# The Frequency Domain, without tears

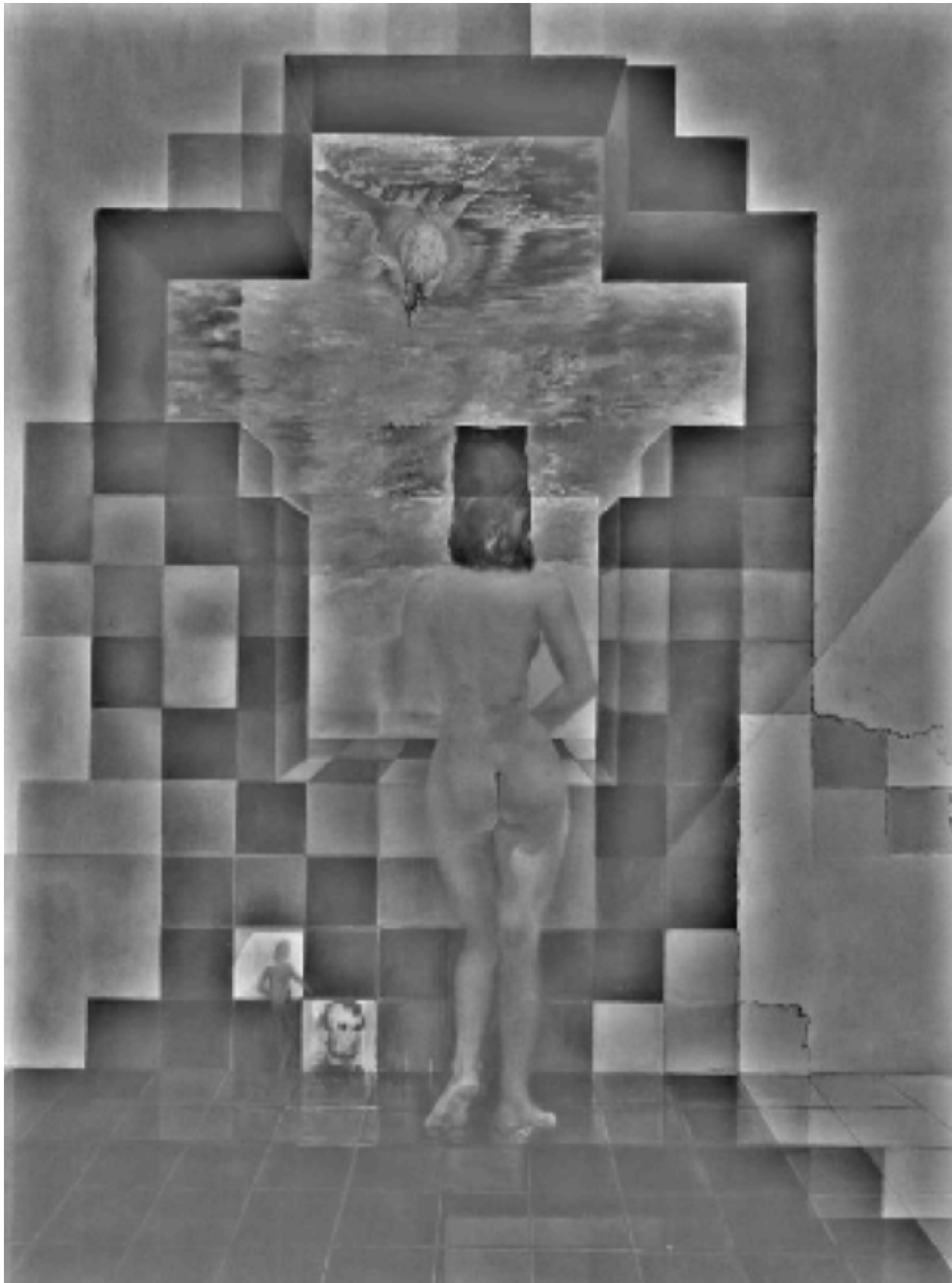
Somewhere in Cinque Terre, May 2005

CS194: Image Manipulation & Computational Photography

Alexei Efros, UC Berkeley, Fall 2017

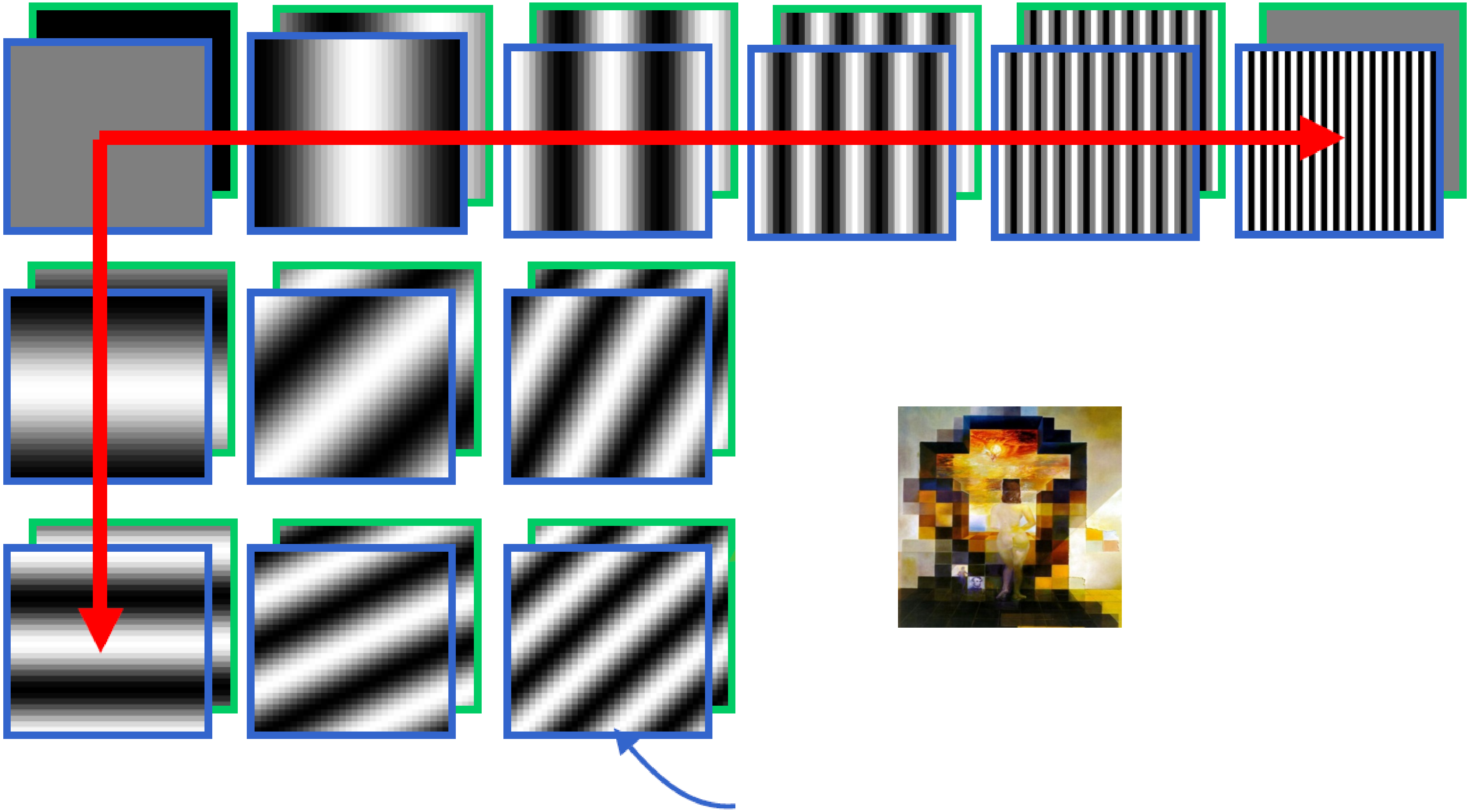
Many slides borrowed from Steve Seitz

**Salvador Dali**
*“Gala Contemplating the Mediterranean Sea, which at 30 meters becomes the portrait of Abraham Lincoln*”, 1976



# A nice set of basis

Teases away fast vs. slow changes in the image.



This change of basis has a special name…

# Jean Baptiste Joseph Fourier (1768-1830)

had crazy idea (1807

***Any*** *univariate function ca*
*be rewritten as a weightec*
*sum of sines and cosines*
*different frequencies.*

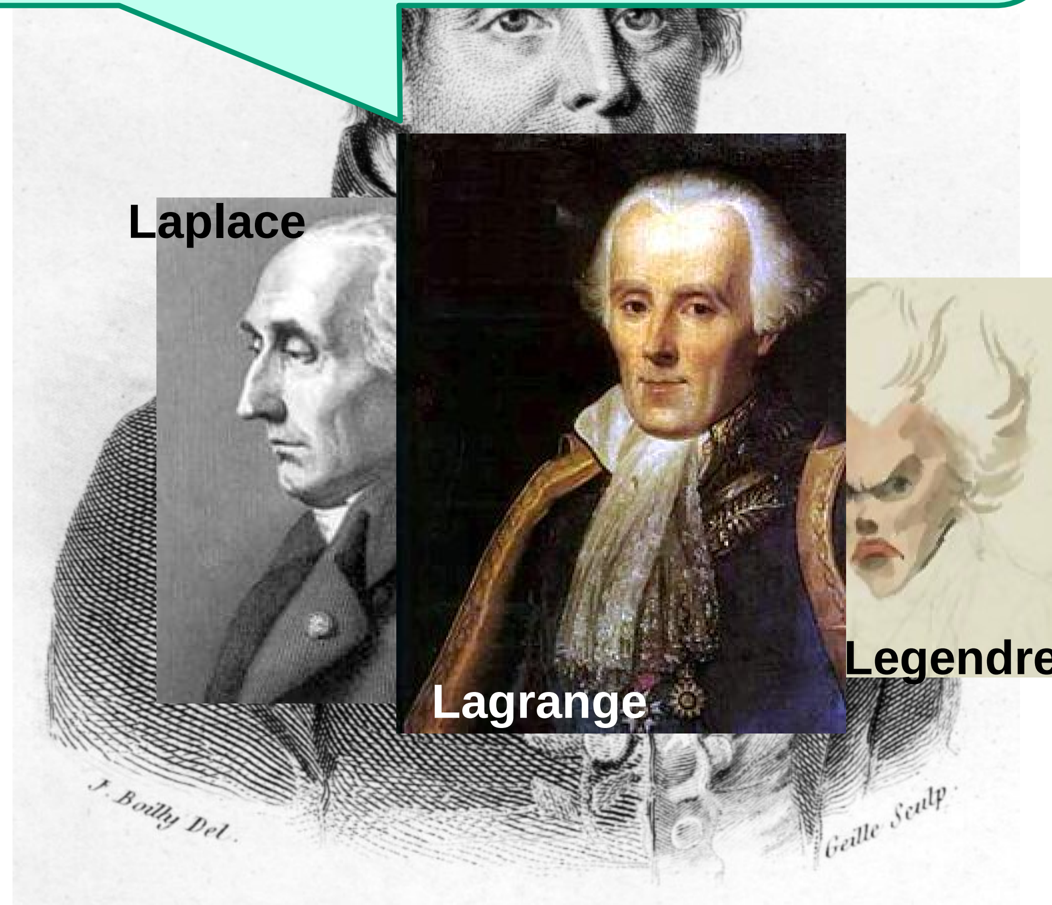
*...the manner in which the author arrives at these equations is not exempt of difficulties and...his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.*

Don't believe it?

- Neither did Lagrange, Laplace, Poisson and other big wigs
- Not translated into English until 1878!

But it's (mostly) true!

- called Fourier Series



# A sum of sines

Our building block:

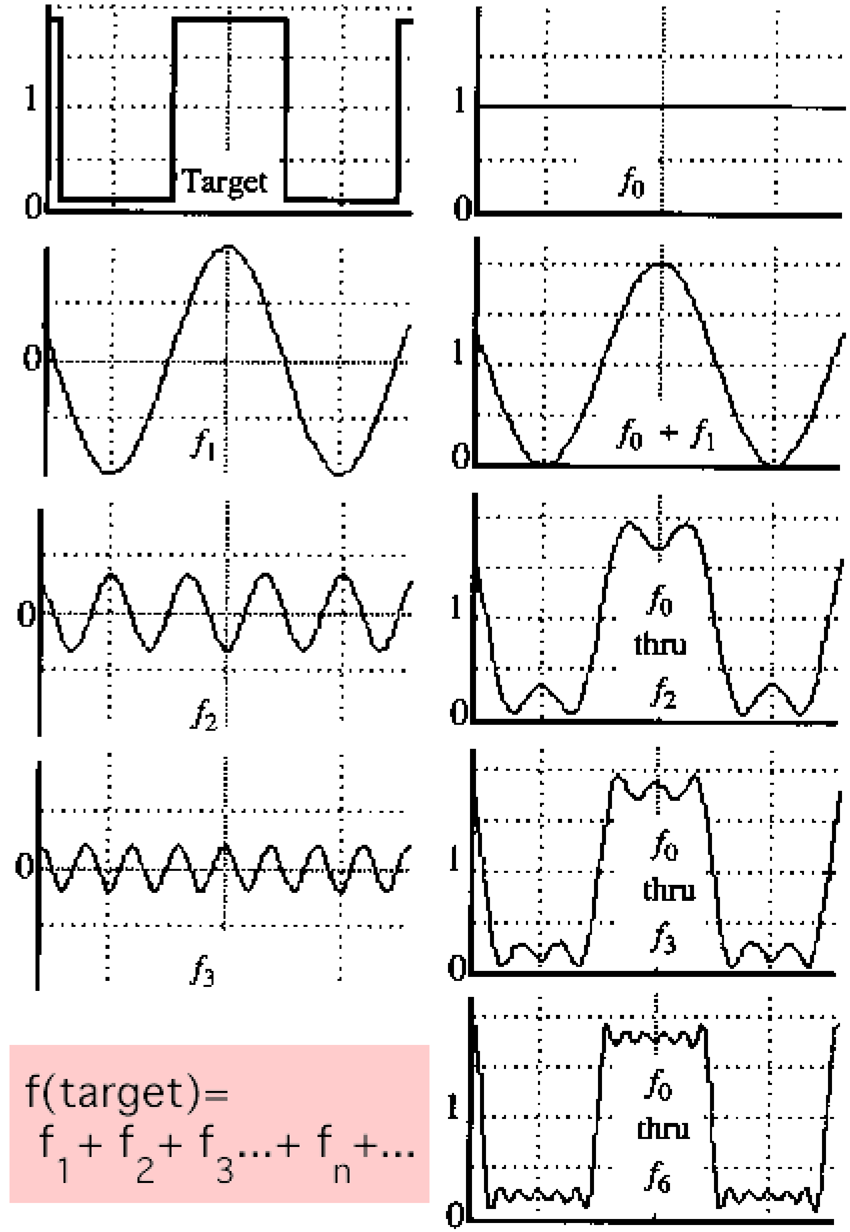
$$A\sin(\omega x + \phi)$$

Add enough of them to get any signal *f(x)* you want!

How many degrees of freedom?

What does each control?

Which one encodes the coarse vs. fine structure of the signal?




f(target)=
$f_1 + f_2 + f_3 ... + f_n + ...$

# Fourier Transform

We want to understand the frequency $\omega$ of our signal. So, let's reparametrize the signal by $\omega$ instead of *x*:

$$\boldsymbol{f(x)} \longrightarrow \boxed{\text{Fourier Transform}} \longrightarrow \boldsymbol{F(\omega)}$$

For every $\omega$ from 0 to inf, ***F(ω)*** holds the amplitude *A* and phase $\phi$ of the corresponding sine $A\sin(\omega x + \phi)$

- How can *F* hold both?

$$F(\omega) = R(\omega) + iI(\omega)$$

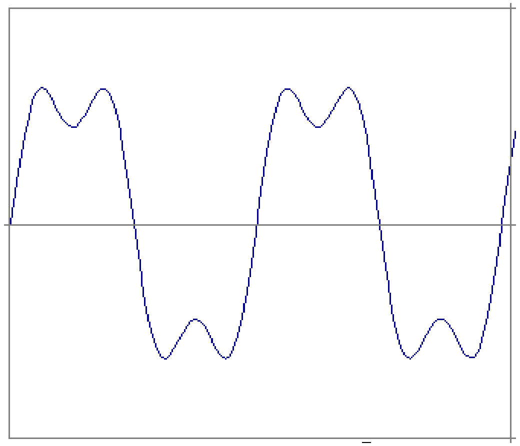
$$A = \pm\sqrt{R(\omega)^2 + I(\omega)^2} \qquad \phi = \tan^{-1}\frac{I(\omega)}{R(\omega)}$$

We can always go back:

$$\boldsymbol{F(\omega)} \longrightarrow \boxed{\text{Inverse Fourier Transform}} \longrightarrow \boldsymbol{f(x)}$$

# Time and Frequency

example : $g(t) = \sin(2pf\ t) + (1/3)\sin(2p(3f)\ t)$



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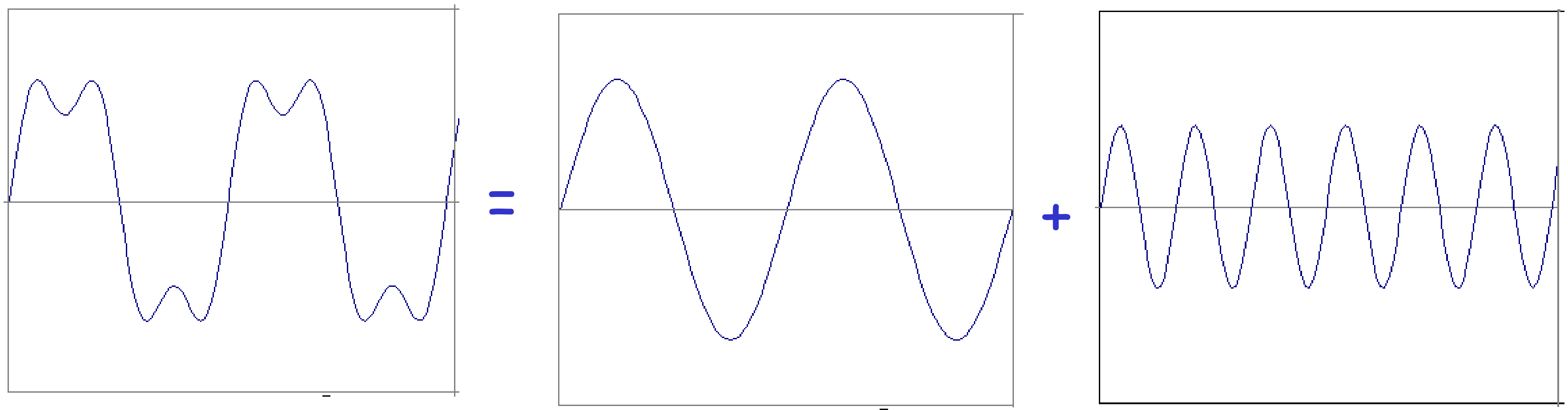

# Frequency Spectra

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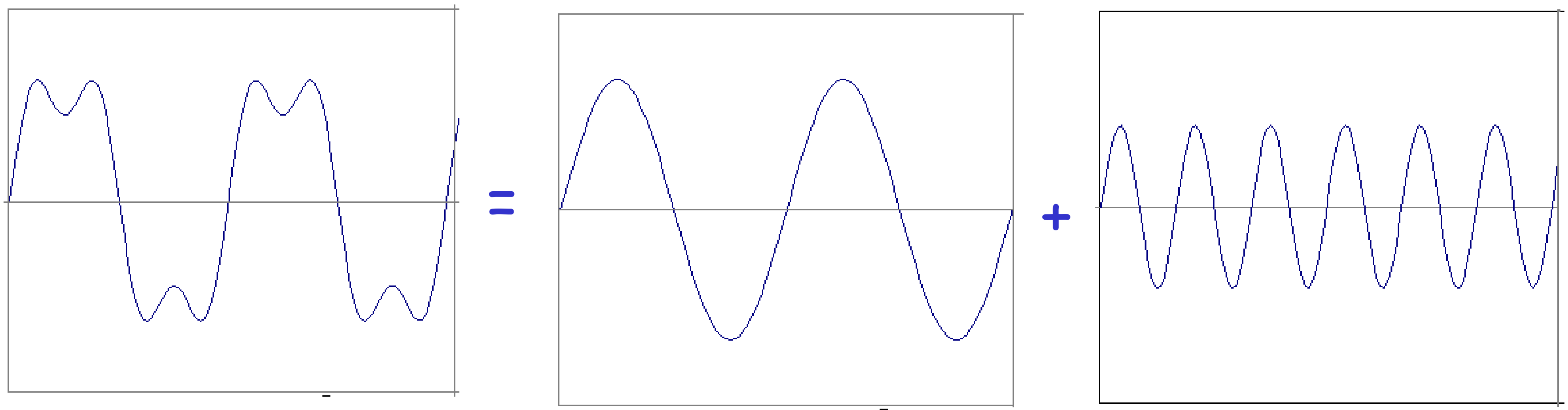
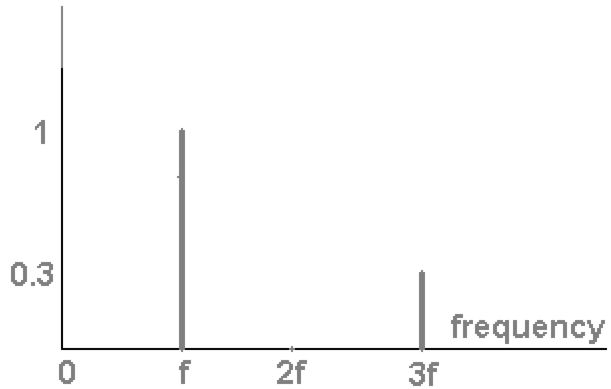
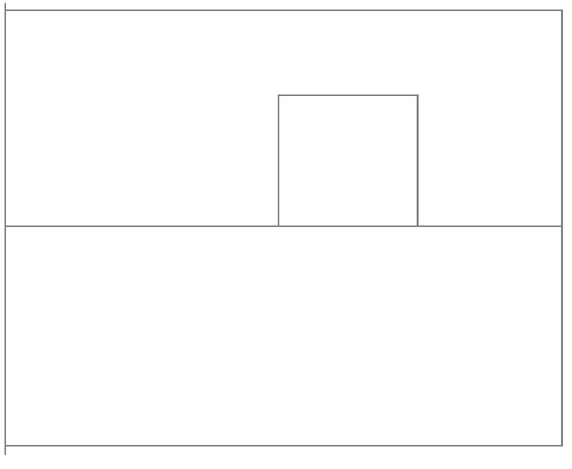


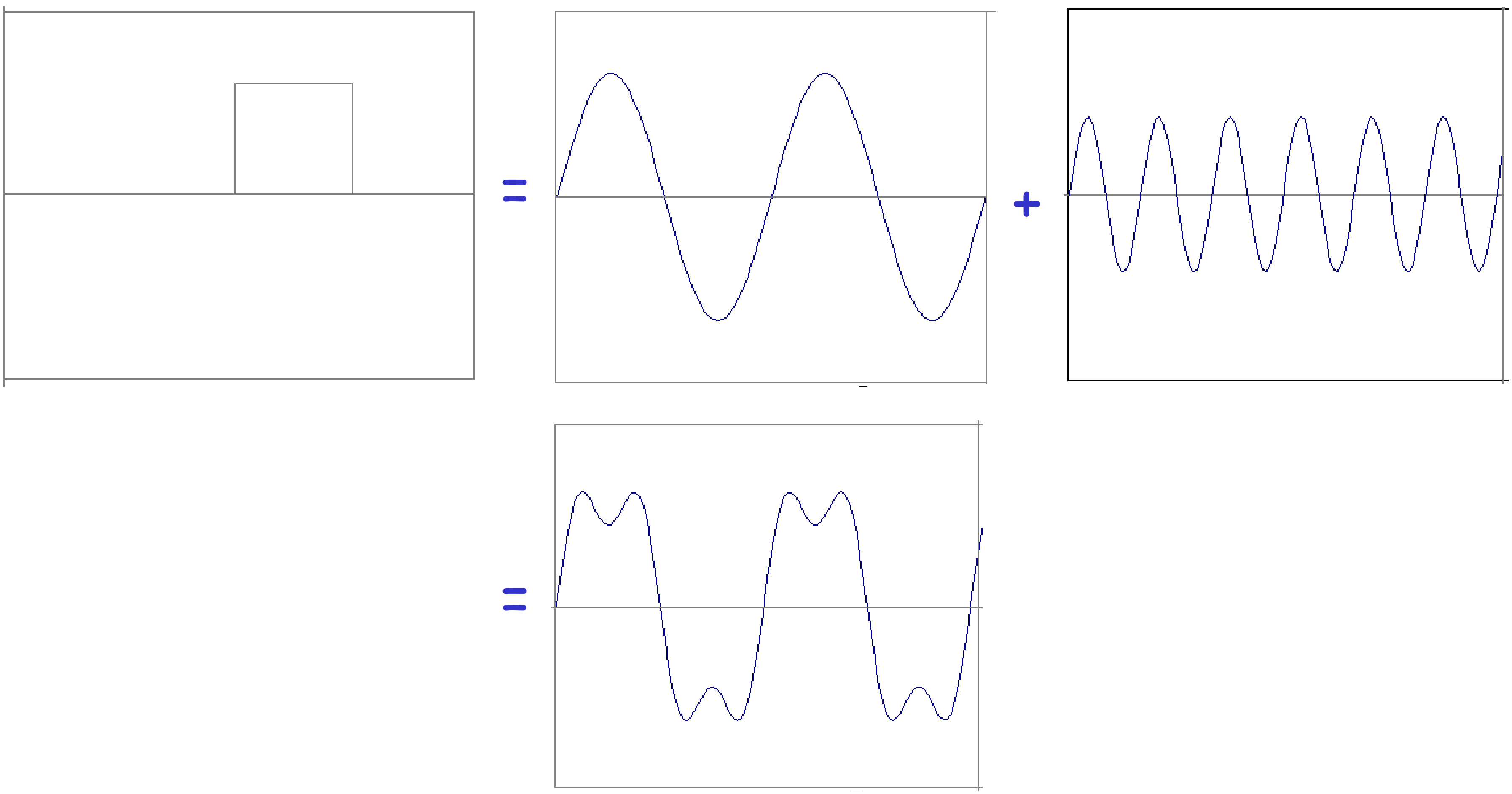



# Frequency Spectra

Usually, frequency is more interesting than the phase



# Frequency Spectra



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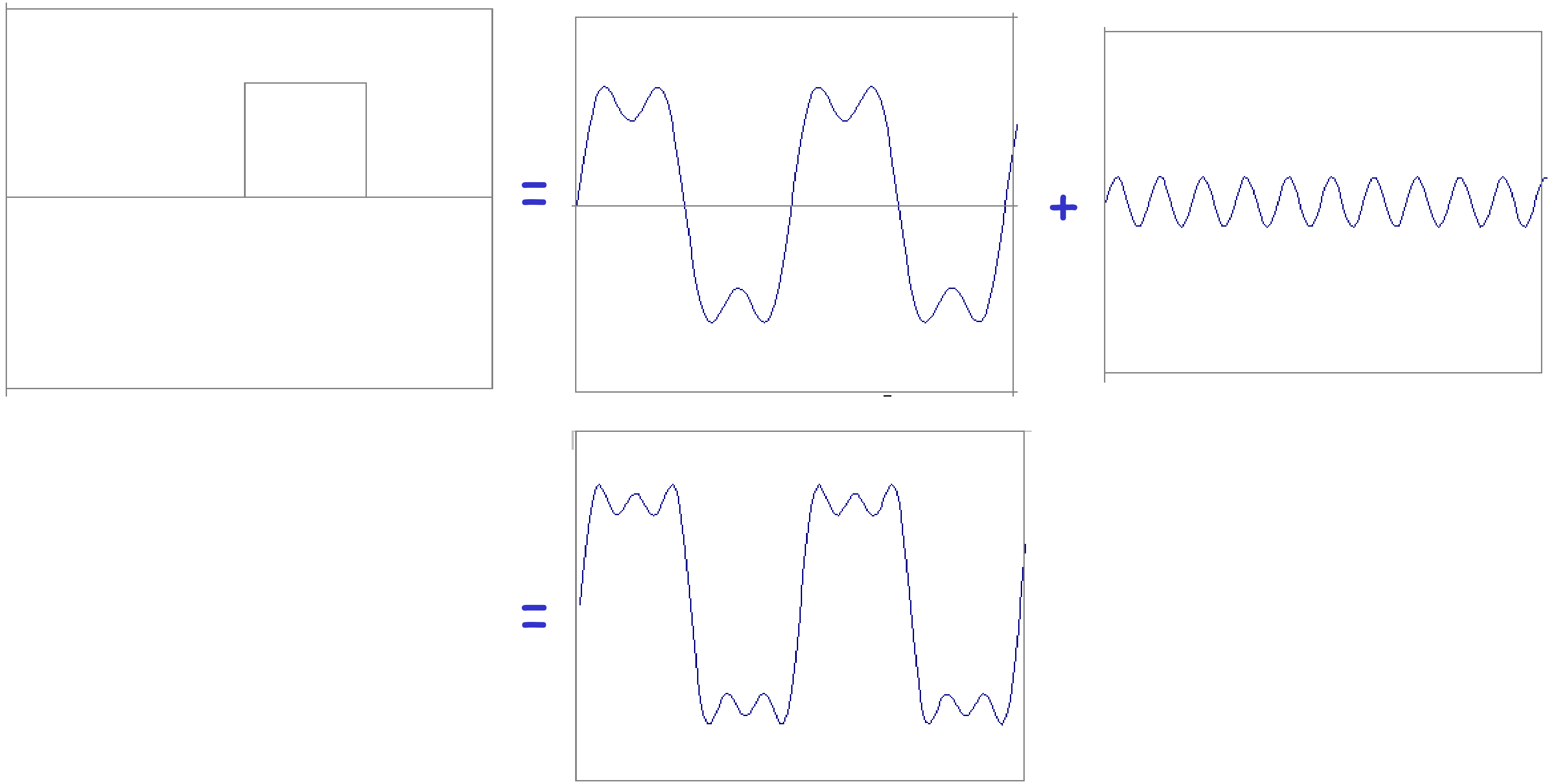

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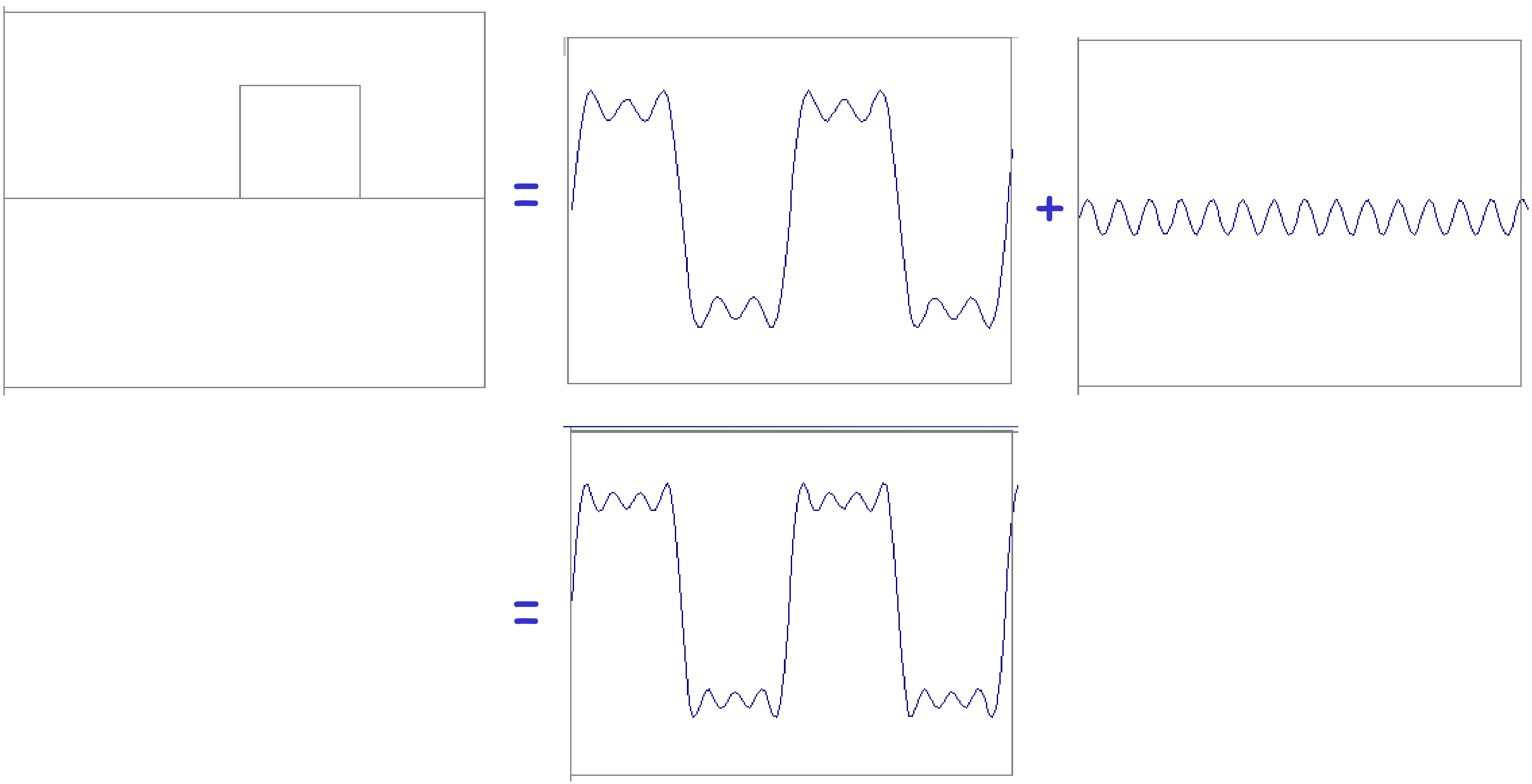

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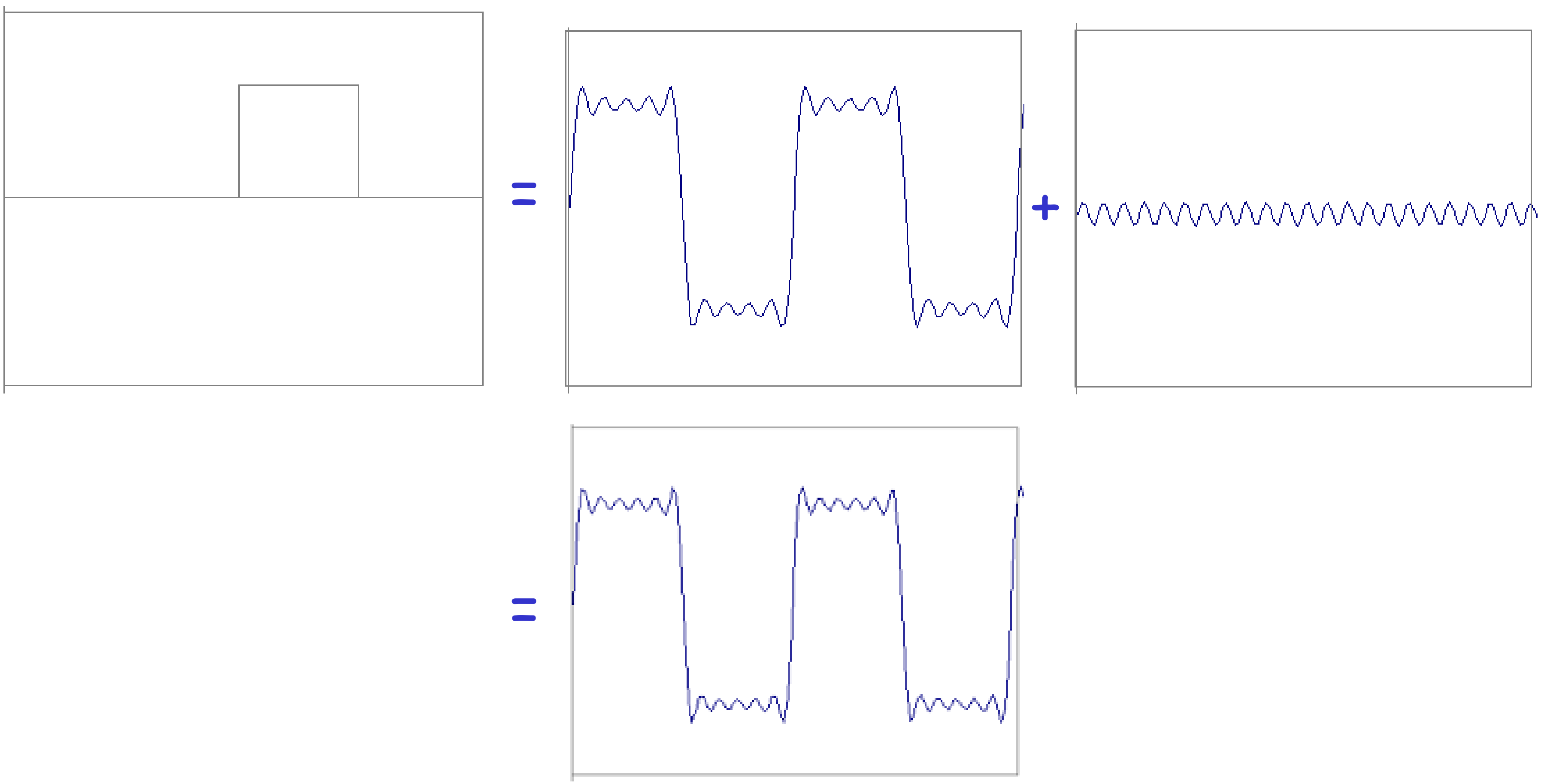
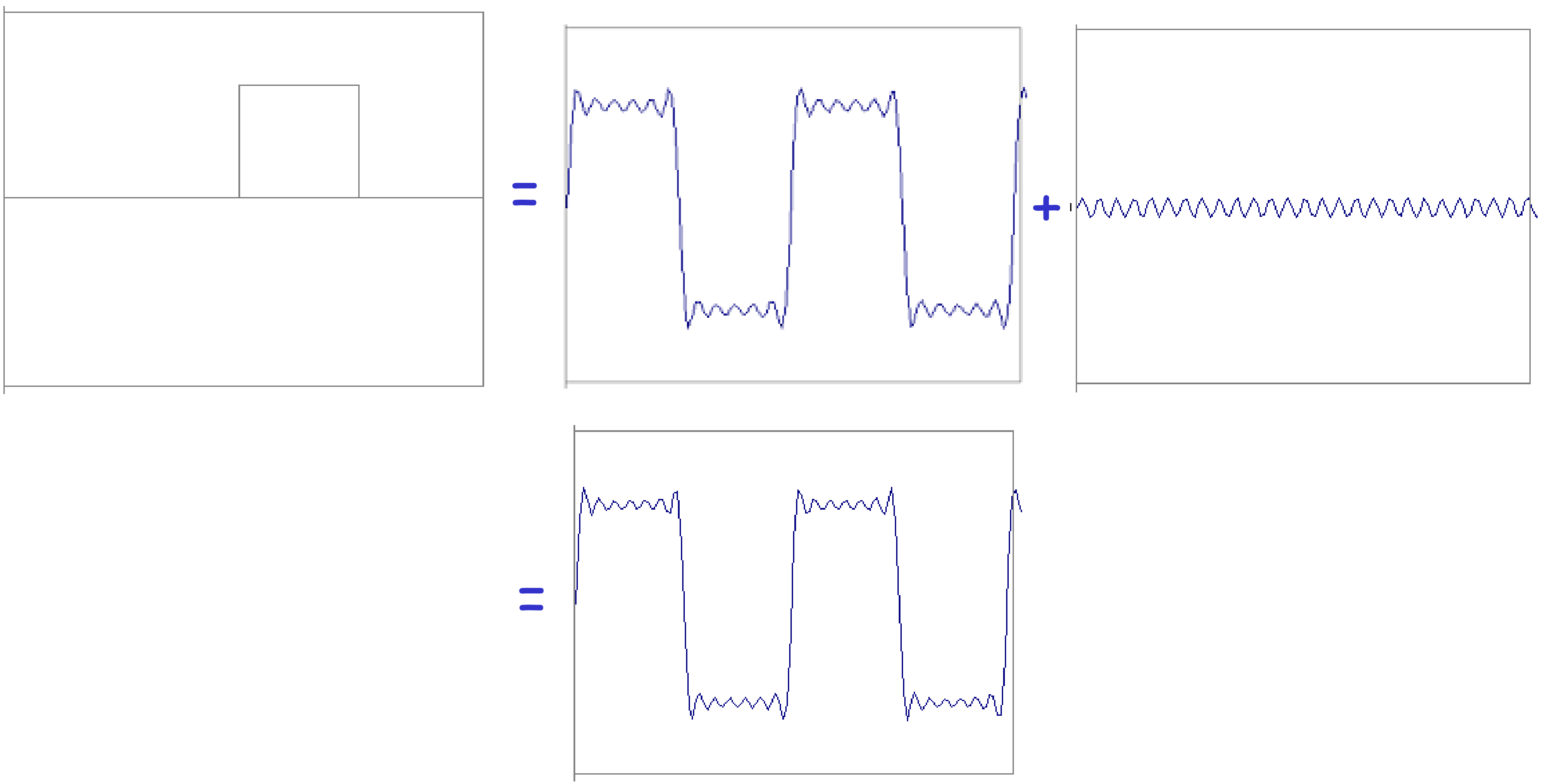
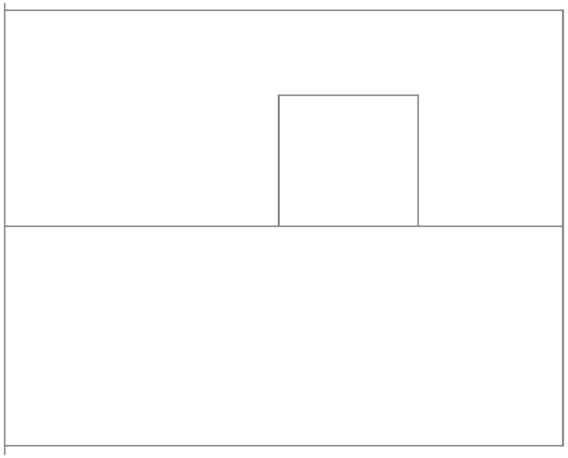


# Frequency Spectra

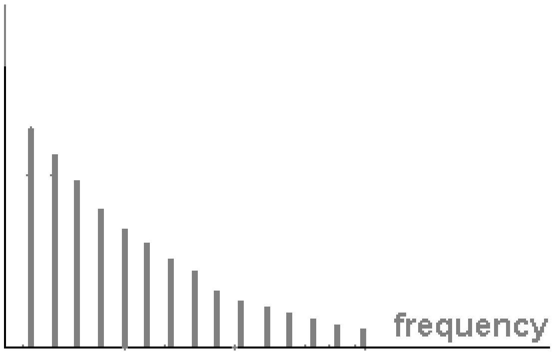


# Frequency Spectra

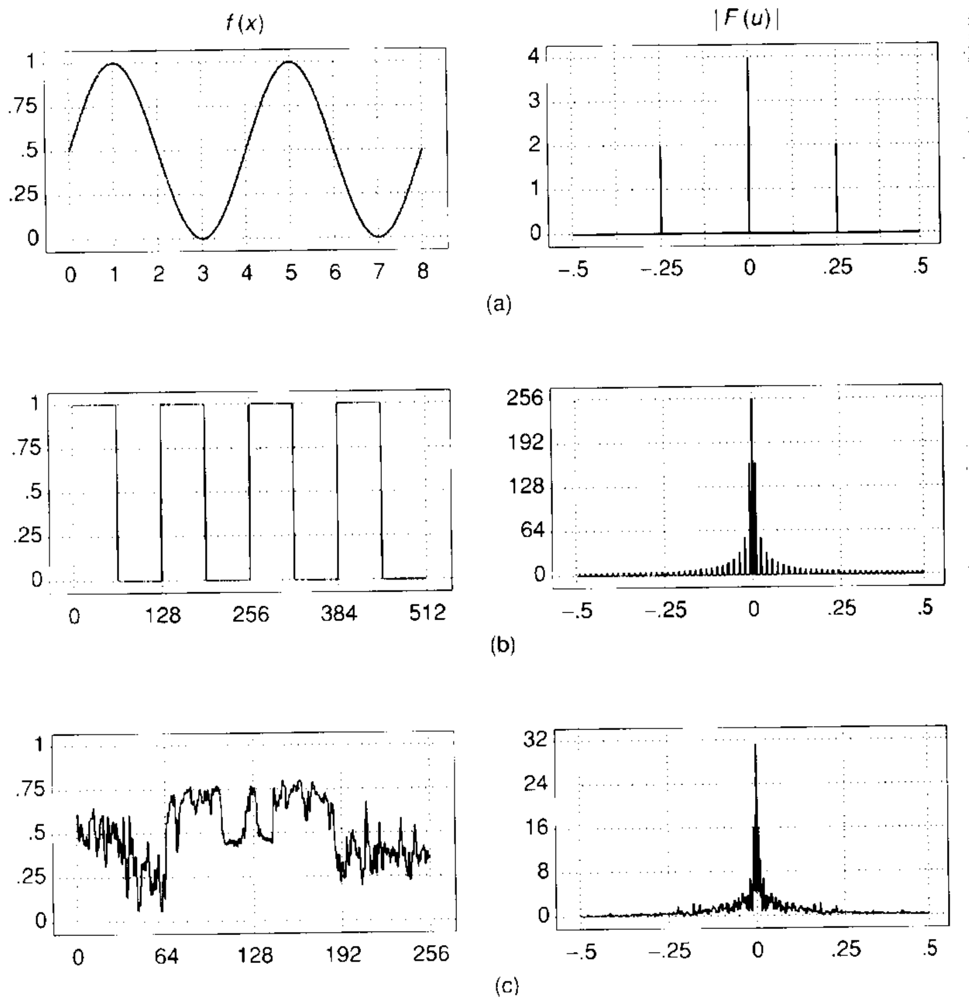


=

$$\boxed{A\sum_{k=1}^{\infty}\frac{1}{k}\sin(2\pi kt)}$$

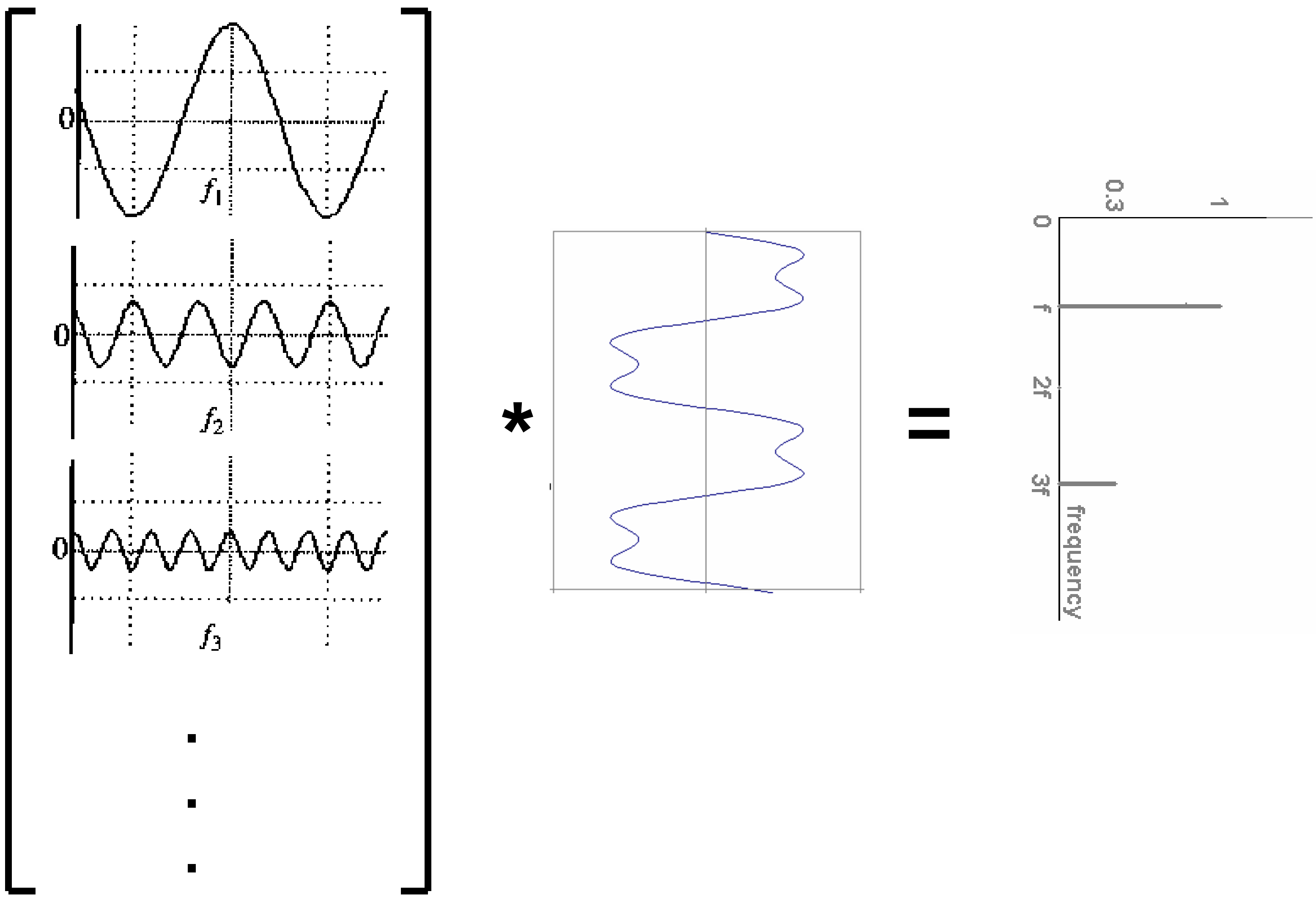


# Frequency Spectra



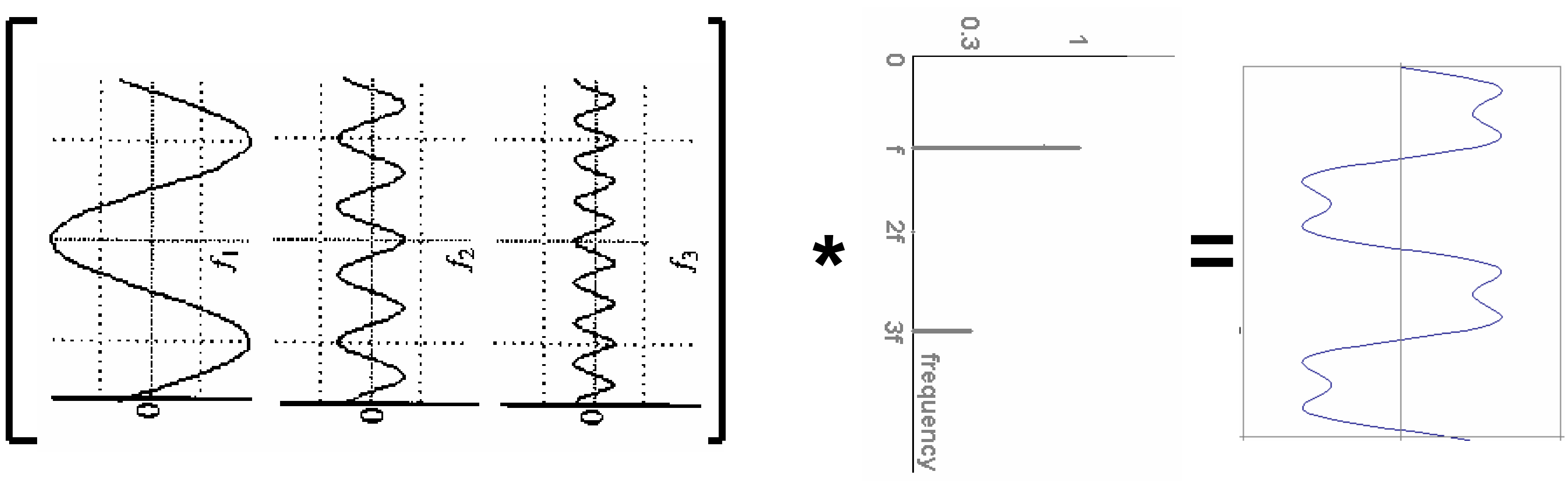
# FT: Just a change of basis

$$M * f(x) = F(\omega)$$



# IFT: Just a change of basis

$$M^{-1} * F(\omega) = f(x)$$



# Finally: Scary Math

$$\text{Fourier Transform :}\quad F(\omega) = \int_{-\infty}^{+\infty} f(x) e^{-i\omega x} dx$$

$$\text{Inverse Fourier Transform :}\quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega x} d\omega$$

# Finally: Scary Math

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…not really scary: $e^{i\omega x} = \cos(\omega x) + i\sin(\omega x)$

is hiding our old friend: $\sin(\omega x + \phi)$

phase can be encoded by sin/cos pair $\rightarrow$

$$P\cos(x) + Q\sin(x) = A\sin(x + \phi)$$

$$A = \pm\sqrt{P^2 + Q^2} \qquad \phi = \tan^{-1}\left(\frac{P}{Q}\right)$$

So it's just our signal *f(x)* times sine at frequency $\omega$

# Extension to 2D

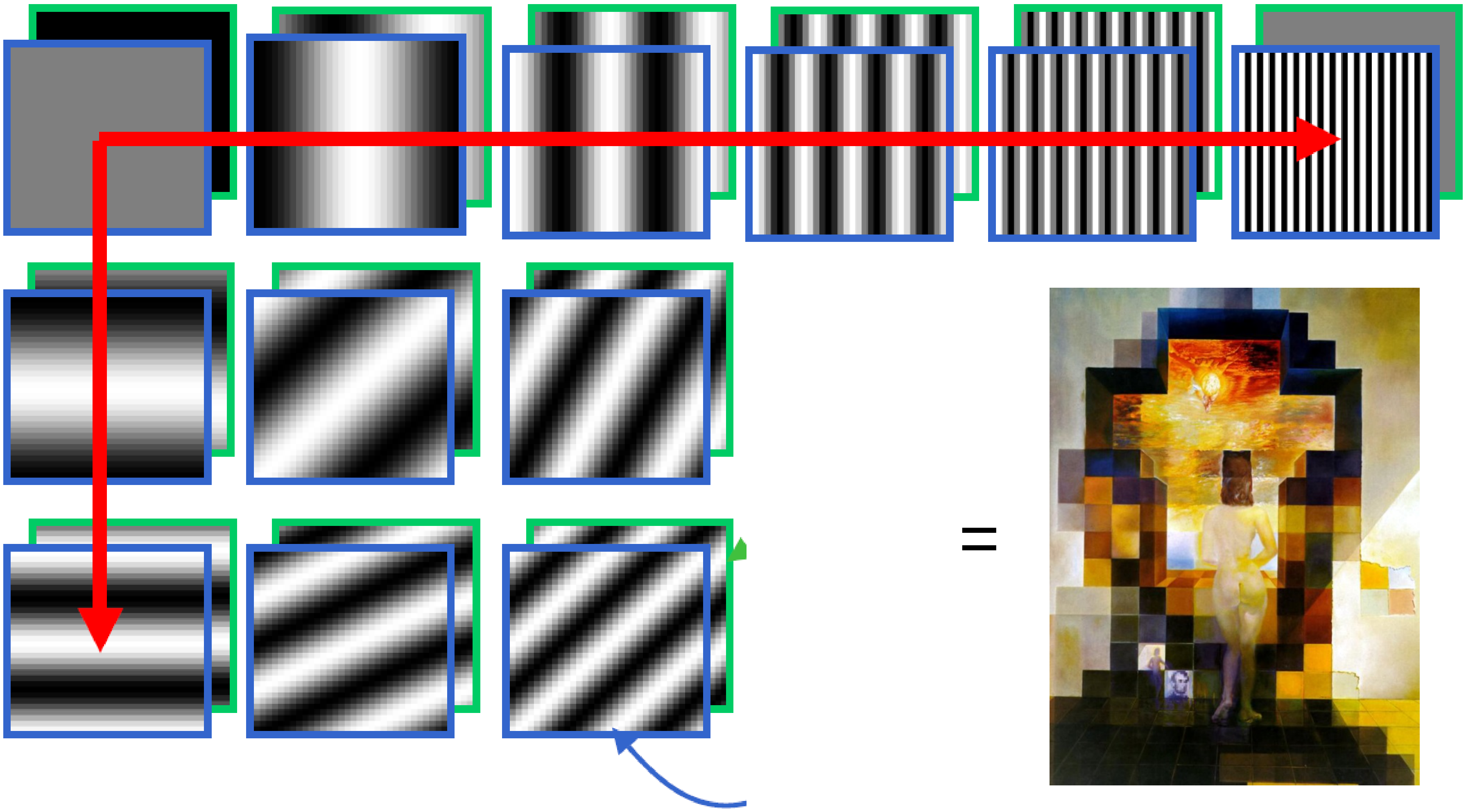
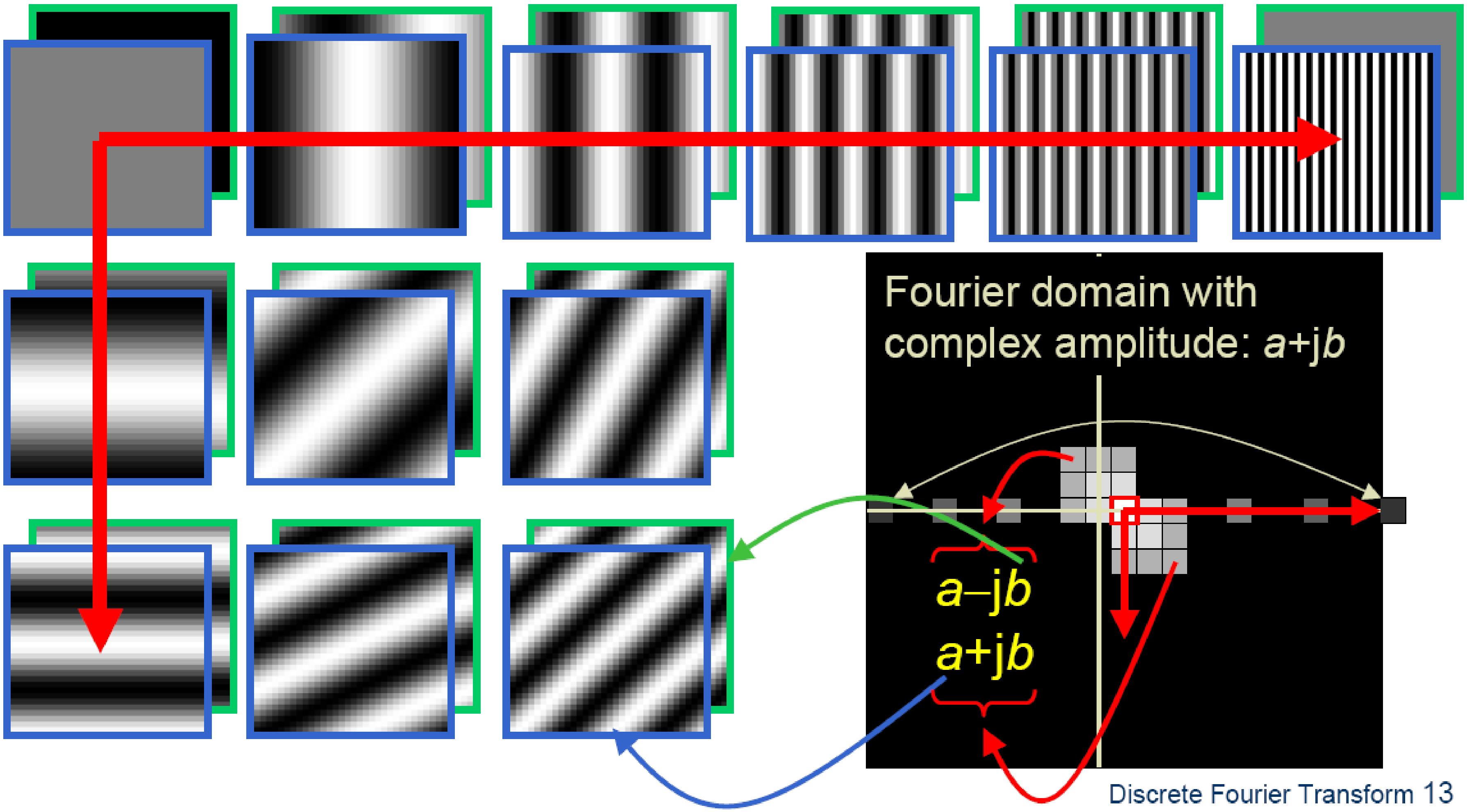


Image as a sum of basis images

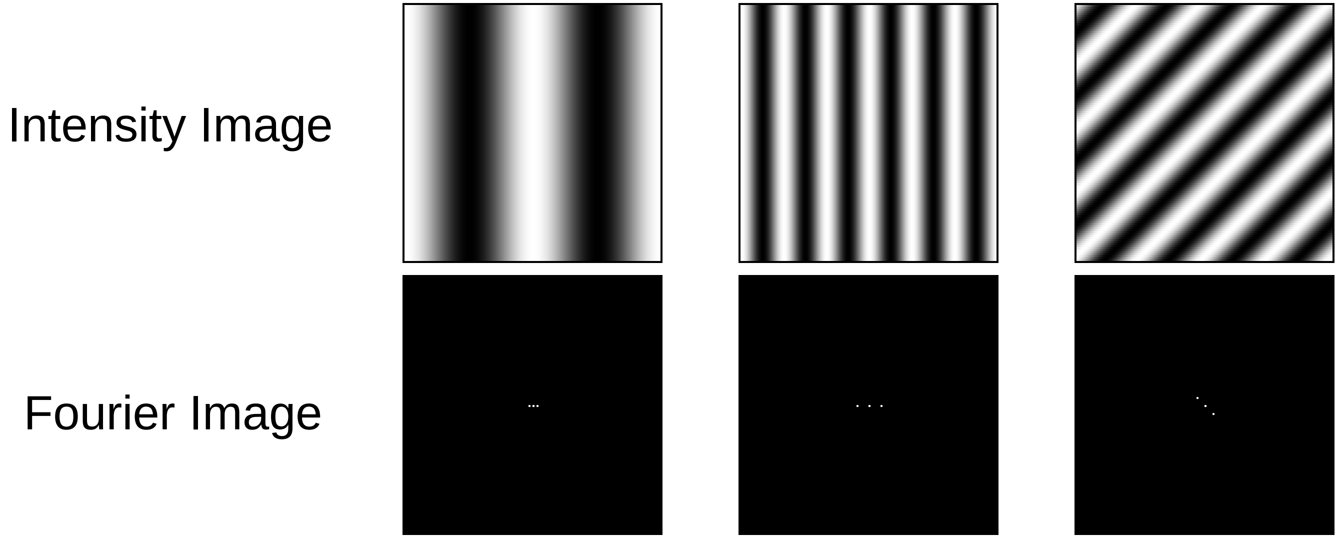
# Extension to 2D





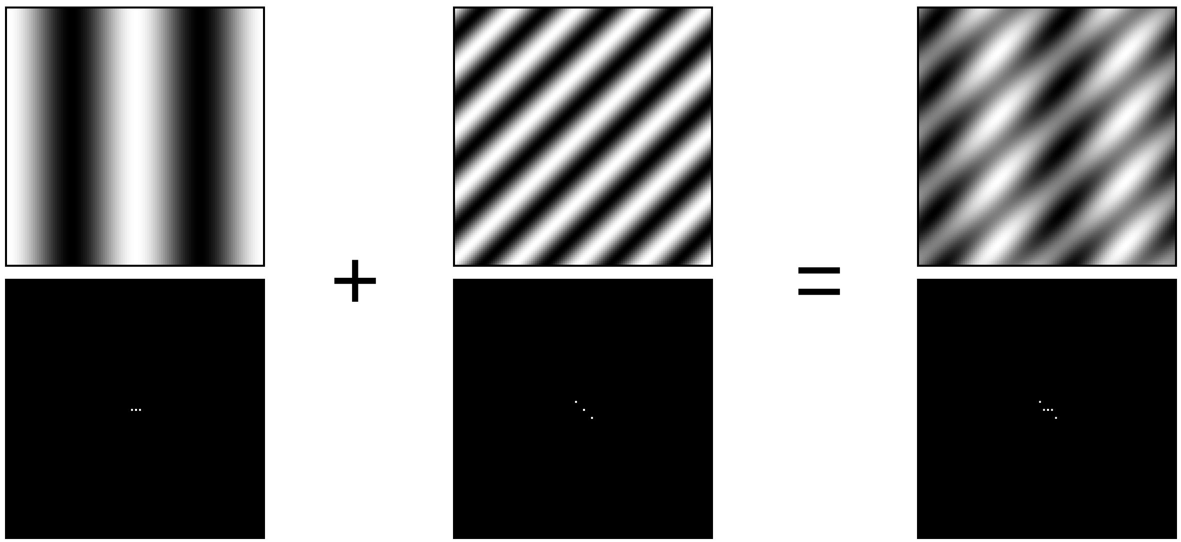

in Matlab, check out: imagesc(log(abs(fftshift(fft2(im)))));

# Fourier analysis in images



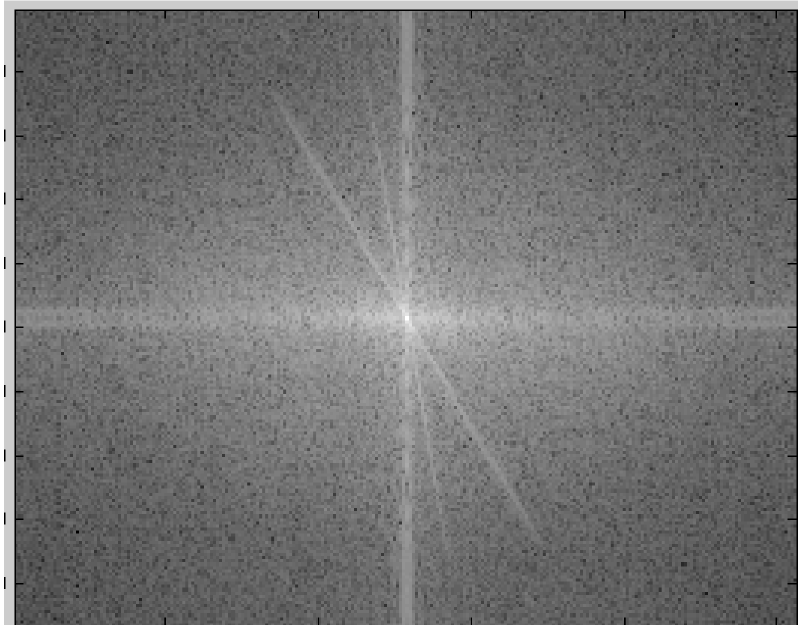

http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering

# Signals can be composed



http://sharp.bu.edu/~slehar/fourier/fourier.html#filtering
More: http://www.cs.unm.edu/~brayer/vision/fourier.html

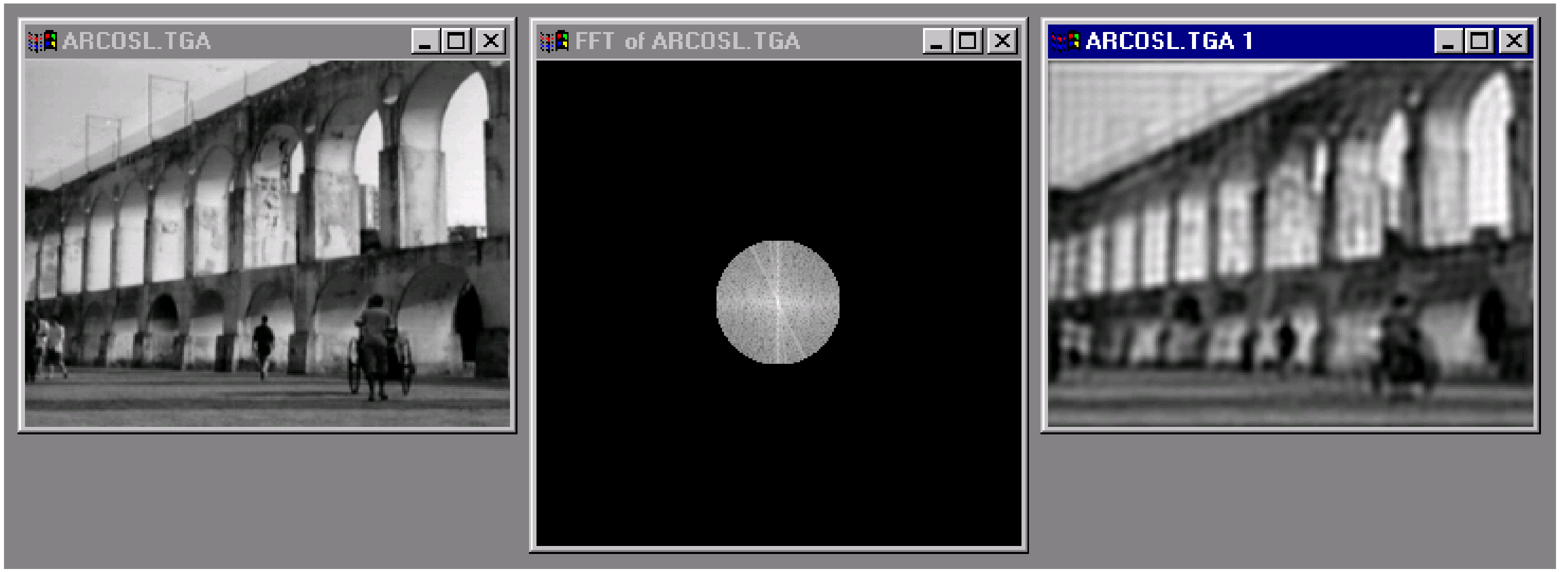
# Man-made Scene

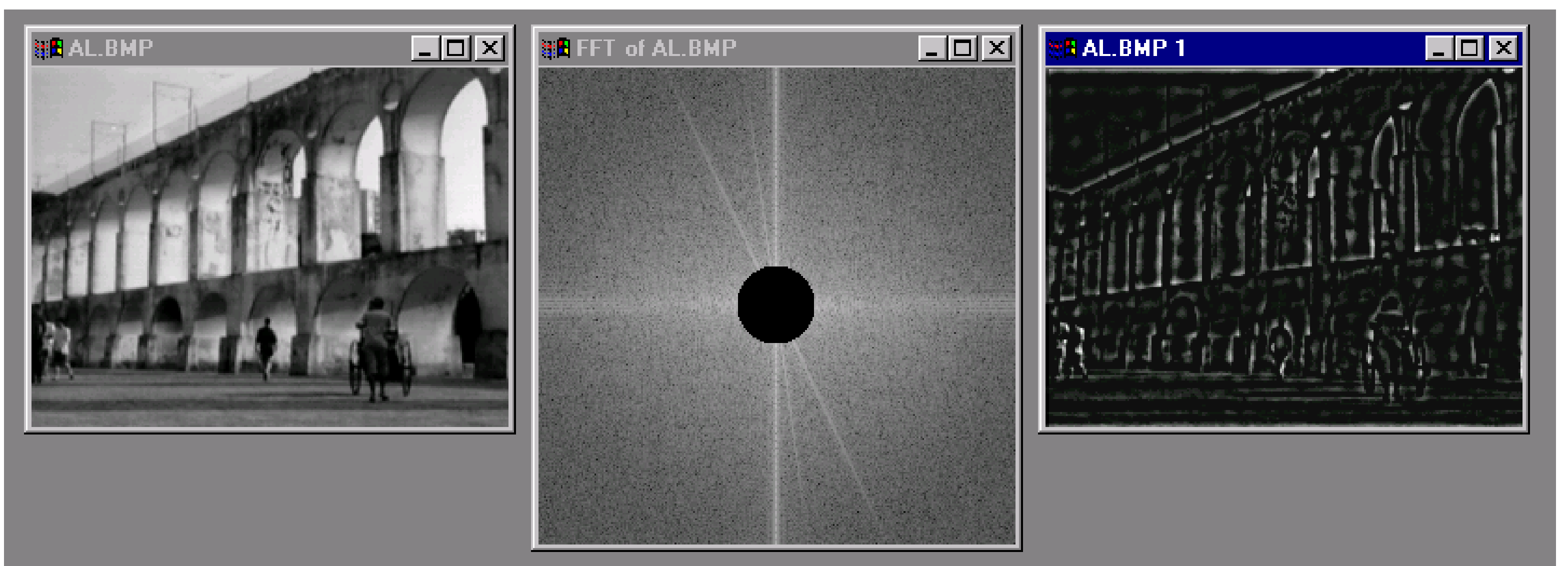


# Can change spectrum, then reconstruct


Local change in one domain, courses global change in the other

# Low and High Pass filtering



# The Convolution Theorem

The greatest thing since sliced (banana) bread!

- The Fourier transform of the convolution of two functions is the product of their Fourier transforms

$$\mathrm{F}[g * h] = \mathrm{F}[g]\,\mathrm{F}[h]$$

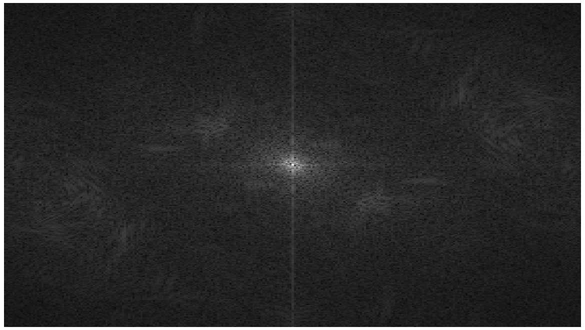
- The inverse Fourier transform of the product of two Fourier transforms is the convolution of the two inverse Fourier transforms

$$\mathrm{F}^{-1}[gh] = \mathrm{F}^{-1}[g] * \mathrm{F}^{-1}[h]$$

- **Convolution** in spatial domain is equivalent to **multiplication** in frequency domain!

# 2D convolution theorem example

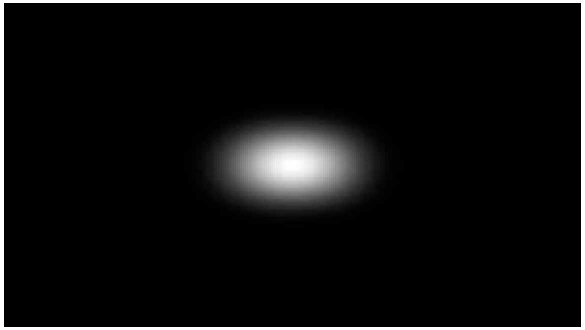
$f(x,y)$



$|F(s_x,s_y)|$

$*$ $\times$

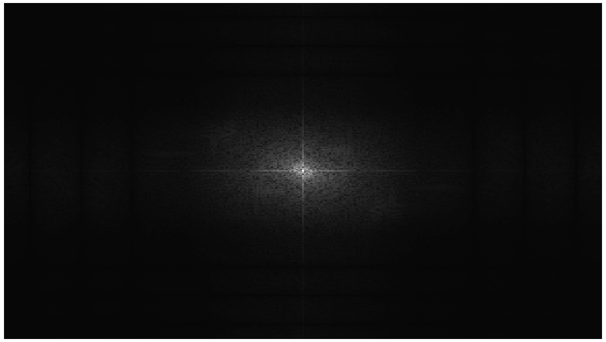
$h(x,y)$



$|H(s_x,s_y)|$

$\Downarrow$ $\Downarrow$

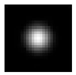
$g(x,y)$

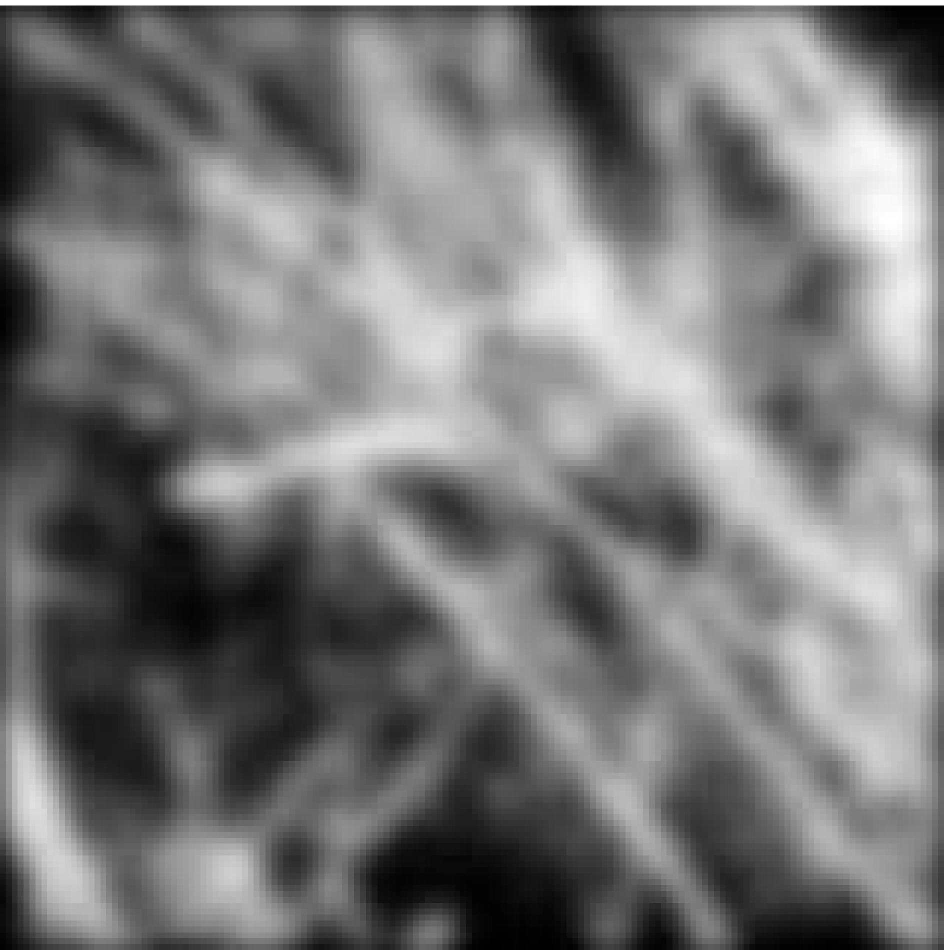


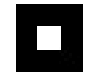
$|G(s_x,s_y)|$

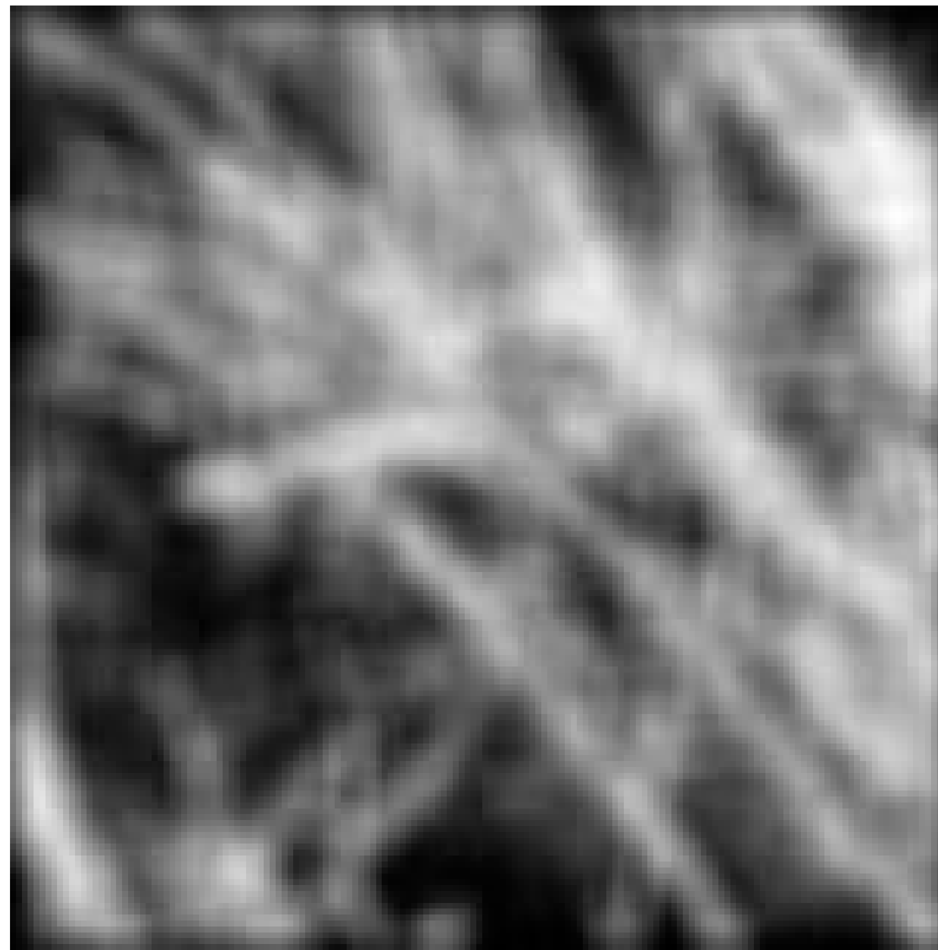
# Filtering

**Why does the Gaussian give a nice smooth image, but the square filter give edgy artifacts?**

Gaussian 



Box filter 



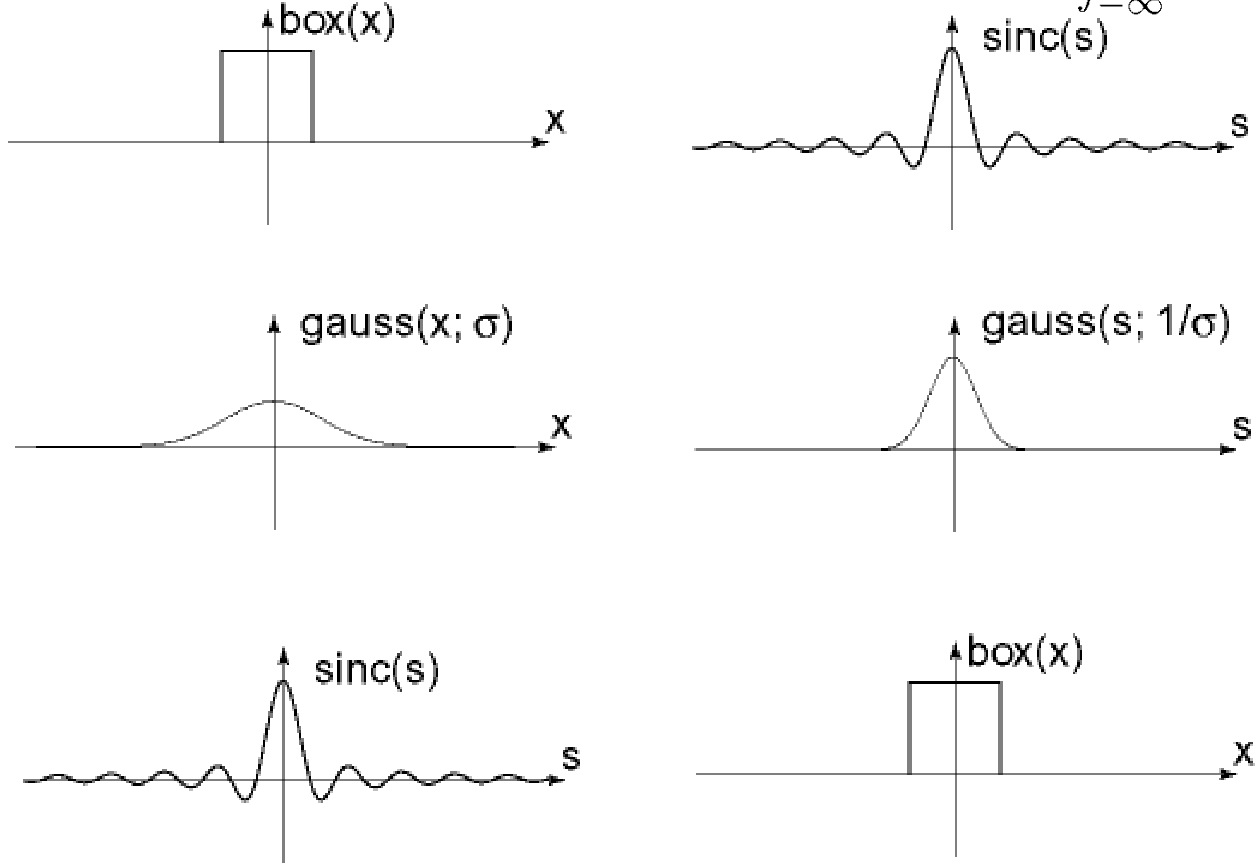
# Fourier Transform pairs

Spatial domain

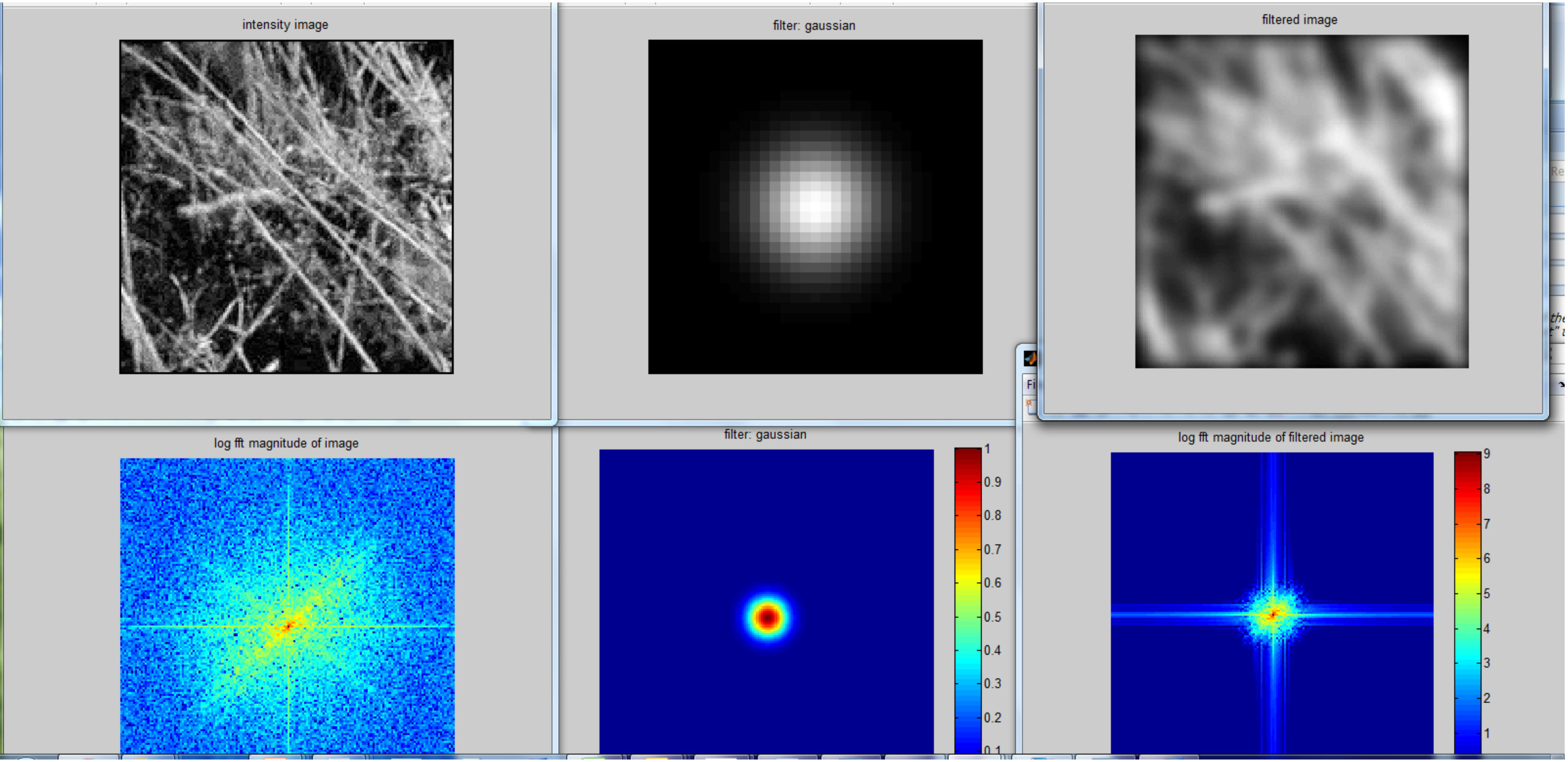
$f(x)$

Frequency domain

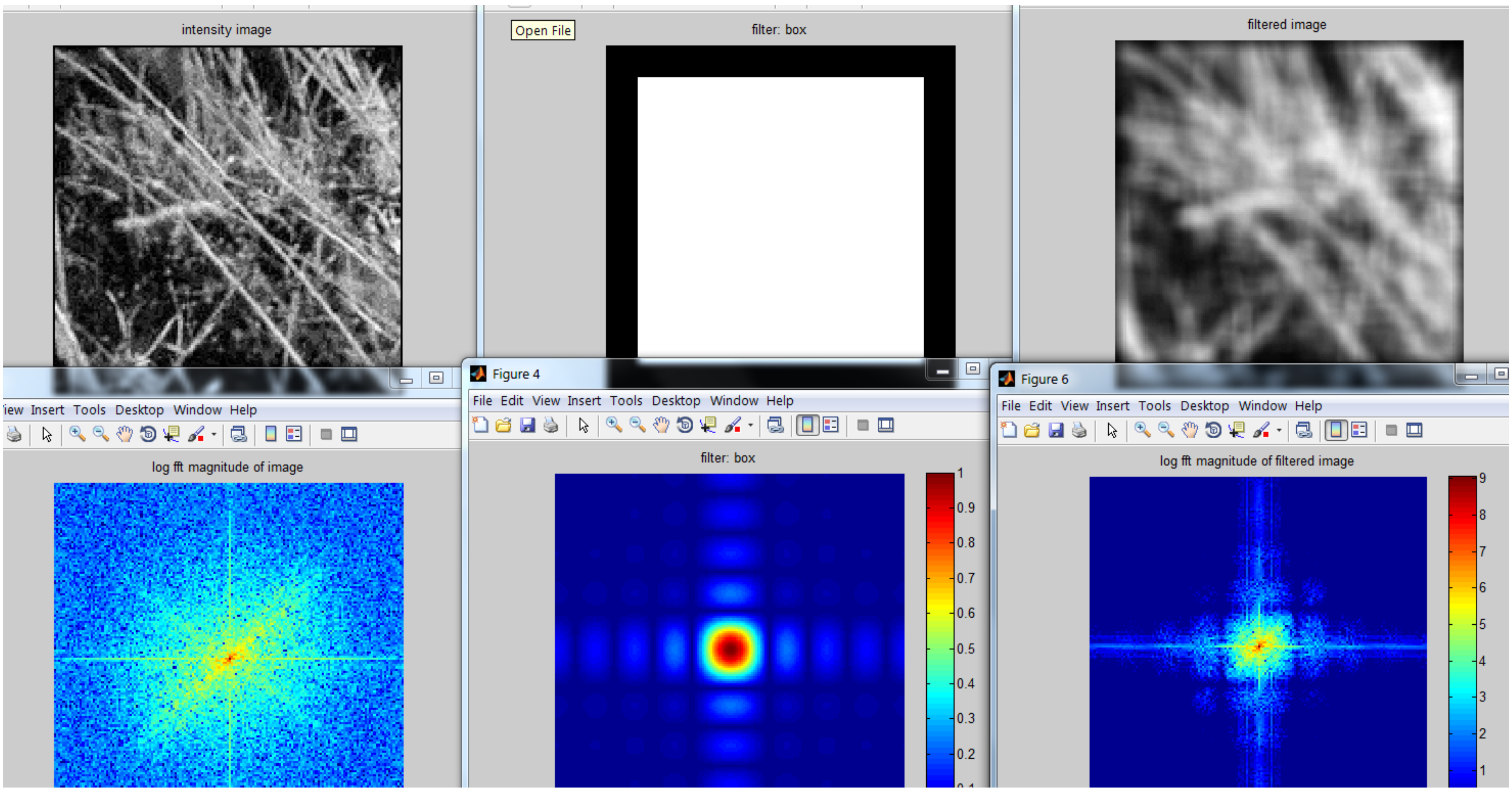
$$F(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi sx} dx$$



# Gaussian


intensity image
filter: gaussian
filtered image
log fft magnitude of image
filter: gaussian
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
log fft magnitude of filtered image
9
8
7
6
5
4
3
2
1

# Box Filter


intensity image
Open File
filter: box
filtered image
Figure 4
Figure 6
File Edit View Insert Tools Desktop Window Help
log fft magnitude of image
filter: box
log fft magnitude of filtered image