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Topic 2: Type Theory
Section 6.2: Type Semantics III

Dana S. Scott
Hillman University Professor (Emeritus)
School of Computer Science
Carnegie Mellon University
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Visiting Professor EECS
Visiting Scientist
Logic & Methodology Program
University of California, Berkeley

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Isomorphisms and higher types

■ A function-space question

As we recall $\mathbb{P} = \mathbb{P} \times \mathbb{P}$ and $\mathcal{P} = \mathcal{I}d(\mathbb{P})$ are ***non-isomorphic types***. (Why?) The next question concerns $(\mathcal{P} \to \mathcal{P})$.

Theorem. There is a $\mathbb{Q} \subseteq \mathbb{P}$ such that $\mathcal{Q} = \mathcal{I}d\,(\mathbb{Q}) \cong (\mathcal{P} \to \mathcal{P})$.

Proof. It is true that every $U : (\mathcal{P} \to \mathcal{P})$. (Why?) But that ***does not*** mean that $(\mathcal{P} \to \mathcal{P})$ is a trivial ***PER***. In fact, $U\,(\mathcal{P} \to \mathcal{P})\,V$ means that $U[X] = V[X]$ for all $X \in \mathbb{P}$. (Why?) But this, in turn, means that $\lambda X.\,U[X] = \lambda X.\,V[X]$. (Why?) And, moreover, $U\,(\mathcal{P} \to \mathcal{P})\,\lambda X.\,U[X]$ holds. (Why?)

OK. Let $\mathbb{Q} = \{\lambda X.\,U[X] \mid U \in \mathbb{P}\}$. Then the ***desired isomorphism*** will follow. **Q.E.D.**

Theorem. The two types $\mathcal{P}$ and $\mathcal{Q}$ are ***not*** isomorphic.

Proof. Suppose they ***were*** isomorphic. We want to derive a contradiction.

Isomorphism means we have operators $P : \mathcal{P} \to Q$ and $Q : Q \to \mathcal{P}$ such that $Q \circ P = \mathbb{1}_{\mathcal{P}}$ and $P \circ Q = \mathbb{1}_{Q}$. (Why?). Because these operators are ***monotone***, it would follow that for all $X, Y \in \mathbb{P}$, we would have

$$X \subseteq Y \textit{ if, and only if, } P[X] \subseteq P[Y].$$

By the same token, it would follow that for all $U, V \in \mathbb{Q}$, we would have

$$U \subseteq V \textit{ if, and only if, } Q[U] \subseteq Q[V].$$

(Why?) Note that this implies $P[\emptyset] = \lambda X. \emptyset$. (Why?)

Now in $\mathbb{P}$ the set $\{0\}$ has a ***very special*** property: for all $X \in \mathbb{P}$, if $X \subseteq \{0\}$, then either $X = \emptyset$ or $X = \{0\}$. (Why?) It then follows that for all $U \in \mathbb{Q}$, if $U \subseteq P[\{0\}]$, then either $U = P[\emptyset]$ or $U = P[\{0\}]$. (Why?) This means that $P[\{0\}]$ has only two subsets in $\mathbb{Q}$. Note also that $P[\{0\}] \neq P[\emptyset]$. (Why?) Therefore, $P[\{0\}][\mathbb{N}] \neq \emptyset$. (Why?) Hence, $P[\{0\}][E] \neq \emptyset$ for some ***finite*** $E \in \mathbb{P}$. (Why?) Let $S = P[\{0\}][E]$.

Consider the operator $W = \lambda X.\{n \in S \mid E \subseteq X\}$. Then $W \in \mathbb{Q}$, $W \subseteq P[\{0\}]$ and $W \neq P[\emptyset]$. (Why?) Now let $e \in \mathbb{N} \setminus E$.

Consider the operator $V = \lambda X.\left\{n \in S \,\middle|\, E \bigcup \{e\} \subseteq X\right\}$. Then $V \in \mathbb{Q}$, $V \subseteq P[\{0\}]$ and $V \neq P[\emptyset]$. (Why?) ***But*** $V \neq W$. (Why?)

Thus, we have a ***contradiction!*** (Why?) **Q.E.D.**

Question. What about Q and $(\mathcal{P} \to Q)$ and $(Q \to \mathcal{P})$?

■ Higher-type spaces

We are going to consider whether the types

$$\mathcal{P}, (\mathcal{P} \to \mathcal{P}), ((\mathcal{P} \to \mathcal{P}) \to \mathcal{P}), (((\mathcal{P} \to \mathcal{P}) \to \mathcal{P}) \to \mathcal{P}), ((((\mathcal{P} \to \mathcal{P}) \to \mathcal{P}) \to \mathcal{P}) \to \mathcal{P}), \ldots$$

naturally have some kind of "***limit***". To do this, we define ***operators*** which can be used to define these types in a convenient way.

$$D_0 = \lambda U. U$$
$$D_{n+1} = \lambda U \lambda X. U[D_n[X]] = \lambda U. (U \circ D_n)$$

$$U \,\mathcal{D}_n\, V \Longleftrightarrow D_n[U] == D_n[V]$$

Whatever the operator D_n is, it is clear that $\mathcal{D}_n$ is a ***(total) equivalence relation***. (Why?)

Note. We can ***unwind*** the definition of D_4 in order to understand better what these operators do. Some ***changes of variables*** will help us follow the uses of the definitions.

$$\begin{aligned} D_4[U_4] &= \lambda U_3. U_4[D_3[U_3]] \\ &= \lambda U_3. U_4[\lambda U_2. U_3[D_2[U_2]]] \\ &= \lambda U_3. U_4[\lambda U_2. U_3[\lambda U_1. U_2[D_1[U_1]]]] \\ &= \lambda U_3. U_4[\lambda U_2. U_3[\lambda U_1. U_2[\lambda U_0. U_1[D_0[U_0]]]]] \\ &= \lambda U_3. U_4[\lambda U_2. U_3[\lambda U_1. U_2[\lambda U_0. U_1[U_0]]]] \end{aligned}$$

Now, if we knew in general that the equation $U = \lambda X. U[X]$ held, then the above expression would ***collapse*** to U_4. But, this is ***not true*** in $\mathbb{P}$. All we can conclude is $U_4 \subseteq D_4[U_4]$.

Theorem. The D-operators form a tower: $D_0 \subseteq D_1 \subseteq D_2 \subseteq \ldots \subseteq D_n \subseteq D_{n+1} \subseteq \ldots$.

Proof. As the first inclusion means that $\lambda U.U \subseteq \lambda U \lambda X.U[X]$, we know this is true. So, we then proceed by ***induction***. Assume that $D_n \subseteq D_{n+1}$, and try to prove that $D_{n+1} \subseteq D_{n+2}$.

By definition $D_{n+2} = \lambda U.(U \circ D_{n+1})$. By the inductive assumption and monotonicity, we see that $D_{n+2} \supseteq \lambda U.(U \circ D_n) = D_{n+1}$. **Q.E.D.**

Theorem. For all $n \in \mathbb{N}$, we have $D_n \circ D_n = D_n$.

Proof. Use induction. $D_0 \circ D_0 = D_0$ is obvious. So, assume $D_n \circ D_n = D_n$, and try to prove $D_{n+1} \circ D_{n+1} = D_{n+1}$.

We use the definitions:

$$\begin{aligned} D_{n+1} \circ D_{n+1} &= \lambda U.D_{n+1}[D_{n+1}[U]] \\ &= \lambda U.D_{n+1}[U \circ D_n] \\ &= \lambda U.(U \circ D_n) \circ D_n \\ &= \lambda U.U \circ (D_n \circ D_n) \\ &= \lambda U.U \circ D_n \\ &= D_{n+1} \end{aligned}$$

Q.E.D.

Theorem. For all $n \in \mathbb{N}$, we have $D_n \circ D_{n+1} = D_{n+1} = D_{n+1} \circ D_n$.

Proof. Because $D_0 \subseteq D_n$, we can argue by monotonicity:

$$D_{n+1} == D_0 \circ D_{n+1} \subseteq D_n \circ D_{n+1} \subseteq D_{n+1} \circ D_{n+1} == D_{n+1}.$$

Hence, the first equation follows. The second equation is proved similarly. **Q.E.D.**

Corollary. For all $n, m \in \mathbb{N}$ if $m \geq n$, we have $D_n \circ D_m = D_m = D_m \circ D_n$.

Proof. This is an easy induction. **Q.E.D.**

Corollary. For all $n \in \mathbb{N}$, we have $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$.

Proof. Assume $U\ \mathcal{D}_n\ V$ holds. This means that $D_n[U] == D_n[V]$. But then:

$$D_{n+1}[D_n[U]] == D_{n+1}[D_n[V]].$$

Then, by our theorem, we see that

$$D_{n+1}[U] == D_{n+1}[V].$$

In other words, $U\ \mathcal{D}_{n+1}\ V$ holds. **Q.E.D.**

We now can derive at once some ***limiting properties*** using the equations for the D_n-operators and continuity. The proofs can safely be left to the reader.

$$D_\infty == \bigcup_{n==0}^{\infty} D_n$$

$$U\ \mathcal{D}_\infty\ V \Longleftrightarrow D_\infty[U] == D_\infty[V]$$

Theorem. $D_\infty \circ D_\infty = D_\infty = \lambda U.U \circ D_\infty$

Theorem. For all $n \in \mathbb{N}$, we have $D_n \circ D_\infty = D_\infty \circ D_n = D_\infty$.

Corollary. For all $n \in \mathbb{N}$, we have $\mathcal{D}_n \subseteq \mathcal{D}_\infty$.

With these $\mathcal{D}$-equivalence relations, the ***larger*** the relation the ***fewer*** the distinctions. This "decrease" in structure can be made clearer by picking out ***canonical*** representatives of each equivalence class.

$$\mathbb{D}_n = \{D_n[U] \mid U \in \mathbb{P}\} \text{ including } n = \infty.$$

Theorem. For all $n \in \mathbb{N} \bigcup \{\infty\}$, we have $\mathcal{Id}(\mathbb{D}_n) \cong \mathcal{D}_n$.

Proof. Take any one of these D-operators. The corresponding $\mathcal{D}$-relation is defined as: $U\, \mathcal{D}\, V \Longleftrightarrow D[U] == D[V]$. Because $D = D \circ D$, we see $U\, \mathcal{D}\, D[U]$, for all U. We also see that $\mathcal{Id}(\mathbb{D}) \subseteq \mathcal{D}$. The desired isomorophisms are thus $D : \mathcal{D} \to \mathcal{Id}(\mathbb{D})$ and $\lambda\, U.\, U : \mathcal{Id}(\mathbb{D}) \to \mathcal{D}$. **Q.E.D.**

Warning. While $\mathcal{D}_n$ is a ***total equivalence relation***, $\mathbb{D}_n$ is just a ***PER***. (Why?)

Theorem. For all $n \in \mathbb{N}$, we have $\mathbb{D}_{n+1} \subseteq \mathbb{D}_n$ and $\mathbb{D}_\infty == \bigcap_{n==0}^{\infty} \mathbb{D}_n$.

Proof. The first inclusion follows from the equation $D_n \circ D_{n+1} = D_{n+1}$. That $\mathbb{D}_\infty \subseteq \mathbb{D}_n$, follows from the equation $D_n \circ D_\infty = D_\infty$. Suppose $U \in \bigcap_{n==0}^{\infty} \mathbb{D}_n$. Because $D_\infty == \bigcup_{n==0}^{\infty} D_n$, we find that $D_\infty[U] == U \in \mathbb{D}_\infty$. **Q.E.D.**

We can now establish a key connection between ***mapping spaces***.

Theorem. $(\mathcal{D}_\infty \to \mathcal{P}) \subseteq \mathcal{D}_\infty$

Proof. Suppose that $U\, (\mathcal{D}_\infty \to \mathcal{P})\, V$. This means that, for all X and Y, if $D_\infty[X] == D_\infty[Y]$, then $U[X] = V[Y]$. We need to show that $D_\infty[U] == D_\infty[V]$. But, the equation $D_\infty[D_\infty[X]] = D_\infty[D_\infty[X]]$ is obviously true. Hence, for all X, we have $U[D_\infty[X]] = V[D_\infty[X]]$. This implies $\lambda\, X.\, U[D_\infty[X]] = \lambda\, X. V[D_\infty[X]]$. But we know that $D_\infty[U] = \lambda\, X.\, U[D_\infty[X]]$ and $D_\infty[V] = \lambda\, X.\, V[D_\infty[X]]$. **Q.E.D.**

Note. It would be nice if we could prove now that $\mathcal{D}_\infty \subseteq (\mathcal{D}_\infty \to \mathcal{P})$, and, hence, $\mathcal{D}_\infty = (\mathcal{D}_\infty \to \mathcal{P})$. But this is ***not true***. We can, however, estabilsh an ***isomorphism***. The problem here is that as a ***PER*** $\mathcal{D}_\infty$ is a ***total*** equivalence relation, but $(\mathcal{D}_\infty \to \mathcal{P})$ is ***not***. What we can prove first is this:

Theorem. $U\, (\mathcal{D}_\infty \to \mathcal{P})\, U$ if, and only if, $\lambda\, X.\, U[X] = D_\infty[U]$.

Proof. We have the following equivalences:

$$U\, (\mathcal{D}_\infty \to \mathcal{P})\, U$$

$$\Longleftrightarrow \forall\, X, Y\, \Big[\, D_\infty[X] = D_\infty[Y] \Longrightarrow U[X] = U[Y] \,\Big]$$

$$\Longleftrightarrow \forall\, X\, .\, U[X] = U[D_\infty[X]] \qquad \text{(Why?)}$$

$$\Longleftrightarrow \lambda\, X.\, U[X] = \lambda\, X.\, U[D_\infty[X]] = D_\infty[U]$$

Q.E.D.

Note. The right-hand side of the theorem is not true when $U = \lambda\, X.\, X$, as we showed in the last section.

Theorem. $\mathcal{D}_\infty \cong (\mathcal{D}_\infty \to \mathcal{P})$

Proof. Remember $\mathbb{1} = \lambda\, X.\, X$. Owing to the ***inclusion*** we did prove, it is clear that $\mathbb{1} : (\mathcal{D}_\infty \to \mathcal{P}) \to \mathcal{D}_\infty$. (Why?)

For a mapping in the other direction, we will show $D_\infty : \mathcal{D}_\infty \to (\mathcal{D}_\infty \to \mathcal{P})$.

We note next these equations:

$$\begin{aligned} & D_\infty[D_\infty[U]] \\ &\quad = D_\infty[U] \\ &\quad = \lambda\, X.\, U[D_\infty[X]] \\ &\quad = \lambda\, X.\, D_\infty[U][X] \,. \end{aligned}$$

And, by our previous theorem, $D_\infty[U]\, (\mathcal{D}_\infty \to \mathcal{P})\, D_\infty[U]$ then holds for all U.

So, suppose $U\ \mathcal{D}_\infty\ V$ is true. This means that $D_\infty[U] = D_\infty[V]$. By what we just proved, $D_\infty[U]\ (\mathcal{D}_\infty \to \mathcal{P})\ D_\infty[V]$ must hold, and so D_∞ has the right mapping property.

To show we have an isomorphism, we need to establish two facts:

$$\forall\ U : \mathcal{D}_\infty.\ D_\infty[U]\ \mathcal{D}_\infty\ U \quad \text{and} \quad \forall\ U : (\mathcal{D}_\infty \to \mathcal{P}).\ D_\infty[U]\ (\mathcal{D}_\infty \to \mathcal{P})\ U.$$

Inasmuch as $D_\infty[D_\infty[U]] = D_\infty[U]$, the first fact is clear.

To prove the second, assume that $U\ (\mathcal{D}_\infty \to \mathcal{P})\ U$. Next, assume $D_\infty[X] == D_\infty[Y]$. We have these equalities:

$$\begin{aligned} &D_\infty[U][X] \\ &\quad = U[D_\infty[X]] \\ &\quad = U[D_\infty[Y]] \\ &\quad = D_\infty[U][Y] \\ &\quad = U[Y]\,. \end{aligned}$$

Thus, $D_\infty[U]\ (\mathcal{D}_\infty \to \mathcal{P})\ U$ holds. **Q.E.D.**

Theorem. $(\mathcal{D}_\infty \times \mathcal{D}_\infty) \cong \mathcal{D}_\infty$

Proof. We recall that $\mathbb{P} \times \mathbb{P} = \mathbb{P}$ and so $\mathcal{P} \times \mathcal{P} = \mathcal{P}$. We then argue:

$$\begin{aligned} (\mathcal{D}_\infty \times \mathcal{D}_\infty) &\cong (\mathcal{D}_\infty \to \mathcal{P}) \times (\mathcal{D}_\infty \to \mathcal{P}) \\ &\cong (\mathcal{D}_\infty \to (\mathcal{P} \times \mathcal{P})) \\ &= (\mathcal{D}_\infty \to \mathcal{P}) \\ &\cong \mathcal{D}_\infty\,. \qquad \textbf{Q.E.D.} \end{aligned}$$

Theorem. $(\mathcal{D}_\infty \to \mathcal{D}_\infty) \cong \mathcal{D}_\infty$

Proof. We can now argue:

$$\begin{aligned} (\mathcal{D}_\infty \to \mathcal{D}_\infty) &\cong (\mathcal{D}_\infty \to (\mathcal{D}_\infty \to \mathcal{P})) \\ &\cong ((\mathcal{D}_\infty \times \mathcal{D}_\infty) \to \mathcal{P}) \\ &\cong (\mathcal{D}_\infty \to \mathcal{P}) \\ &\cong \mathcal{D}_\infty\,. \qquad \textbf{Q.E.D.} \end{aligned}$$

Note. The existence of isomorphisms $(\mathcal{D}_\infty \to \mathcal{D}_\infty) \cong \mathcal{D}_\infty \cong (\mathcal{D}_\infty \times \mathcal{D}_\infty)$ can be used to show that $\mathcal{D}_\infty$ is a model of the **$\lambda\eta$-calculus**. This will be spelled out in a later lecture.