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Topic 1: The Lambda Calculus
Section 4.1: Semantics III

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Set-theoretic Properties of the Models

- **Those combinators (in case we need them)**
- **Sequences and operators (in case we need them)**
- **Connections with inclusion properties**
- **The least fixed point**

A the very basis of the idea of recursion is the principle that recursive definitions can be achieved by ***iteration***. In the case of of ***continuous operators***, this can be made very explicit in a form much promoted by S.C. Kleene in his work on the theory of recursive functions

Theorem. Any continuous operator $\Phi : \mathbb{P} \to \mathbb{P}$ has a ***least fixed point*** given by

$$P = \bigcup_{i=0}^{\infty} \Phi^i(\emptyset)$$

Proof. The notation $\Phi^i(X)$ means Φ iterated i-times starting with X.

Now, if we set $X = \emptyset$, then in the ***beginning*** $\emptyset \subseteq \Phi(\emptyset)$. Because Φ is monotone, it then easily follows that in all the ***later stages*** of iteration $\Phi^i(\emptyset) \subseteq \Phi^{i+1}(\emptyset)$. (Why?)

Note, too, that if we knew that Φ had a ***fixed point*** Q, then because $\emptyset \subseteq Q$, we could conclude that $\Phi^i(\emptyset) \subseteq Q$ holds for all i. (Why?) So this means that for the P of the theorem, we would have $P \subseteq Q$.

Therefore, ***if*** P can be shown to be a fixed point, it must be the ***least***.

Well, take the P of the theorem and calculate the action of Φ on it using what we know about continuity. We find:

$$\Phi(P) = \Phi(\bigcup_{i=0}^{\infty} \Phi^i(\emptyset)) = \bigcup_{i=0}^{\infty} \Phi^{i+1}(\emptyset) = \bigcup_{i=1}^{\infty} \Phi^i(\emptyset) = \bigcup_{i=0}^{\infty} \Phi^i(\emptyset) == P. \text{ (Why?)}$$

This shows that P ***is*** a fixed point. **Q.E.D.**

Question. Can this last argument be used to show that the ***following theorem*** holds?

Theorem. The least fixed point of a ***computable*** operator $\Phi : \mathbb{P} \to \mathbb{P}$ is $\mathbb{RE}$.

Proof. We give an argument by ***combinators***. First, we are assuming that

$$F = \lambda X. \Phi(X) \in \mathbb{RE}.$$

Let $D = \lambda X. F[X[X]]$. This is in $\mathbb{RE}$. Then, so is $P = D[D]$. (Why?)

From our knowledge of the ***formal use*** of reductions rules we know that $P = F[P]$ holds.

But, by definition, $F[P] = \Phi(P)$. So P ***is*** a fixed point of Φ in $\mathbb{RE}$.

Ah, ha! But is our ***combinator-created*** fixed point the least one? ***Some proof is needed.***

Let Q ***any*** fixed point of Φ in $\mathbb{P}$ (not just in $\mathbb{RE}$). We have to show $P \subseteq Q$.

Because the operator $X[X]$ is continuous, we can write: $P = \bigcup_{x \in D^*} \S\, x[\S\, x]$. (Why?)

So we need to prove that for all x, if $x \in D^*$, then $\S\, x[\S\, x] \subseteq Q$.

And we do that by ***Strong Induction.***

So, assume that $x \in D^*$ and for integers ***less than*** x, the ***implication*** holds.

Assume $y \in \S\, x[\S\, x]$. We have to prove $y \in Q$.

By ***definition of application***, we know that there is a pair $(z, y) \in \S\, x$ with $\S\, z \subseteq \S\, x$.

Inasmuch as the pair is a ***term*** of the sequence with number x, we know that $z < x$.

Now because $\S\, x \subseteq D$, we have $\S\, z \subseteq D$ and so $z \in D^*$. Hence, $\S\, z[\S\, z] \subseteq Q$ by the ***inductive assumption.***

Also, because $\S\, x \subseteq D$, we have $(z, y) \in D$. Hence, $y \in F[\S\, z[\S\, z]]$ by the definition of D.

By monotonicity, $F[\S\, z[\S\, z]] \subseteq F[Q] == Q$. Hence, $y \in Q$. **Q.E.D.**

Note. This proof is adapted from an old proof of the late ***David Park*** (Warwick University, UK) for a different model.

■ A note on defining fixed points

Note. This proof is added for those students who might be interested. It goes beyond what we need to know about definitions of ***computable*** objects, however.

In a more general set-theoretical setting, ***every*** monotone operator on a powerset has fixed points.

Knaster-Tarski Theorem (1928). If the operator $\mathbf{\Phi} : \mathcal{P}\,\mathbb{A} \to \mathcal{P}\,\mathbb{A}$ is ***monotone***, then the sets

$$\mathbf{LFP}_{\Phi} = \bigcap \{X \subseteq \mathbb{A} \mid \Phi(X) \subseteq X\} \text{ and}$$

$$\mathbf{GFP}_{\Phi} = \bigcup \{X \subseteq \mathbb{A} \mid X \subseteq \Phi(X)\}$$

are, respectively, the ***least*** and ***greatest*** fixed point of $\mathbf{\Phi}$.

Proof. Remark first that the definition of $\mathbf{LPF}_{\Phi}$ is a ***proper intersection*** over a non-empty class, because we have $\mathbf{\Phi}(\mathbb{A}) \subseteq \mathbb{A}$. Hence, $\mathbf{LPF}_{\Phi} \in \mathcal{P}\,\mathbb{A}$. Of course, $\mathbf{GPF}_{\Phi} \in \mathcal{P}\,\mathbb{A}$.

Note too, that all fixed points $\boldsymbol{P}$ of $\mathbf{\Phi}$ belong to the classes in the two definitions, thus

$$\mathbf{LFP}_{\Phi} \subseteq P \subseteq \mathbf{GFP}_{\Phi}.$$

However, this does not prove that the two sets ***are*** fixed points.

Suppose that $\mathbf{\Phi}(X) \subseteq X \subseteq \mathbb{A}$ is any one of those sets in the first definition. Then we have:

$$\mathbf{LFP}_{\Phi} \subseteq X.$$

By monotonicity we find:

$$\mathbf{\Phi}(\mathbf{LFP}_{\Phi}) \subseteq \mathbf{\Phi}(X) \subseteq X.$$

Because $\mathbf{\Phi}(\mathbf{LFP}_{\Phi})$ is included in ***all*** the sets in the intersection, we conclude:

$$\mathbf{\Phi}(\mathbf{LFP}_{\Phi}) \subseteq \mathbf{LFP}_{\Phi}.$$

Again, by monotonicity we have:

$$\mathbf{\Phi}(\mathbf{\Phi}(\mathbf{LFP}_{\Phi})) \subseteq \mathbf{\Phi}(\mathbf{LFP}_{\Phi}).$$

In other words, $\mathbf{\Phi}(\mathbf{LFP}_{\Phi})$ ***is one*** of the sets in the intersection, and so:

$$\mathbf{LFP}_{\Phi} \subseteq \mathbf{\Phi}(\mathbf{LFP}_{\Phi}).$$

Thus, we have proved:

$$\mathbf{LFP}_{\Phi} = \mathbf{\Phi}\,(\mathbf{LFP}_{\Phi}).$$

And, by what we noted at the beginning, it must be the ***least*** fixed point.

The argument for $\mathbf{GFP}_{\Phi}$ is analogous. **Q.E.D.**

Note. The proof for $\mathbf{GFP}_{\Phi}$ can also be given by defining the operator:

$$\mathbf{\Psi}(X) = \mathbb{A} \setminus \mathbf{\Phi}(\mathbb{A} \setminus X),$$

and noting that $\mathbf{\Psi}$ is also monotone, and so $\mathbf{GFP}_{\Phi} == \mathbb{A} \setminus \mathbf{LFP}_{\Psi}$.

Note. Here are two original references:

B. Knaster (1928). "Un théorème sur les fonctions d'ensembles". Ann. Soc. Polon. Math., vol. 6, pp. 133-134.

A. Tarski (1955). "A lattice-theoretical fixpoint theorem and its applications". Pacific Jour. Math., vol. 5. pp. 285-309.