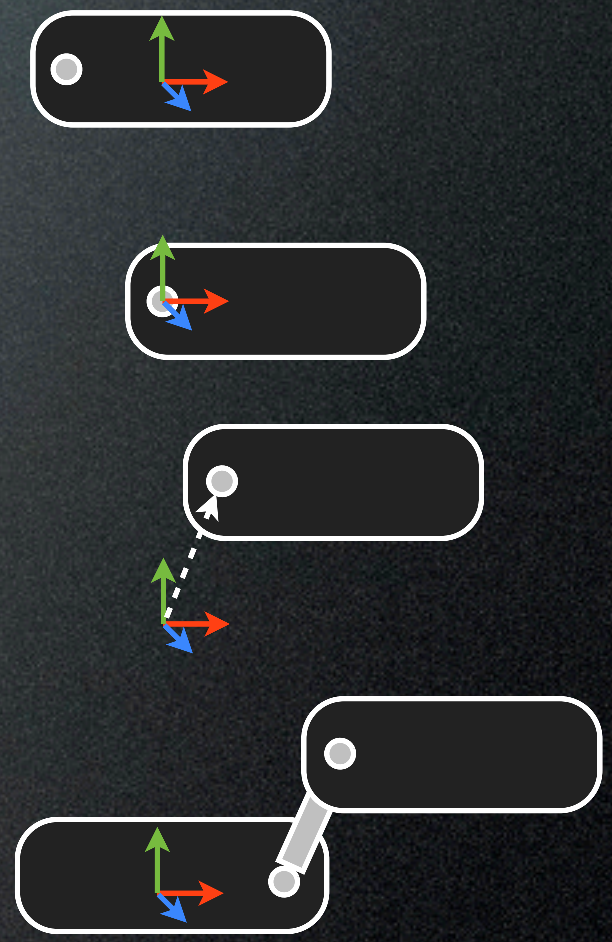
- Pin joints
    - Translate inboard joint to origin
    - Apply rotation about axis
    - Translate origin to location of outboard joint on parent body



$$\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{t}_{\text{parent}} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}(\theta) & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{t}_{\text{child}} \\ \mathbf{0} & 1 \end{bmatrix}$$

Forward kinematics

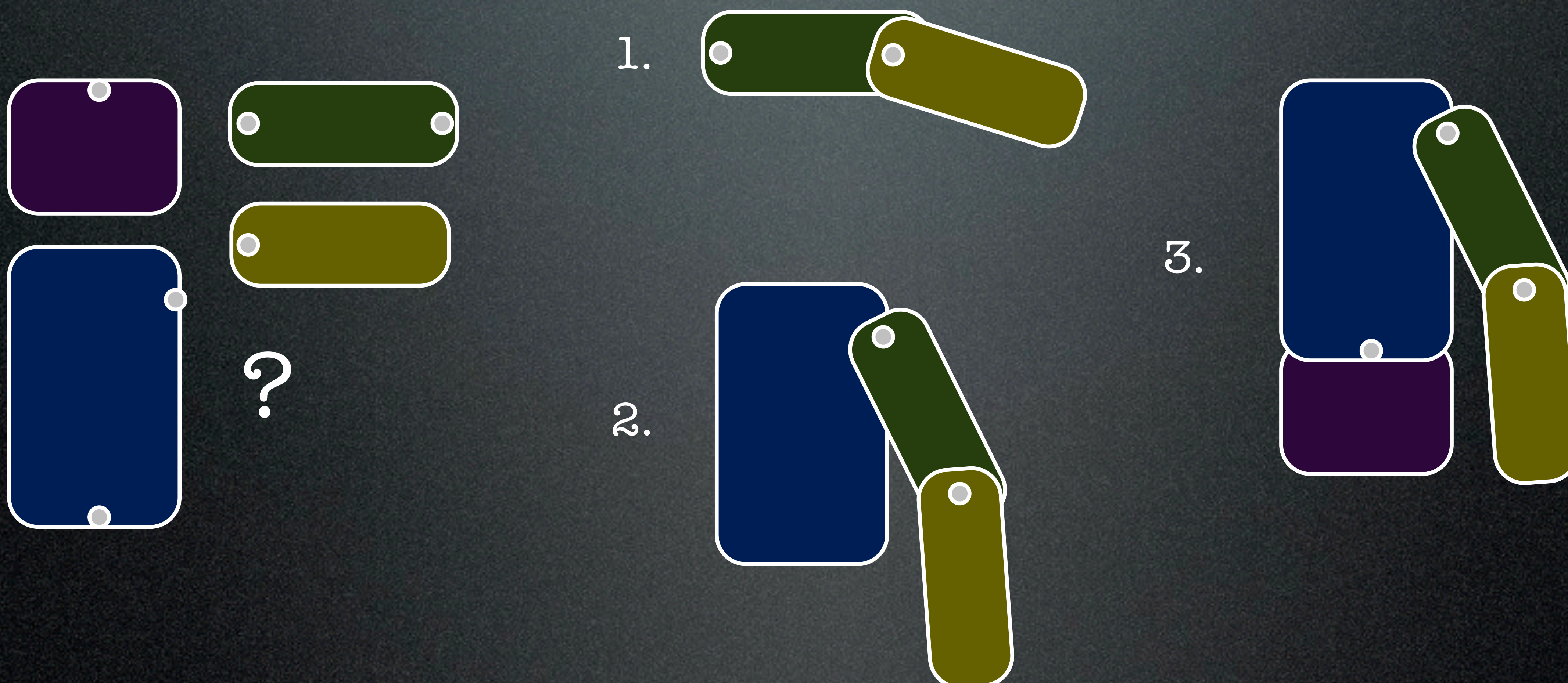
- Prism joints
  - Translate inboard joint to origin
  - Translate along axis
  - Translate origin to location of outboard joint on parent body



$$M = \begin{bmatrix} \mathbf{I} & \mathbf{t}_{\text{parent}} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & t\mathbf{a} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{t}_{\text{child}} \\ \mathbf{0} & 1 \end{bmatrix}$$

Forward kinematics

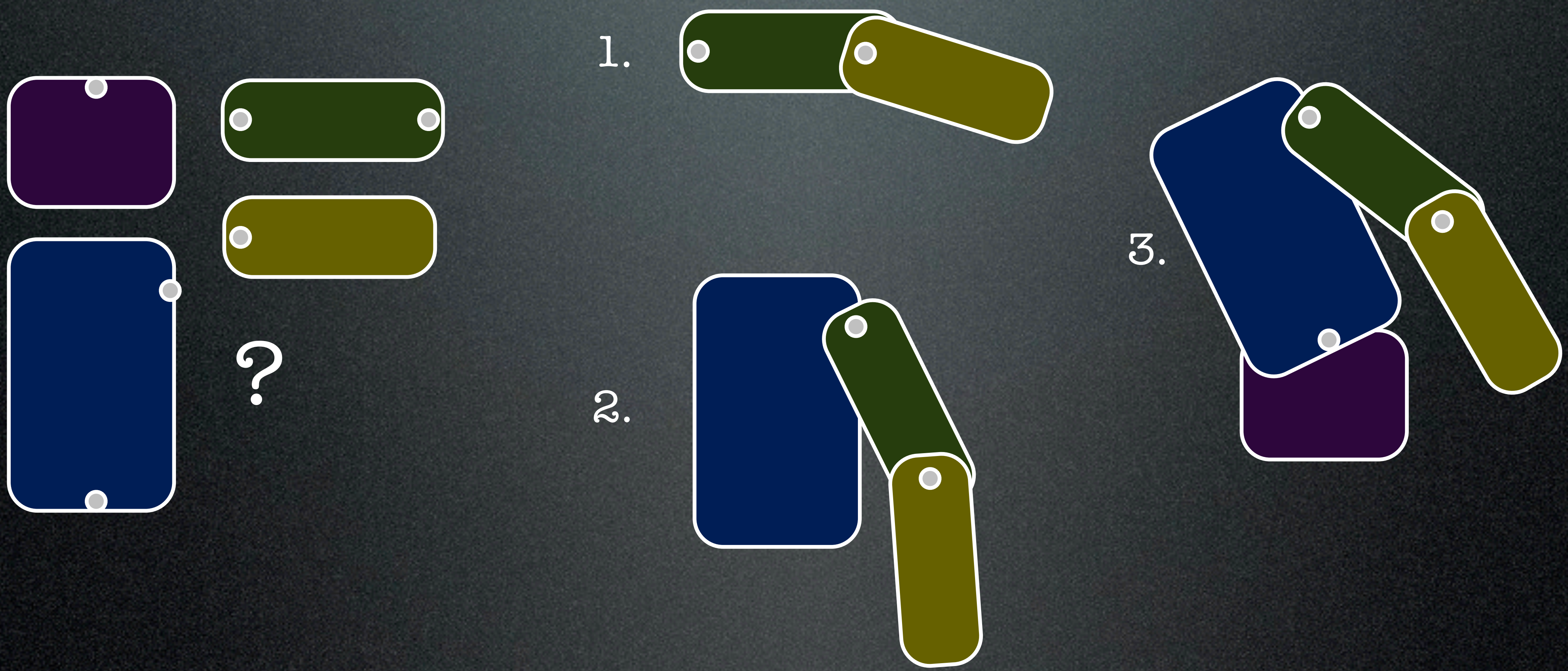
- Composite transformations up the hierarchy



- $\mathbf{M}_{\text{world}\leftarrow\text{forearm}} = \mathbf{M}_{\text{world}\leftarrow\text{hip}} \cdot \mathbf{M}_{\text{hip}\leftarrow\text{torso}} \cdot \mathbf{M}_{\text{torso}\leftarrow\text{upperarm}} \cdot \mathbf{M}_{\text{upperarm}\leftarrow\text{forearm}}$

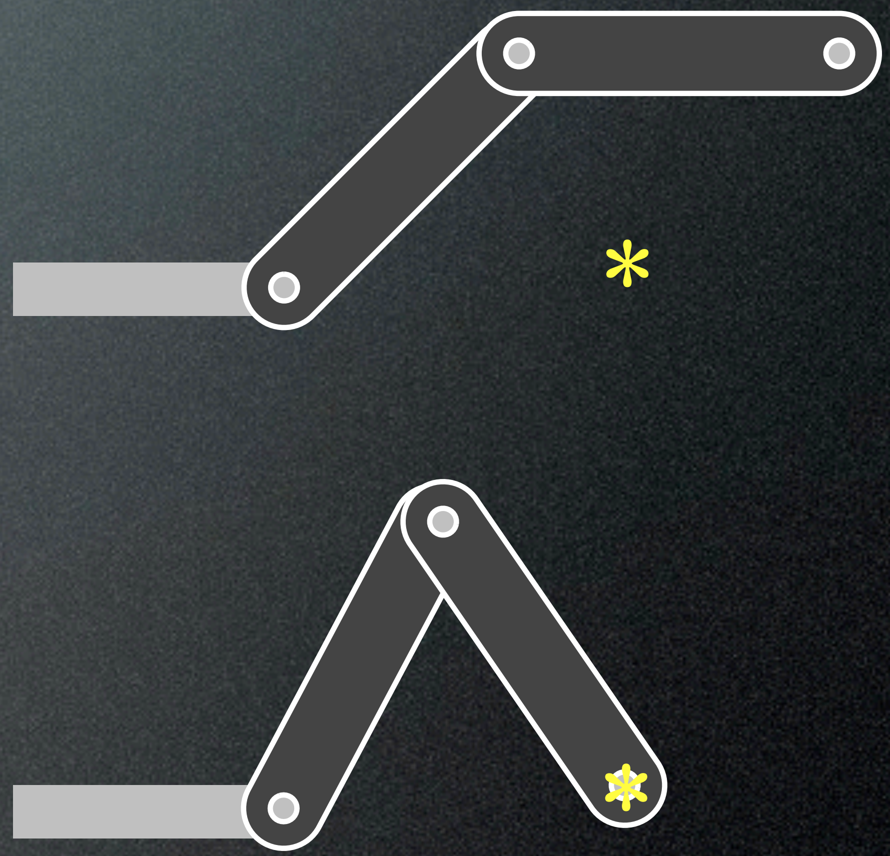
Forward kinematics

- Composite transformations up the hierarchy



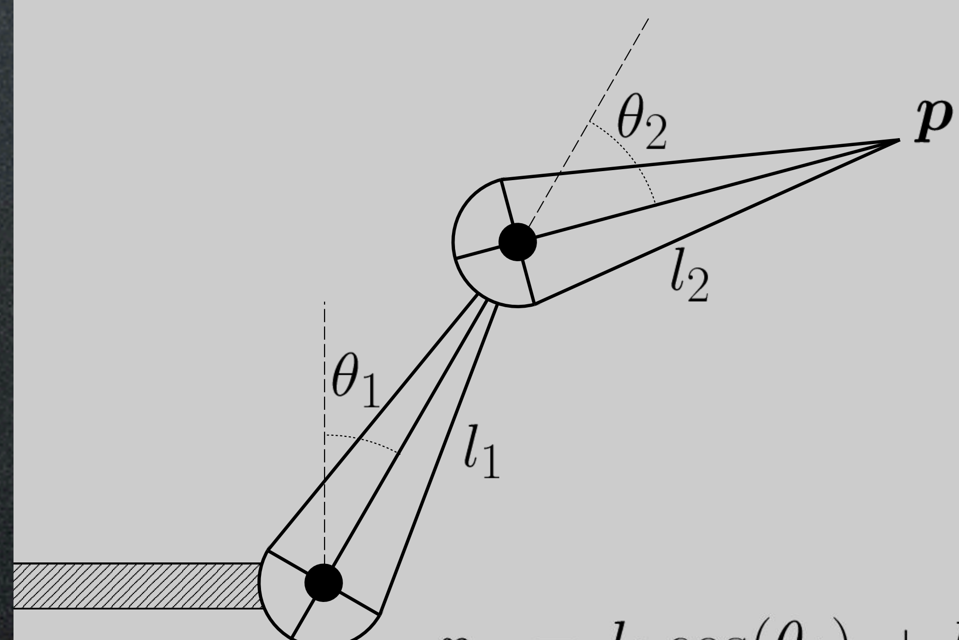
Inverse kinematics

- Given
  - Root transformation
  - Initial configuration
  - Desired location of end point
- Find
  - Internal parameter settings



Inverse kinematics

- A simple two segment arm in 2D



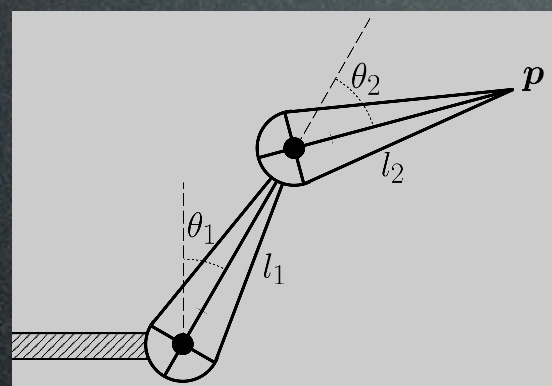
$$p_z = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$

$$p_x = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

James O'Brien

Direct IK

- Just solve for the parameters! What's the problem?

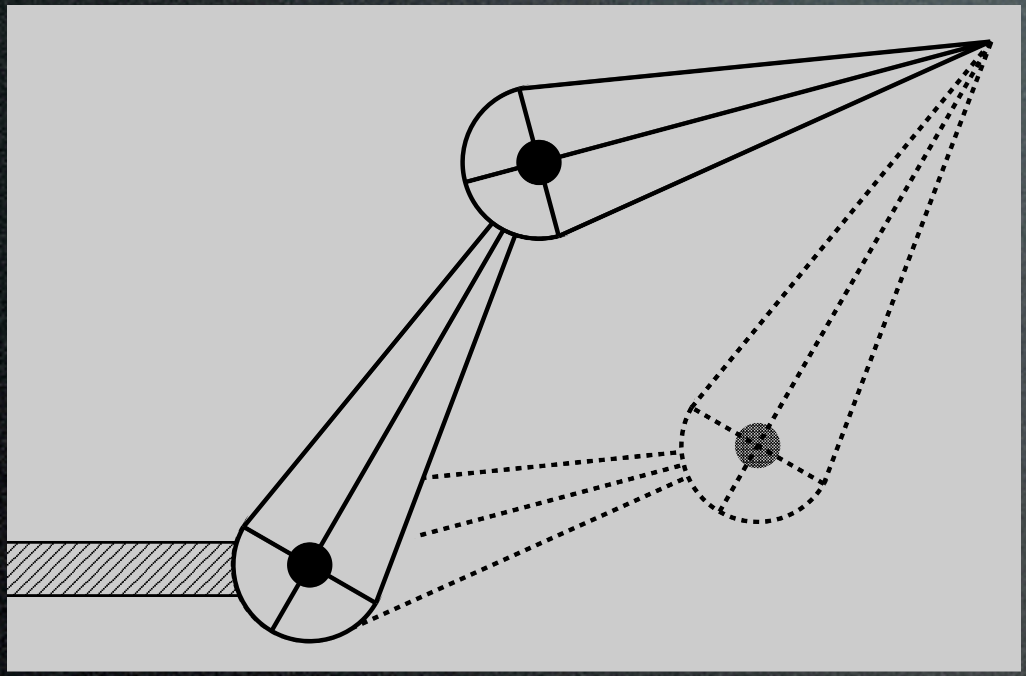


$$\theta_2 = \cos^{-1}\left(\frac{p_z^2 + p_x^2 - l_1^2 - l_2^2}{2 l_1 l_2}\right)$$

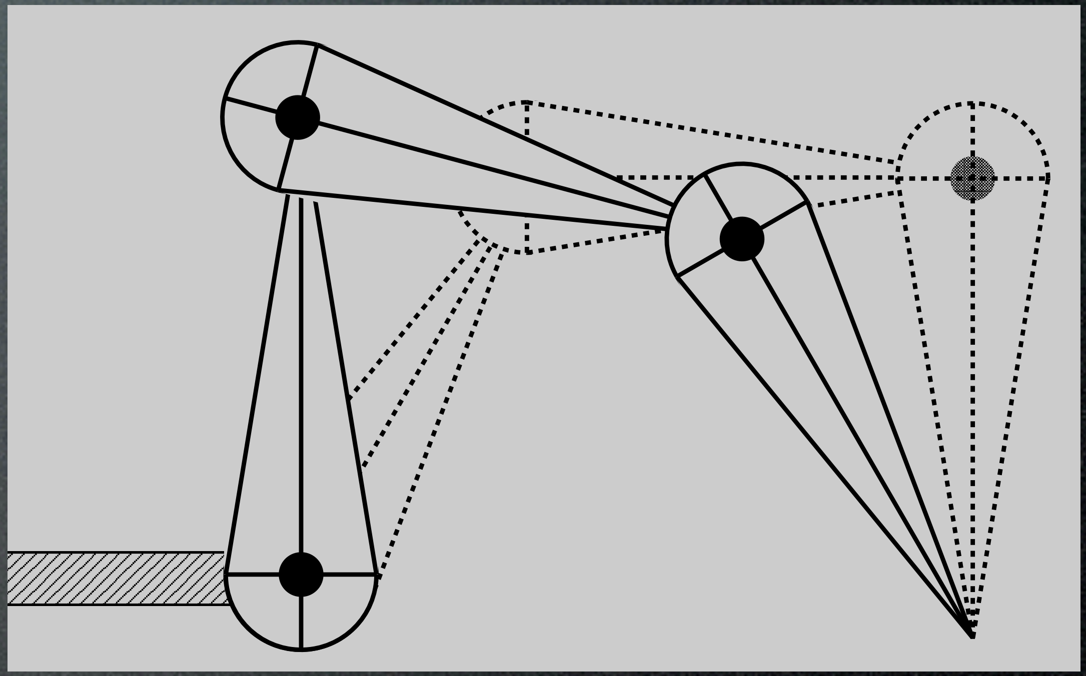
$$\theta_1 = \frac{-p_z l_2 \sin(\theta_2) + p_x(l_1 + l_2 \cos(\theta_2))}{p_x l_2 \sin(\theta_2) + p_z(l_1 + l_2 \cos(\theta_2))}$$

Why is this hard?

Multiple disconnected solutions

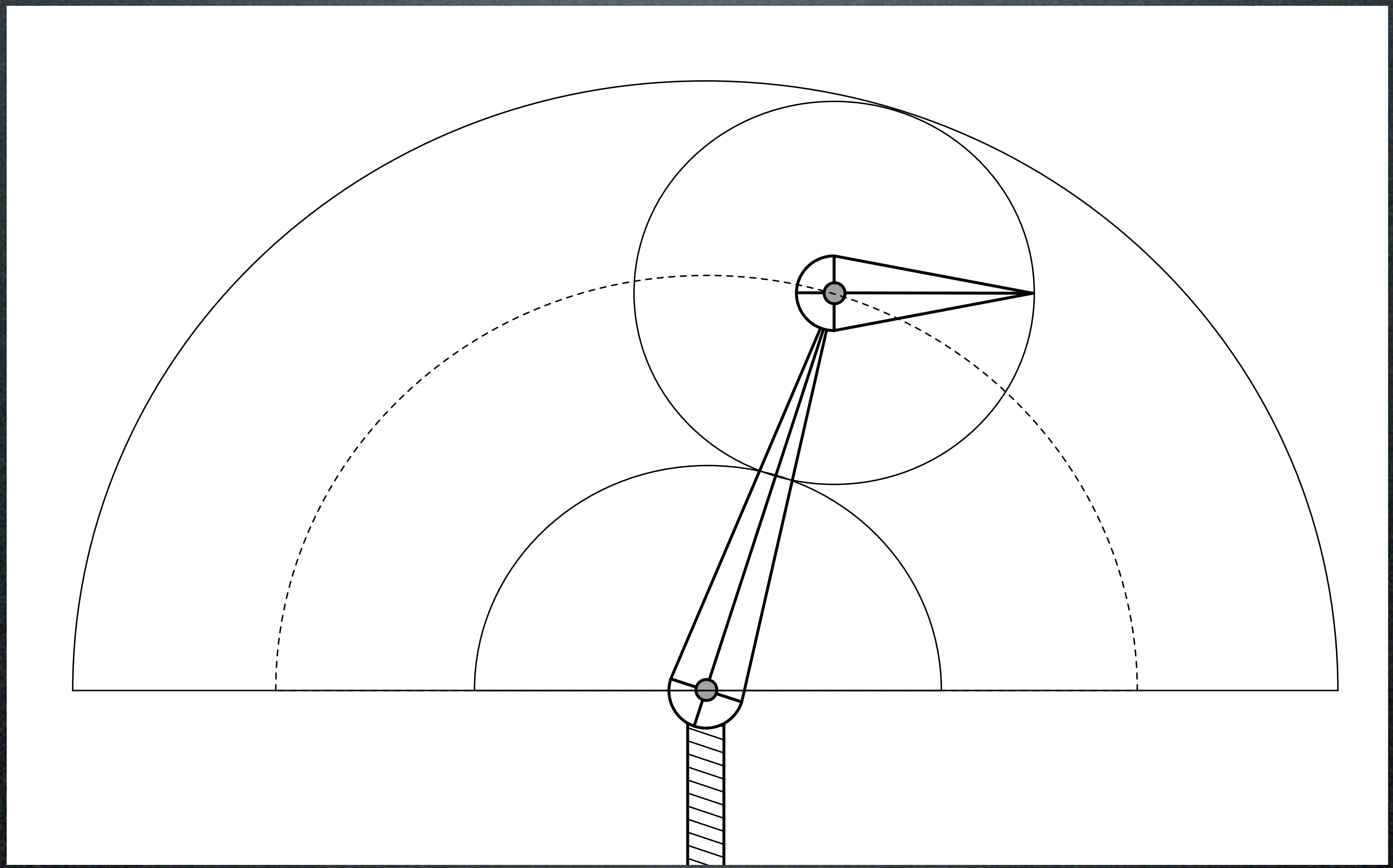


Multiple **connected** solutions



Why is this hard?

Solutions don't always exist

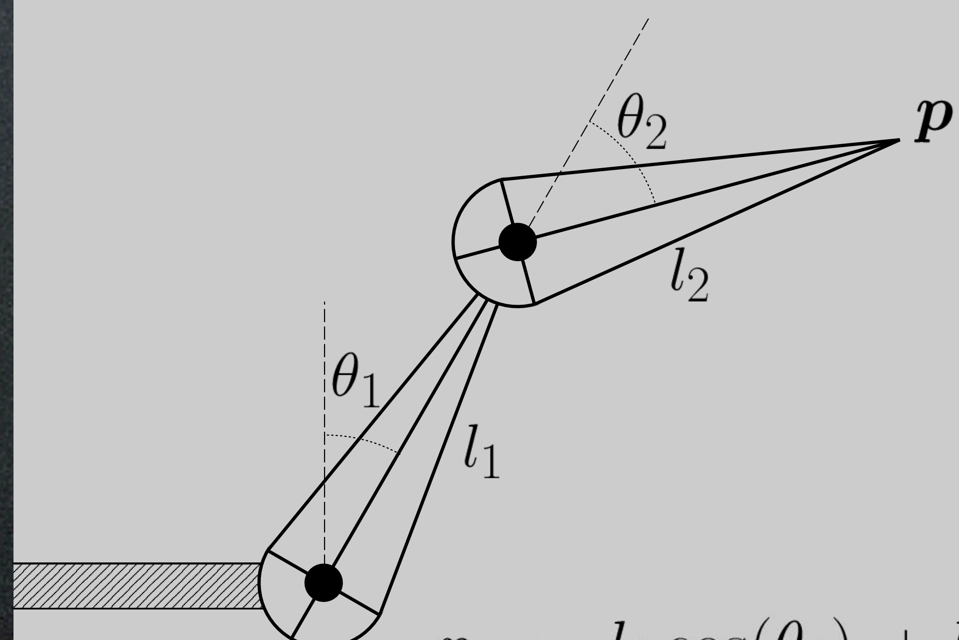


Numerical IK

- Start in some initial configuration
- Define an error metric (e.g. $\mathbf{p}_{goal} - \mathbf{p}_{current}$)
- Compute Jacobian of error w.r.t joint angles $\boldsymbol{\theta}$
- Apply Newton's method (or other procedure)
- Iterate...

Inverse kinematics

- Recall the simple two segment arm:

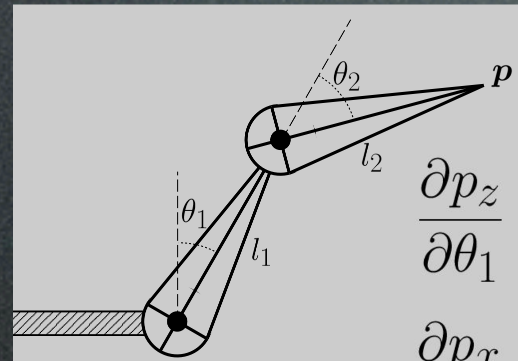


$$p_z = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$

$$p_x = l_1 \sin(\theta_1) + l_2 \sin(\theta_1 + \theta_2)$$

Numerical IK

- We can write the derivatives



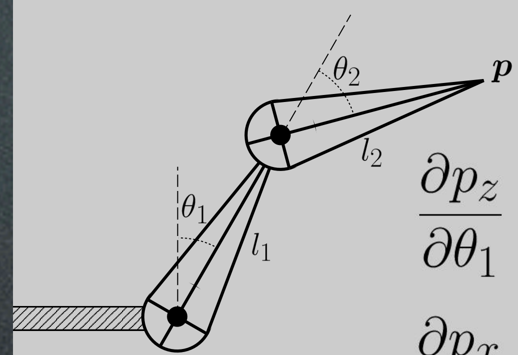
$$\frac{\partial p_z}{\partial \theta_1} = -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2)$$

$$\frac{\partial p_x}{\partial \theta_1} = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$

$$\frac{\partial p_z}{\partial \theta_2} = - l_2 \sin(\theta_1 + \theta_2)$$

$$\frac{\partial p_x}{\partial \theta_2} = + l_2 \cos(\theta_1 + \theta_2)$$

Numerical IK



$$\frac{\partial p_z}{\partial \theta_1} = -l_1 \sin(\theta_1) - l_2 \sin(\theta_1 + \theta_2)$$

$$\frac{\partial p_x}{\partial \theta_1} = l_1 \cos(\theta_1) + l_2 \cos(\theta_1 + \theta_2)$$

$$\frac{\partial p_z}{\partial \theta_2} = - l_2 \sin(\theta_1 + \theta_2)$$

$$\frac{\partial p_x}{\partial \theta_2} = + l_2 \cos(\theta_1 + \theta_2)$$

- If we change the angles by a small amount $d\theta_1$ and $d\theta_2$, this tells us how p_x and p_z change.

The Jacobian

- Matrix of partial derivatives $J_{ij} = \frac{\partial p_i}{\partial \theta_j}$
- For a two segment arm in 2D,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} \\ \frac{\partial p_z}{\partial \theta_1} & \frac{\partial p_z}{\partial \theta_2} \end{bmatrix}$$

The Jacobian

- A small change in $\boldsymbol{\theta}$ leads to a small change in $\mathbf{p}$

$$\mathrm{d}p_x = \frac{\partial p_x}{\partial \theta_1}\mathrm{d}\theta_1 + \frac{\partial p_x}{\partial \theta_2}\mathrm{d}\theta_2$$

$$\mathrm{d}p_z = \frac{\partial p_z}{\partial \theta_1}\mathrm{d}\theta_1 + \frac{\partial p_z}{\partial \theta_2}\mathrm{d}\theta_2$$

$$\mathrm{d}\mathbf{p} = \begin{bmatrix} \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} \\ \frac{\partial p_z}{\partial \theta_1} & \frac{\partial p_z}{\partial \theta_2} \end{bmatrix} \cdot \begin{bmatrix} \mathrm{d}\theta_1 \\ \mathrm{d}\theta_2 \end{bmatrix} = \mathbf{J} \cdot \mathrm{d}\theta$$

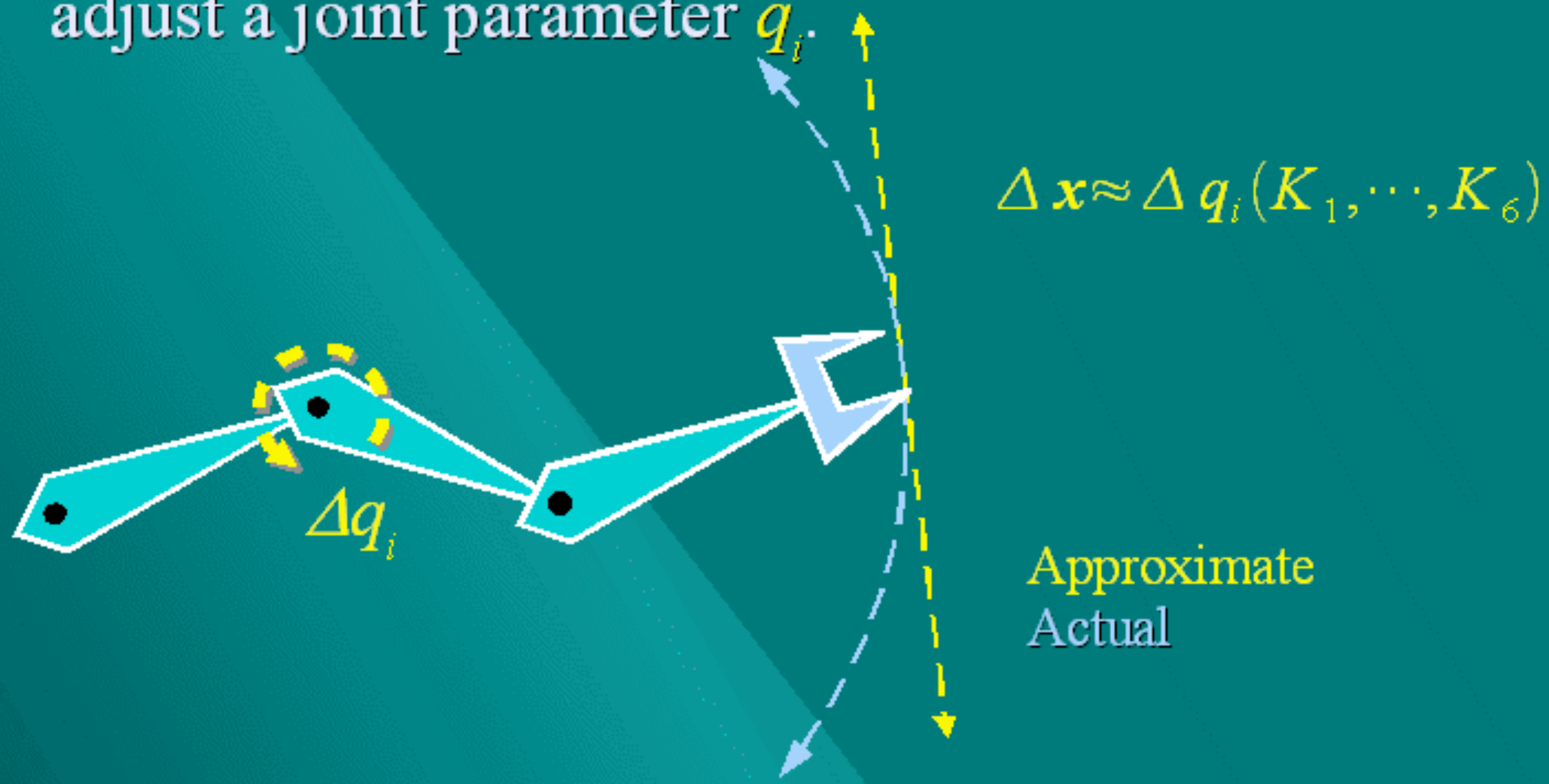
- So... if we want to change $\mathbf{p}$, this tells us how to change $\boldsymbol{\theta}$?

$$\mathrm{d}\mathbf{p} = \mathbf{J} \cdot \mathrm{d}\theta$$

$$\mathrm{d}\theta = \mathbf{J}^{-1} \cdot \mathrm{d}\mathbf{p}?$$

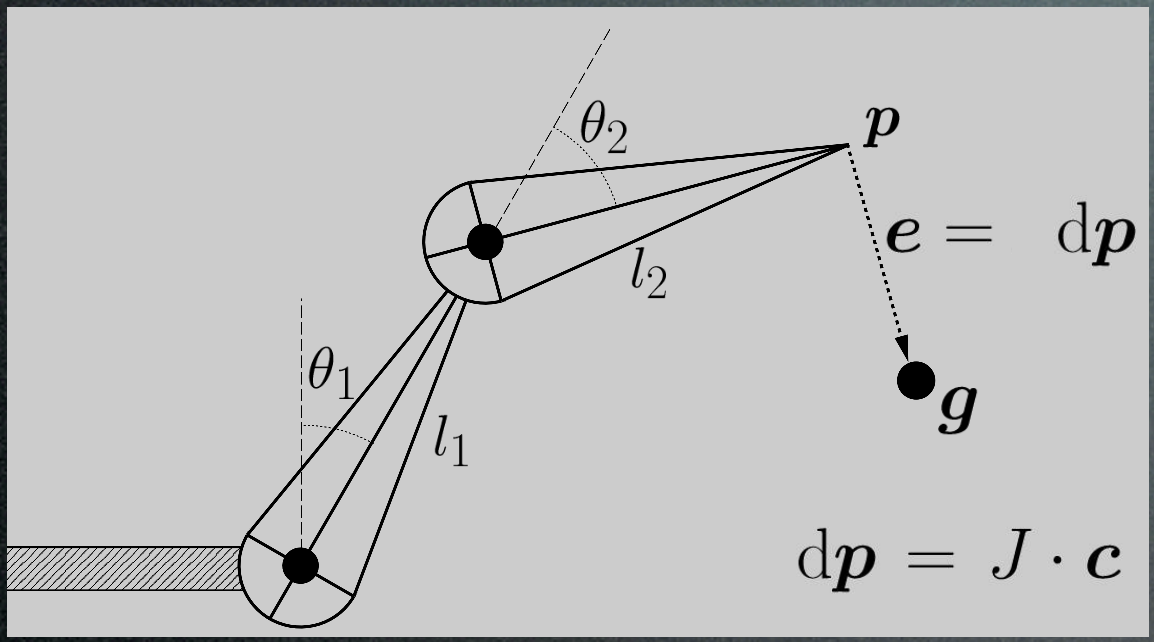
The Jacobian

- Put another way, J tells us approximately how much x will change in world space when we adjust a joint parameter q_i.



Bill Baxter

Back to inverse kinematics



$= \mathbf{J} \cdot \mathrm{d}\theta$

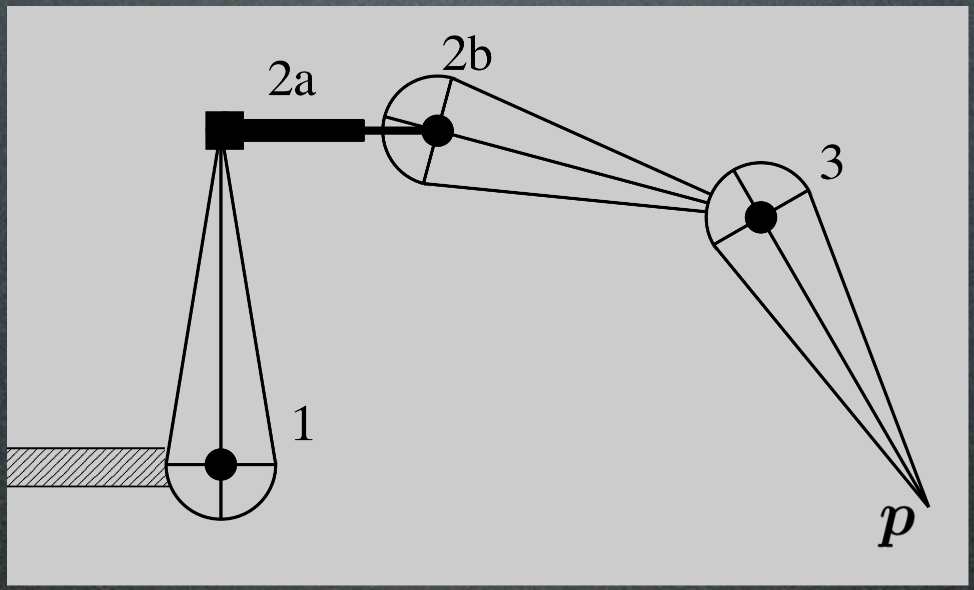
- We want **p** to change by d**p**
- Can we simply change **θ** by d**θ** = $\mathbf{J}^{-1}$ d**p**?
  - ...Is **J** invertible?

Inverse kinematics

- Problems:
  - Jacobian may (will!) not be invertible → use pseudo-inverse, or use more robust numerical method
  - Jacobian is not constant → take small steps
- Nonlinear optimization, but (mostly) well-behaved

Multiple links

- We need a generic way of building the Jacobian



$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ d_{2a} \\ \theta_{2b} \\ \theta_3 \end{bmatrix} \qquad \mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{p}}{\partial \theta_1} & \frac{\partial \mathbf{p}}{\partial d_{2a}} & \frac{\partial \mathbf{p}}{\partial \theta_{2b}} & \frac{\partial \mathbf{p}}{\partial \theta_3} \end{bmatrix} = ?$$

Remember forward kinematics

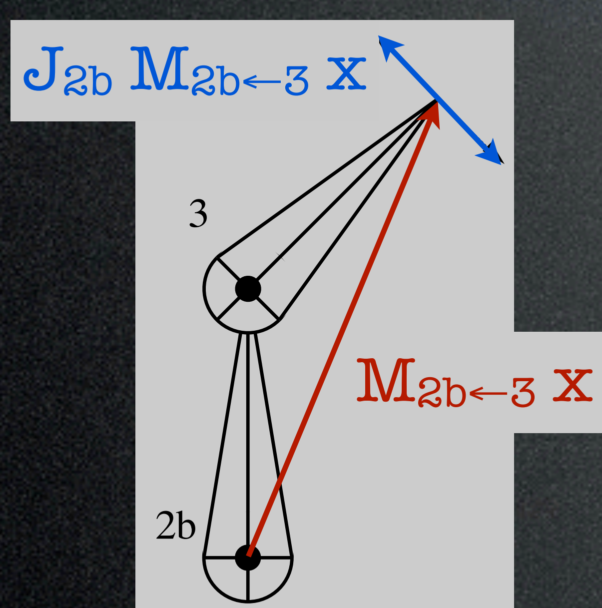
- World position of point is given by composition of transformations

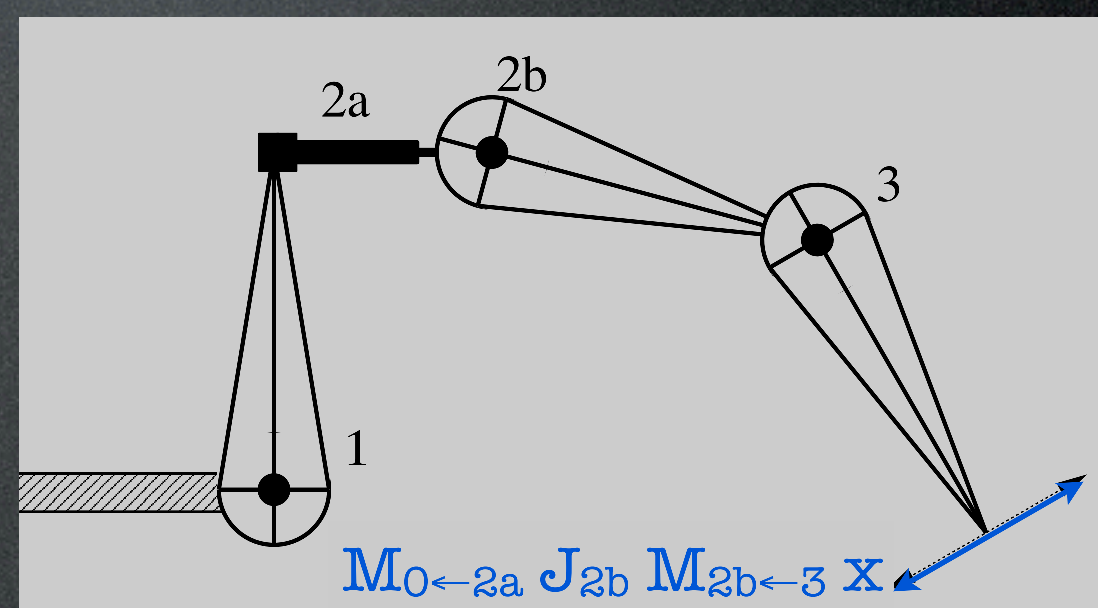
$$\mathbf{p} = \mathbf{M}_{0\leftarrow 1} \cdot \mathbf{M}_{1\leftarrow 2a} \cdot \mathbf{M}_{2a\leftarrow 2b} \cdot \mathbf{M}_{2b\leftarrow 3} \cdot \mathbf{x}$$

- If joint 2b moves, only $\mathbf{M}_{2a\leftarrow 2b}$ changes

$$\frac{\partial \mathbf{p}}{\partial \theta_{2b}} = \mathbf{M}_{0\leftarrow 1} \cdot \mathbf{M}_{1\leftarrow 2a} \cdot \frac{\partial}{\partial \theta_{2b}} \mathbf{M}_{2a\leftarrow 2b} \cdot \mathbf{M}_{2b\leftarrow 3} \cdot \mathbf{x}$$

$$= \mathbf{M}_{0\leftarrow 1} \cdot \mathbf{M}_{1\leftarrow 2a} \cdot \mathbf{J}_{2b}(\theta_{2b}) \cdot \mathbf{M}_{2b\leftarrow 3} \cdot \mathbf{x}$$





Multiple links

- Compute each joint's Jacobian locally (between outboard and inboard bodies)

$$\frac{\partial \mathbf{p}}{\partial \theta_1} = \mathbf{J}_1(\theta_1)\ \mathbf{M}_{1\leftarrow 3}\mathbf{x}$$

$$\frac{\partial \mathbf{p}}{\partial d_{2a}} = \mathbf{M}_{0\leftarrow 1}\mathbf{J}_{2a}(d_{2a})\mathbf{M}_{2a\leftarrow 3}\mathbf{x}$$

$$\frac{\partial \mathbf{p}}{\partial \theta_{2b}} = \mathbf{M}_{0\leftarrow 2a}\mathbf{J}_{2b}(\theta_{2b})\mathbf{M}_{2b\leftarrow 3}\mathbf{x}$$

$$\frac{\partial \mathbf{p}}{\partial \theta_3} = \mathbf{M}_{0\leftarrow 2b}\ \mathbf{J}_3(\theta_3)\ \mathbf{x}$$

$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ d_{2a} \\ \theta_{2b} \\ \theta_3 \end{bmatrix}$$

$$\mathbf{J} = \begin{bmatrix} \frac{\partial \mathbf{p}}{\partial \theta_1} & \frac{\partial \mathbf{p}}{\partial d_{2a}} & \frac{\partial \mathbf{p}}{\partial \theta_{2b}} & \frac{\partial \mathbf{p}}{\partial \theta_3} \end{bmatrix}$$

$$\mathrm{d}\mathbf{p} = \mathbf{J} \cdot \mathrm{d}\mathbf{q}$$

Multiple links

- Compute each joint's Jacobian locally (between outboard and inboard bodies)

$$\mathbf{J} = \begin{bmatrix} \cdot\ \mathbf{J}_1(\theta_1)\ \cdot \mathbf{M}_{1\leftarrow 3}\mathbf{x}, \\ \mathbf{M}_{0\leftarrow 1}\cdot \mathbf{J}_{2a}(d_{2a})\cdot \mathbf{M}_{2a\leftarrow 3}\mathbf{x}, \\ \mathbf{M}_{0\leftarrow 2a}\cdot \mathbf{J}_{2b}(\theta_{2b})\cdot \mathbf{M}_{2b\leftarrow 3}\mathbf{x}, \\ \mathbf{M}_{0\leftarrow 2b}\cdot\ \mathbf{J}_3(\theta_3)\ \cdot \mathbf{x} \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} \theta_1 \\ d_{2a} \\ \theta_{2b} \\ \theta_3 \end{bmatrix}$$

(each entry here is a column of the matrix)

$$\mathrm{d}\mathbf{p} = \mathbf{J} \cdot \mathrm{d}\mathbf{q}$$