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Topic 2: Type Theory

Section 7.1: Type Semantics IV

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Modelling $(\lambda\eta\gamma)$-calculus

■ $\mathbb{D}$-continuity

All we really need ***to know*** about D_∞ can be stated as follows :

$$\mathbb{1} \subseteq D = D \circ D = \lambda U.\, U \circ D$$

And note we have dropped the ***∞-subscript*** to simplify writing. We also introduced a ***PER*** and a ***special set*** associated to this D:

$$U\, \mathcal{D}\, V \Longleftrightarrow U \circ D = V \circ D$$

$$\mathbb{D} = \{U \in \mathbb{P} \mid U = U \circ D\} = \{U \circ D \mid U \in \mathbb{P}\}$$

Concerning these notions we also proved earlier:

Theorem. $\mathcal{Id}(\mathbb{D}) \cong \mathcal{D} \cong (\mathcal{D} \to \mathcal{P}) \cong (\mathcal{D} \times \mathcal{D}) \cong (\mathcal{D} \to \mathcal{D})$

Before turning to showing how these properties lead us to a ***new model*** of λ-calculus, we need to consider how continuous functions behave on $\mathbb{D}$.

Suppose $\Phi(X_0, X_1, \ldots, X_{n-1})$ is a continuous operator of n-variables.

Φ is said to be $\mathbb{D}$-***continuous*** provided that

$$\Phi(U_0, U_1, \ldots, U_{n-1}) \in \mathbb{D} \text{ whenever } U_0, U_1, \ldots, U_{n-1} \in \mathbb{D}.$$

A very broad class of operators is $\mathbb{D}$-continuous:

Theorem. The operator $\lambda X.\Phi(U_0[X], U_1[X], \ldots, U_{n-1}[X])$ is always $\mathbb{D}$-continuous.

Proof. It will be sufficient to consider just ***two variables*** and to show:

$$\lambda X.\Phi(U[X], V[X]) \in \mathbb{D} \text{ whenever } U, V \in \mathbb{D}.$$

Assume we have $U, V \in \mathbb{D}$. We then calculate:

$$(\lambda X.\Phi(U[X], V[X])) \circ D$$

$$= \lambda X.\Phi(U[D[X]], V[D[X]])$$

$$= \lambda X.\Phi(U \circ D[X], V \circ D[X])$$

$$= \lambda X.\Phi(U[X], V[X]) \in \mathbb{D}.$$

Q.E.D.

We can conclude at once from this result that ***many operators*** have the desired continuity property.

■ Pairing, application and abstraction

Theorem. The following operators are $\mathbb{D}$-continuous:

$$((U, V)) = \lambda X.\lfloor U[X], V[X] \rfloor,$$

$$\mathbb{1st}(W) = \lambda X.1\,\mathrm{st}[W[X]], \text{ and}$$

$$\mathbb{2nd}(W) = \lambda X.2\,\mathrm{nd}[W[X]].$$

Note. Everything here has been written as ***operators*** rather than as ***combinators*** in $\mathbb{P}$.

Recall that $\mathbb{P} \times \mathbb{P} = \mathbb{P}$. Properties of the underlying pairing can now be lifted from $\mathbb{P}$ to $\mathbb{D}$. The proofs should be obvious. (Why?)

Theorem. For $U, V, W \in \mathbb{D}$ we have:

$$\mathbb{1st}(((U, V))) = U,$$

$$\mathbb{2nd}(((U, V))) = V, \text{ and}$$

$$((\mathbb{1st}(W), \mathbb{2nd}(W))) = W.$$

Note. Any continuous operator $\Phi(U_0, U_1, \ldots, U_{n-1})$ can be made into a $\mathbb{D}$-continuous one by changing it to $\Phi(U_0, U_1, \ldots, U_{n-1}) \circ D$. This ***cheap trick*** does not necessarily mean that the new operator has ***interesting properties***, however. We need to use the device in the next two definitions, nevertheless.

Take note of a special property of the elements of $\mathbb{D}$.

Theorem. If $U \in \mathbb{D}$, then $U == \lambda X.U[X]$.

Proof. Under the assumption we find:

$$\begin{aligned}&\lambda X.U[X]\\ &== \lambda X.(U\circ D)[X]\\ &== \lambda X.U[D[X]]\\ &== U\circ D == U\end{aligned}$$

Q.E.D.

We can now give the two basic definitions of the new λ-calculus. And read the strange symbol £ as "***fancy λ***" to distinguish it from the ordinary λ we have used for ℙ. We have do this work since it is ***not the case*** that 𝔻 is closed under the ordinary application operator $F[U]$. (Why?)

$$F[\![U]\!] = (\lambda V.F[((U, V))])\circ D$$

$$£\, U.\Phi(U, V_0, V_1, \ldots) = (\lambda W.\Phi(\mathbb{1st}(W), V_0, V_1, \ldots)[\mathbb{2nd}(W)])\circ D$$

Note. The D in the above equations can be ***brought inside***. For example, the first equation could have been written as: $F[\![U]\!] = \lambda V.F[((U, D[V]))]$.

Note, too, it does not matter which variable we use in defining $£\, U$. (Why?)

■ The conversion rules

Our first main task is to verify that 𝔻 models ***β-conversion*** under these definitions.

Note. In the next theorem we write $\Phi(U)$ instead of $\Phi(U, V_0, V_1, \ldots)$ just to save space and to save writing in the formulae. The ***longer form*** is what is intended, however.

Theorem. Assuming that $\Phi(U)$ is 𝔻-continuous and $W \in \mathbb{D}$, we have

$$(£\, U.\Phi(U))[\![W]\!] = \Phi(W).$$

Proof. We have to compute using the definitions:

$$\begin{aligned}&(£\, U.\Phi(U))[\![W]\!]\\ &== (\lambda V.(£\, U.\Phi(U))[((W, V))])\circ D\\ &== (\lambda V.((\lambda Z.\Phi(\mathbb{1st}(Z))[\mathbb{2nd}(Z)])\circ D)[((W, V))])\circ D\\ &== (\lambda V.((\lambda Z.\Phi(\mathbb{1st}(D[Z]))[\mathbb{2nd}(D[Z])]))[((W, D[V]))])\\ &== \lambda V.\Phi(\mathbb{1st}(D[((W, D[V]))]))[\mathbb{2nd}(D[((W, D[V]))])]\\ &== \lambda V.\Phi(\mathbb{1st}(((W, D[V]))))[\mathbb{2nd}(((W, D[V])))]\\ &== \lambda V.\Phi(W)[D[V]]\\ &== (\lambda V.\Phi(W)[V])\circ D\\ &== \Phi(W)\circ D\\ &== \Phi(W)\end{aligned}$$

Q.E.D.

Our next main task is to verify that 𝔻 models ***η-conversion*** under our definitions.

Theorem. Assuming that $F \in \mathbb{D}$, we have $£\, U.F[\![U]\!] = F$.

Proof. We have to compute using the definitions:

$$
\begin{aligned}
&\text{£}\, U.\, F[\![U]\!] \\
&== (\lambda\, W.\, F[\![\mathbb{1st}(W)]\!][\mathbb{2nd}(W)]) \circ D \\
&== (\lambda\, W.((\lambda\, V.\, F[((\mathbb{1st}(W),\, V))]) \circ D)[\mathbb{2nd}(W)]) \circ D \\
&== (\lambda\, W.(\lambda\, V.\, F[((\mathbb{1st}(W),\, D[V]))])[\mathbb{2nd}(W)]) \circ D \\
&== (\lambda\, W.\, F[((\mathbb{1st}(W),\, D[\mathbb{2nd}(W)]))]) \circ D \\
&== \lambda\, W.\, F[((\mathbb{1st}(D[W]),\, D[\mathbb{2nd}(D[W])]))] \\
&== \lambda\, W.\, F[((\mathbb{1st}(D[W]),\, \mathbb{2nd}(D[W])))] \\
&== \lambda\, W.\, F[D[W]] \\
&== F \circ D \\
&== F
\end{aligned}
$$

Q.E.D.

Our next main task is to verify that $\mathbb{D}$ models ***γ-conversion*** under our definitions.

Theorem. Assuming that $U,\, V,\, W \in \mathbb{D}$, we have $((U,\, V))[\![W]\!] = ((U[\![W]\!],\, V[\![W]\!]))$.

Proof. We have to compute using the definitions:

$$
\begin{aligned}
&((U,\, V))[\![W]\!] \\
&== (\lambda\, Z.\, ((U,\, V))[((W,\, Z))]) \circ D \\
&== (\lambda\, Z.\, \lfloor U[((W,\, Z))],\, V[((W,\, Z))] \rfloor) \circ D \\
&== \lambda\, Z.\, \lfloor U[((W,\, D[Z]))],\, V[((W,\, D[Z]))] \rfloor \\
&== \lambda\, Z.\, \lfloor U[\![W]\!][D[Z]],\, V[\![W]\!][D[Z]] \rfloor \\
&== (\lambda\, Z.\, \lfloor U[\![W]\!][Z],\, V[\![W]\!][Z] \rfloor) \circ D \\
&== ((U[\![W]\!],\, V[\![W]\!])) \circ D \\
&== ((U[\![W]\!],\, V[\![W]\!]))
\end{aligned}
$$

Q.E.D.

As a direct consequence of this theorem we can prove another tidy equation.

Theorem. Assuming that $\Phi(U)$ and $\Psi(U)$ are $\mathbb{D}$-continuous, we have:

$$\text{£}\, U.\, ((\Phi(U),\, \Psi(U))) = ((\text{£}\, U.\, \Phi(U),\, \text{£}\, U.\, \Psi(U))).$$

Proof. We have these equations:

$$
\begin{aligned}
&((\text{£}\, U.\, \Phi(U),\, \text{£}\, U.\, \Psi(U))) \\
&= \text{£}\, V.\, ((\text{£}\, U.\, \Phi(U),\, \text{£}\, U.\, \Psi(U)))[\![V]\!] \\
&= \text{£}\, V.\, (((\text{£}\, U.\, \Phi(U))[\![V]\!],\, (\text{£}\, U.\, \Psi(U))[\![V]\!])) \\
&= \text{£}\, V.\, ((\Phi(V),\, \Psi(V))) \\
&= \text{£}\, U.\, ((\Phi(U),\, \Psi(U)))
\end{aligned}
$$

Q.E.D.

Here is another useful theorem following a definition.

$$U \bullet V = \text{£}\, Z.\, U[\![V[\![Z]\!]]\!]$$

Theorem. Assuming that $U,\, V,\, W \in \mathbb{D}$, we have $((U,\, V)) \bullet W = ((U \bullet W,\, V \bullet W))$.

Proof. We have these equations:

$$
\begin{aligned}
&((U, V)) \bullet W \\
&= £\, Z.\, ((U, V))\, [\![W[\![Z]\!]]\!] \\
&= £\, Z.\, ((U\, [\![W[\![Z]\!]]\!],\, V\, [\![W[\![Z]\!]]\!])) \\
&= ((£\, Z.\, U\, [\![W[\![Z]\!]]\!],\, £\, Z.\, V\, [\![W[\![Z]\!]]\!])) \\
&= ((U \bullet W,\, V \bullet W))
\end{aligned}
$$

Q.E.D.

Bibliographical Notes. The ($\lambda\beta\gamma$)-calculus was introduced and discussed in the book ***Lambda-Calculus, Combinators, and Functional Programming*** by G.E. Revesz, x + 188 pp., Cambridge University Press, 1988, ISBN 0-521-34589-8. The book discusses LISP and other list-processing ideas, including the language FP of John Backus described most fully in his article *Can programming be liberated from the von Neumann style? A functional style and its algebra of programs,* pp. 63-130 in ***1966 to 1985: ACM Turing Award Lectures, The First Twenty Years***, xviii + 483 pp., ACM Press and Addison-Wesley Publishing Co., 1987, ISBN 0-201-07794-9. The model presented here is a simplified form of that worked out as a M.Sc. Thesis by Glenn Durfee in a Carnegie-Mellon Techincal Report, *A model for a list-oriented extension of the lambda calculus*, 29 pp., CMU-CS-97-151, May 1997.

■ Lattice-theoretic properties

The model $\mathbb{D}$ has many properties under inclusion ($\subseteq$) similar to those of $\mathbb{P}$. We note without proof, for example, that $K[\emptyset]$ is the smallest element of $\mathbb{D}$, while $K[\mathbb{N}] == \mathbb{N}$ is the largest. (Why?)

Theorem. Assuming that $U, V \in \mathbb{D}$, then $U \bigcup V \in \mathbb{D}$.

Proof. We have these equations:

$$
\begin{aligned}
&U \bigcup V \\
&= (U \circ D) \bigcup (V \circ D) \\
&= (\lambda\, X.\, U[D[X]]) \bigcup (\lambda\, X.\, V[D[X]]) \\
&= \lambda\, X.\, (U[D[X]] \bigcup V[D[X]]) \\
&= \lambda\, X.\, (U \bigcup V)[D[X]] \\
&= (U \bigcup V) \circ D
\end{aligned}
$$

Q.E.D.

Note. This theorem can easily be generalized to ***infinite unions***. Hence, $\mathbb{D}$-continuous operators have their ***least fixed points*** in $\mathbb{D}$. The author has not checked whether the so-called ***paradoxical combinator*** defined in terms of ***£-abstraction*** and $[\![\,]\!]$-***application*** gives the expected least fixed point. It seems reasonable to conjecture that it does.

Theorem. Assuming that $U, V \in \mathbb{D}$, then $U \bigcap V \in \mathbb{D}$.

Proof. We have these equations:

$$
\begin{aligned}
&U \bigcap V \\
&= \lambda\, X.\, U[X] \bigcap \lambda\, X.\, V[X] \\
&= \lambda\, X.\, (U[X] \bigcap V[X]) \\
&= \lambda\, X.\, (U[D[X]] \bigcap V[D[X]]) \\
&= (\lambda\, X.\, (U[X] \bigcap V[X])) \circ D
\end{aligned}
$$

But that is enough to give the conclusion. (Why?) **Q.E.D.**

Theorem. Assuming that $\Phi(U)$ and $\Psi(U)$ are $\mathbb{D}$-continuous, we have:

$$\pounds\, U.\Phi(U) \subseteq \pounds\, U.\Psi(U) \Longleftrightarrow \forall\, U \in \mathbb{D}.\Phi(U) \subseteq \Psi(U).$$

Proof. We have these equivalences:

$$\begin{aligned}
&\forall\, U \in \mathbb{D}.\Phi(U) \subseteq \Psi(U) \\
&\Longleftrightarrow \forall\, U \in \mathbb{D}.\forall\, X \in \mathbb{P}.\Phi(U)[X] \subseteq \Psi(U)[X] \\
&\Longleftrightarrow \forall\, U \in \mathbb{D}.\forall\, X \in \mathbb{P}.\Phi(U)[D[X]] \subseteq \Psi(U)[D[X]] \\
&\Longleftrightarrow \forall\, U \in \mathbb{D}.\forall\, V \in \mathbb{D}.\Phi(U)[V] \subseteq \Psi(U)[V] \\
&\Longleftrightarrow \forall\, W \in \mathbb{D}.\ \Phi(\mathbb{1st}(W))[\mathbb{2nd}(W)] \subseteq \Psi(\mathbb{1st}(W))[\mathbb{2nd}(W)] \\
&\Longleftrightarrow \forall\, W.\ \Phi(\mathbb{1st}(D[W]))[\mathbb{2nd}(D[W])] \subseteq \Psi(\mathbb{1st}(D[W]))[\mathbb{2nd}(D[W])] \\
&\Longleftrightarrow \lambda W.\ \Phi(\mathbb{1st}(D[W]))[\mathbb{2nd}(D[W])] \subseteq \lambda W.\Psi(\mathbb{1st}(D[W]))[\mathbb{2nd}(D[W])] \\
&\Longleftrightarrow (\lambda W.\ \Phi(\mathbb{1st}(W))[\mathbb{2nd}(W)]) \circ D \subseteq (\lambda W.\Psi(\mathbb{1st}(W))[\mathbb{2nd}(W)]) \circ D \\
&\Longleftrightarrow \pounds\, U.\Phi(U) \subseteq \pounds\, U.\Psi(U)
\end{aligned}$$

Q.E.D.