CS 294-13
Advanced Computer Graphics

Differential Geometry Basics

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Topics

- Vector and Tensor Fields
  - Divergence, curl, *etc.*
- Parametric Curves
  - Tangents, curvature, and *etc.*
- Parametric Surfaces
  - Normals, tangents, curvature, *etc.*
- Implicit Surfaces
  - Normals, curvature, *etc.*

Vectors

- A vectors defines a magnitude and direction
  - Not just a list of numbers $||\mathbf{v}||$ $\hat{\mathbf{v}} = \mathbf{v}/||\mathbf{v}||$
  - Particular numbers are an artifact of the coordinate system we chose
    - Not all coordinate systems are orthonormal $\mathbf{v} = [v_x, v_y, v_z]$
  - Nearly everything that is useful can be defined w/o coordinate system
- Vectors transform like vectors $\mathbf{v}' = \mathbf{A} \cdot \mathbf{v}$
- No set location (e.g. no root)
  - But may be functions of location $\mathbf{v} = \mathbf{v}(u)$

$$\mathbf{v} = \mathbf{v}(x, y)$$

Tensors

- Tensors transform like tensors e.g. $\mathbf{T}' = \mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^{\mathsf{T}}$
- Tensors used to define oriented quantities
  - Independent of coordinate system
  - Specific realization will depend on coordinate system
    - Cartesian tensors -- orthonormal coordinate system
    - General tensors -- non-orthonormal coordinate system → $\mathbf{T}' = \mathbf{A} \cdot \mathbf{T} \cdot \mathbf{A}^{-1}$
- Tensors have *rank*
  - Not related to dimension of space
  - Rank 0 → scalars
  - Rank 1 → vectors
  - Rank 2 → matrices
  - Rank 3 → don't work well in matrix-vector notation

Tensors

- Examples

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{a}^{\mathsf{T}}\cdot\mathbf{b} = ||\mathbf{a}||\,||\mathbf{b}||\mathrm{Cos}(\angle\mathbf{ab})$$

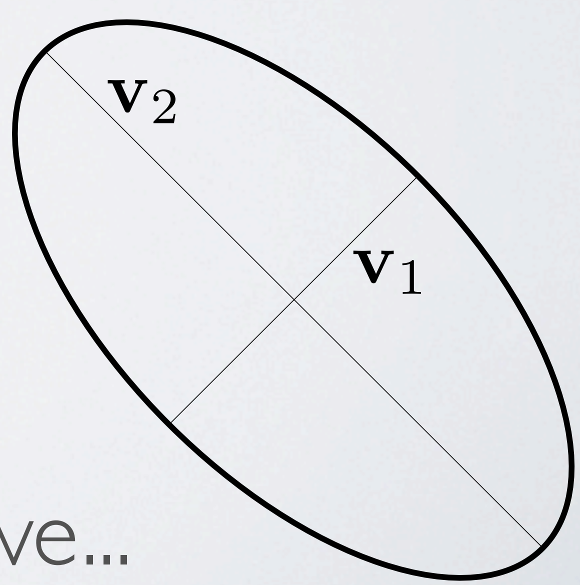
$$\mathbf{a}\cdot\mathbf{b}^{\mathsf{T}} \quad = \quad \mathbf{P} \quad \dashrightarrow \quad (\mathbf{A}\cdot\mathbf{a})\cdot(\mathbf{A}\cdot\mathbf{b})^{\mathsf{T}} \quad =$$

$$\mathbf{A}\cdot(\mathbf{a}\cdot\mathbf{b}^{\mathsf{T}})\cdot\mathbf{A}^{\mathsf{T}} \quad = \quad \mathbf{A}\cdot\mathbf{P}\cdot\mathbf{A}^{\mathsf{T}}$$

$$\mathbf{R} = \mathbf{x}'\cdot\mathbf{x}^{\mathsf{T}} + \mathbf{y}'\cdot\mathbf{y}^{\mathsf{T}} + \mathbf{z}'\cdot\mathbf{z}^{\mathsf{T}}$$

$$\mathbf{v}_1\cdot\mathbf{v}_1^{\mathsf{T}} + \mathbf{v}_2\cdot\mathbf{v}_2^{\mathsf{T}} = \mathbf{S}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$



Note the way inner and otter products behave...

Summation Notation

- Notation due to Einstein
  - Makes life much easier
  - Takes a while to get used to
  - Useful in other contexts as well
- Free index
  - Appears on both sides
  - Unique in each term
  - Implied "for all"
- Dummy index
  - Appears exactly twice in each term
  - Implied "sum over"
- Different for general tensors

$$\mathbf{a} \longrightarrow a_i \qquad \mathbf{A} \longrightarrow A_{ij}$$

$$s = \mathbf{a} \cdot \mathbf{b} \longrightarrow s = a_i b_i$$

$$\mathbf{A} = \mathbf{a} \cdot \mathbf{b}^\mathsf{T} \longrightarrow A_{ij} = a_i b_j$$

$$\mathbf{c} = \mathbf{A} \cdot \mathbf{b} \longrightarrow c_i = A_{ij} b_j$$

$$\mathbf{c}^\mathsf{T} = \mathbf{b}^\mathsf{T} \cdot \mathbf{A}^\mathsf{T} \longrightarrow c_i = b_j A_{ij}$$

$$\mathbf{c} = \mathbf{A} \cdot \mathbf{b} \longrightarrow c_i = b_j A_{ij}$$

$$\mathbf{A}' = \mathbf{R}\mathbf{A}\mathbf{R}^\mathsf{T} \longrightarrow A'_{ij} = A_{kl} R_{ik} R_{jl}$$

Summation Notation

- Two special symbols
  - Delta δ_{ij}
  - Permutation ε_{ijk}

$$a_i \delta_{ij} = a_j$$

$$\varepsilon_{kij}\varepsilon_{kab} = \delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}$$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

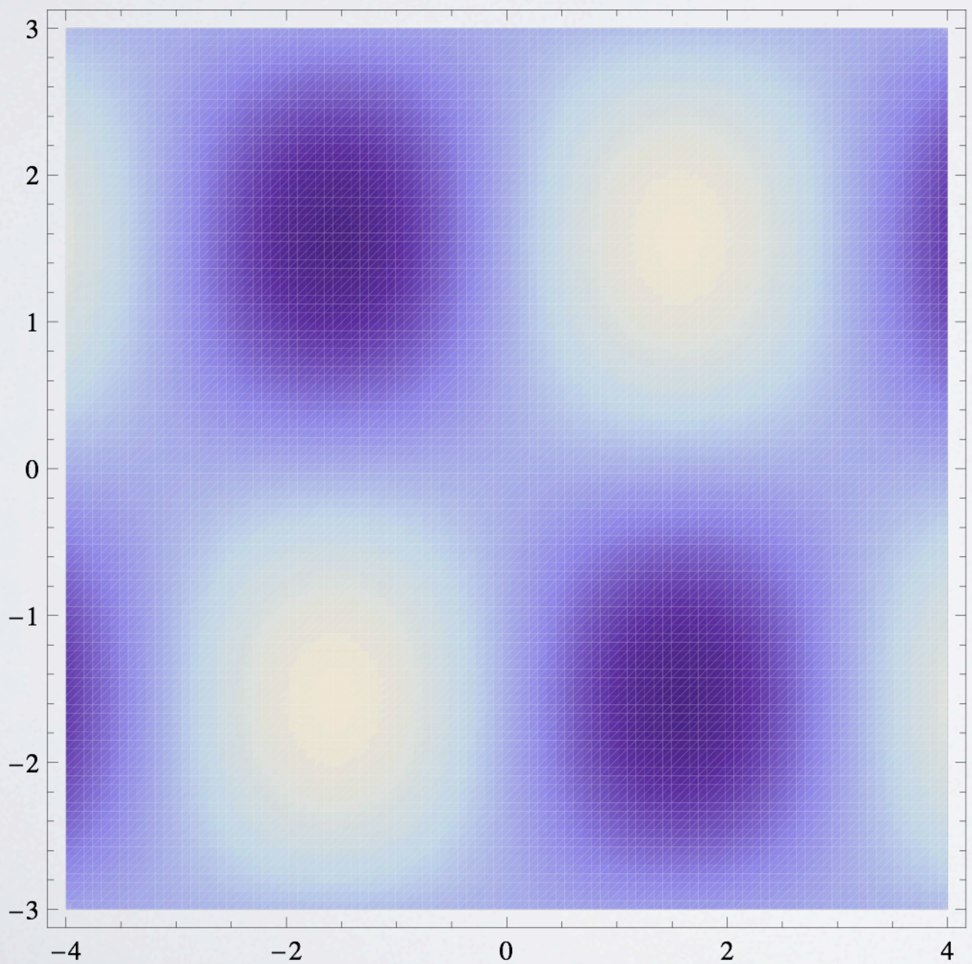
$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } i,j,k \text{ are } \textbf{even} \text{ permutation of } 1,2,3 \\ -1 & \text{if } i,j,k \text{ are } \textbf{odd} \text{ permutation of } 1,2,3 \\ 0 & \text{else} \end{cases}$$

If you're slumming in $\Re^2$ $\quad \varepsilon_{ij} = \begin{cases} 1 & \text{if } i,j \text{ are } 1,2 \\ -1 & \text{if } i,j \text{ are } 2,1 \\ 0 & \text{else} \end{cases}$

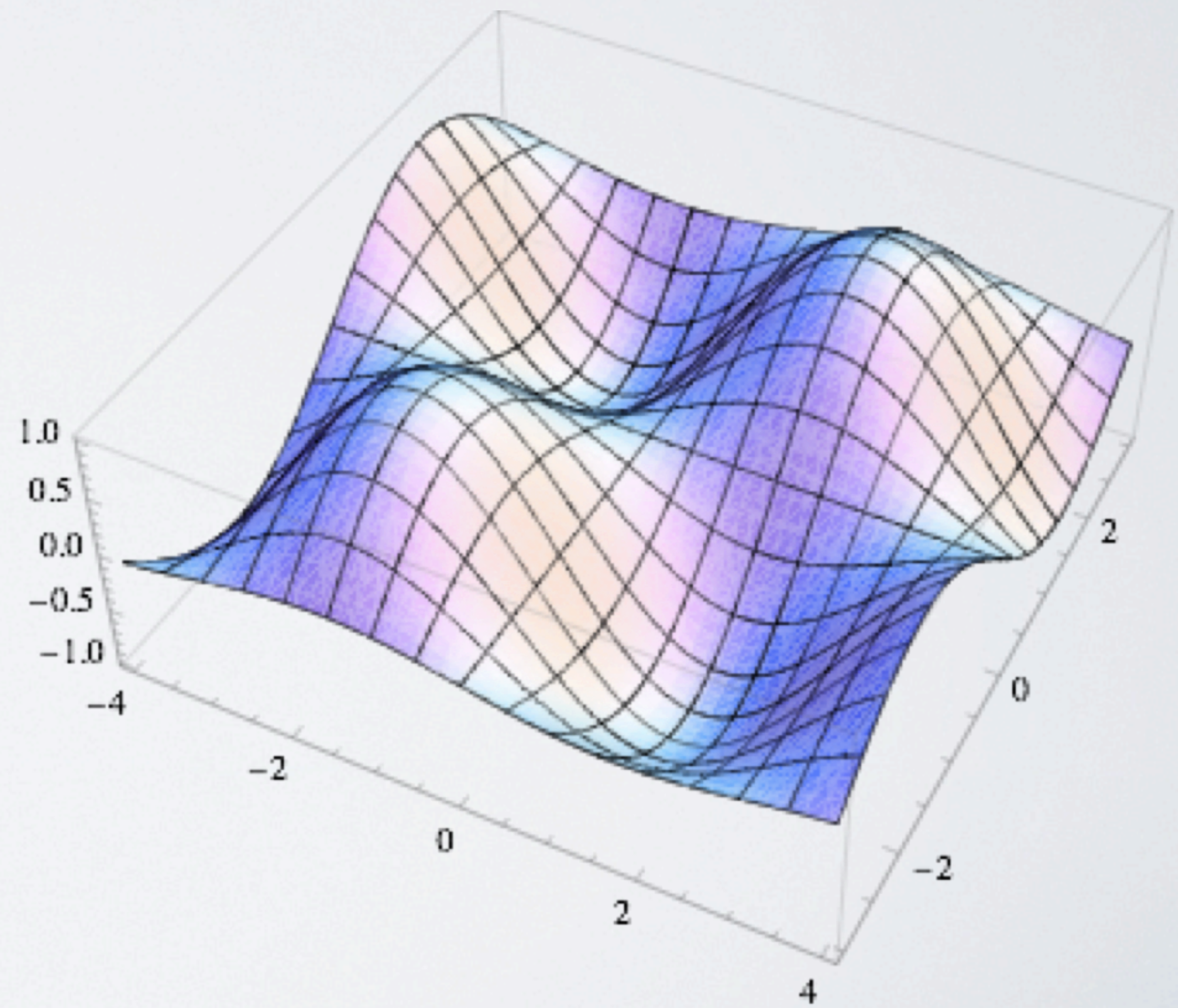
Scalar Fields

- Scalar as function of some spatial variable(s)
  - e.g.:

$$f(x, y) = f(\mathbf{x}) = \mathrm{Sin}(x)\mathrm{Sin}(y)$$



Density Plot

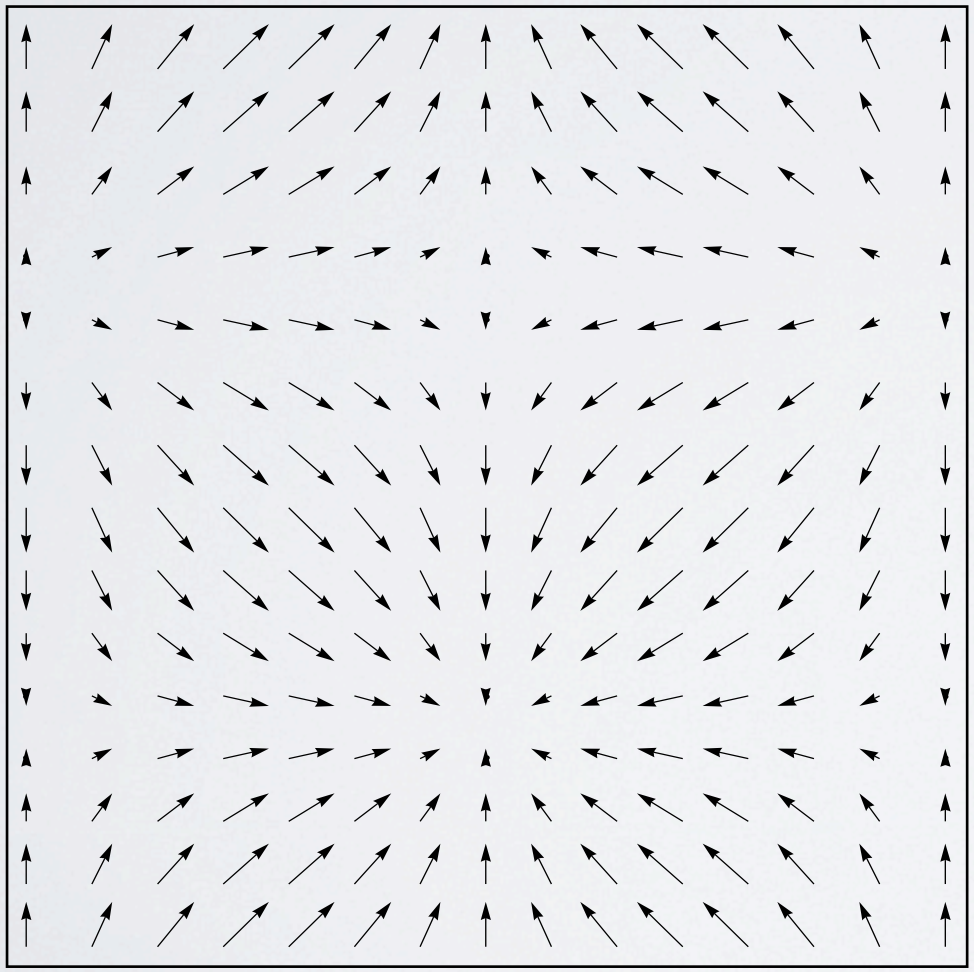


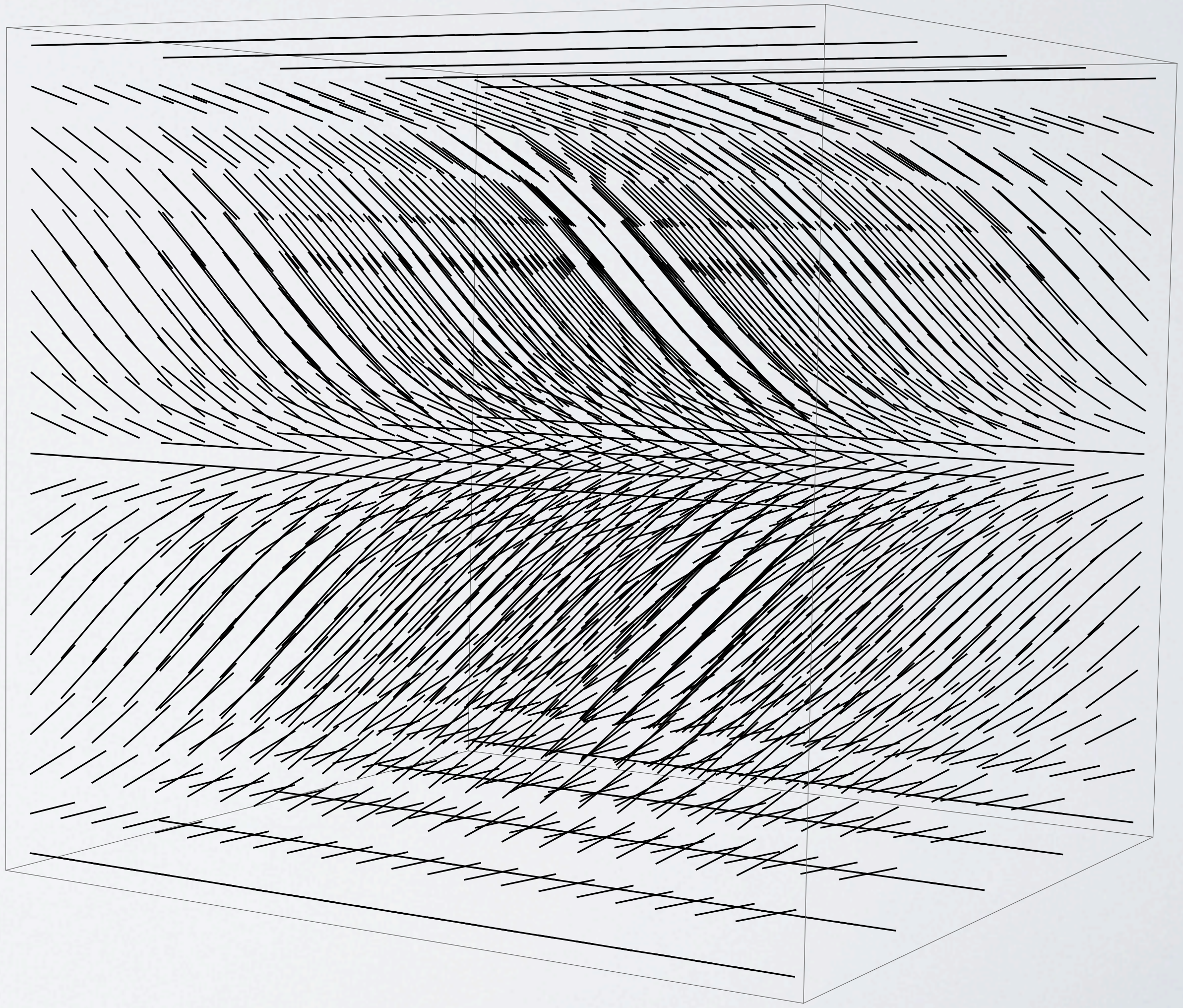
Height-field Plot

Vector Fields

- Vector as function of some spatial variable(s)
  - e.g.:

$$\mathbf{v}(x, y) = \mathbf{v}(\mathbf{x}) = [\mathrm{Sin}(x), \mathrm{Cos}(y)]$$





$$\mathbf{v}(x, y) = \mathbf{v}(\mathbf{x}) = [1, \mathrm{Sin}(y), 0]$$

Differential Operators on Fields

- Derivatives of field w.r.t. spatial coordinates
  - Coordinates implicit given field parameterization
  - Linear operators on the field
  - Not tied to any particular coordinate system
- Basic operators
  - Gradient
  - Divergence
  - Curl
  - Laplacian
- All expressed with $\boldsymbol{\nabla}$ (*a.k.a.* Nabla or del)

$$\boldsymbol{\nabla} = \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i}$$

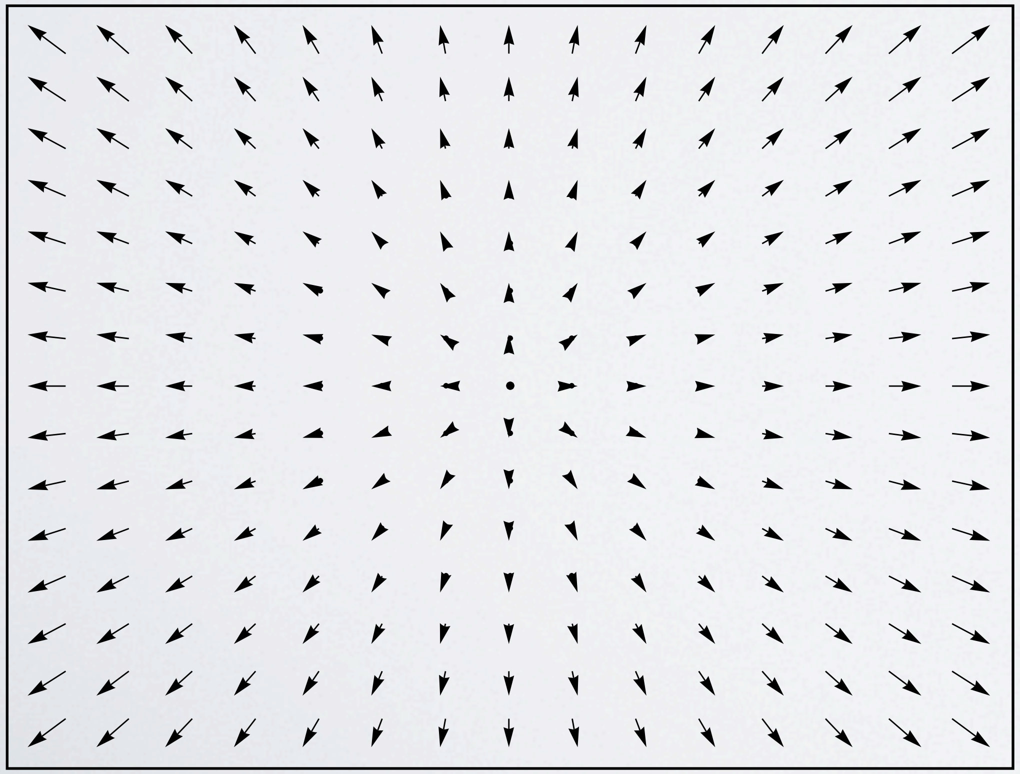
$$\boldsymbol{\nabla} = [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}]$$

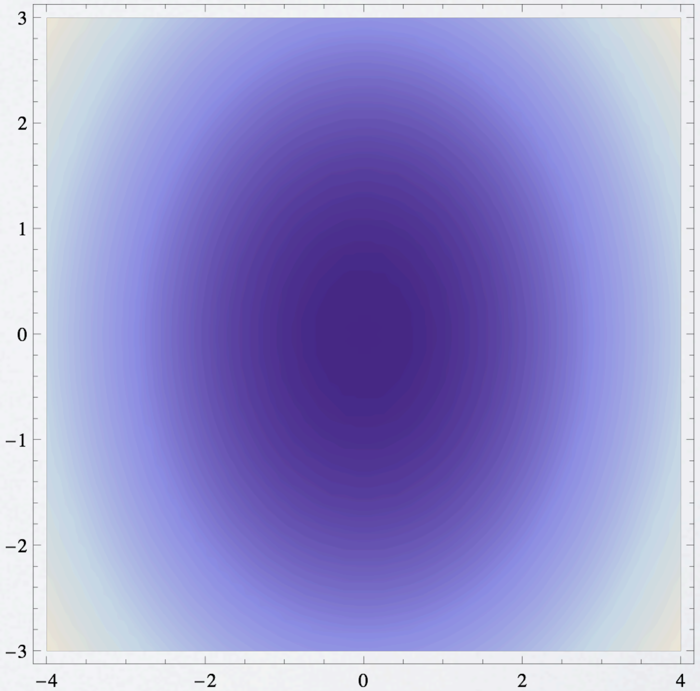
$$\nabla_i = \partial_i = \frac{\partial}{\partial x_i}$$

Gradient

- Often applied to scalar fields
  - Gives direction of steepest accent
- Also has meaning for higher rank fields
  - Elevates rank by one
  - e.g. velocity gradient of a Newtonian fluid gives the strain rate

$$\mathrm{grad} f(\mathbf{x}) = \boldsymbol{\nabla} f(\mathbf{x}) = \left[\frac{\partial f(\mathbf{x})}{\partial x_1}, \frac{\partial f(\mathbf{x})}{\partial x_2}, \frac{\partial f(\mathbf{x})}{\partial x_3}, \right]$$



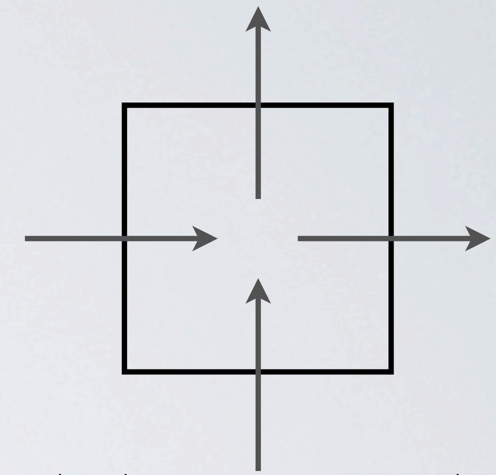


$$f(\mathbf{x}) = x^2 + y^2$$

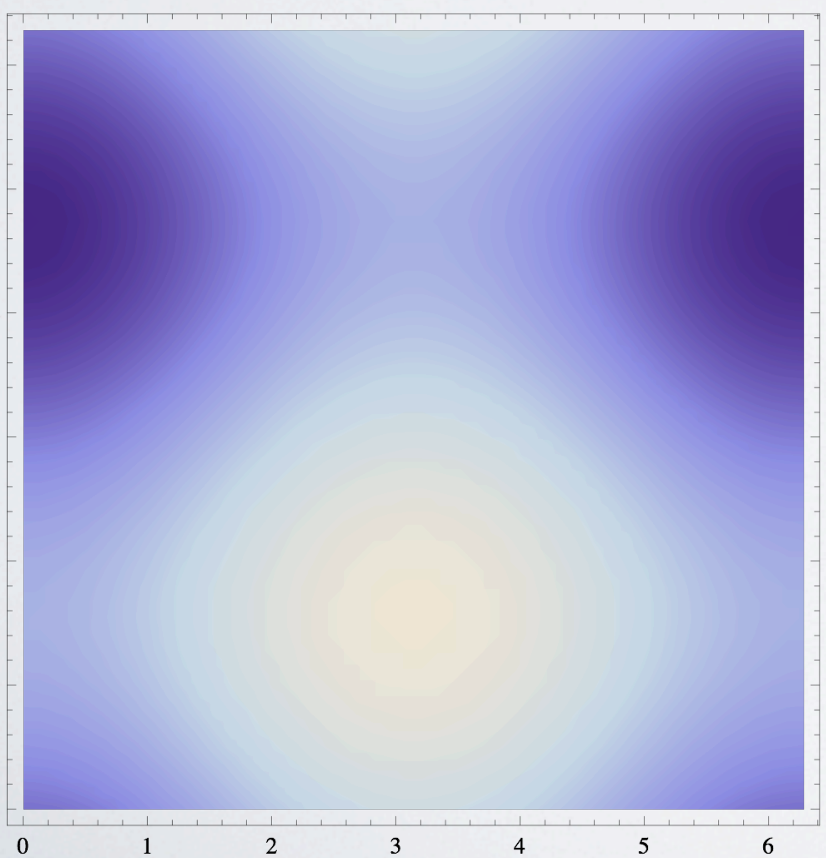
$$\boldsymbol{\nabla} f(\mathbf{x}) = [2x, 2y]$$

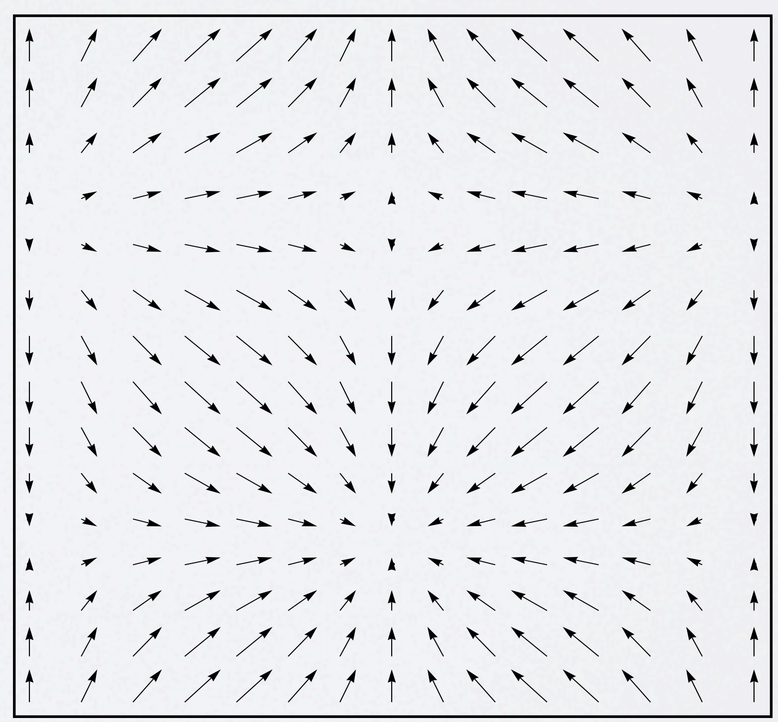
Divergence

- For a vector field it describes the net expansion or contraction
- Lowers rank by one
  - Divergence of vector field is a scalar
  - An inner product of derivatives with the field



$$\operatorname{div}\mathbf{v}(\mathbf{x}) = \boldsymbol{\nabla}\cdot\mathbf{v}(\mathbf{x}) = \boldsymbol{\nabla}^{\mathsf{T}}\cdot\mathbf{v}(\mathbf{x}) = \frac{\partial\mathbf{v}_x(\mathbf{x})}{\partial x_1} + \frac{\partial\mathbf{v}_y(\mathbf{x})}{\partial x_2} + \frac{\partial\mathbf{v}_z(\mathbf{x})}{\partial x_3}$$

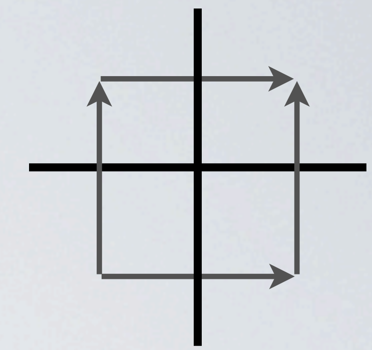




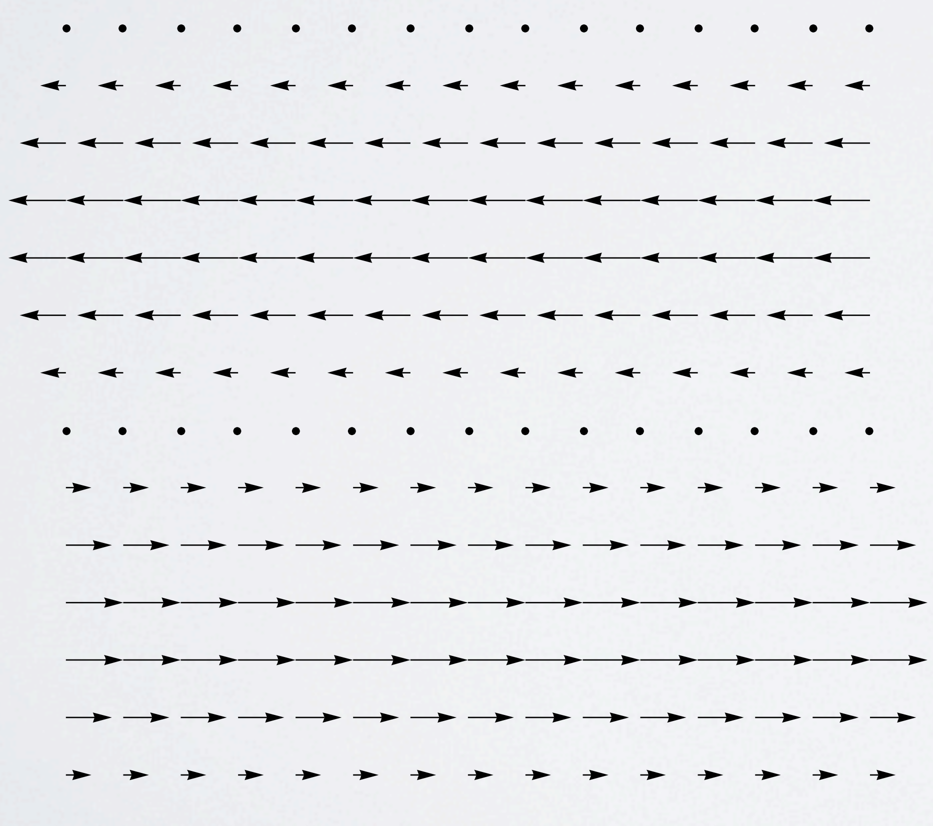
$$\boldsymbol{\nabla}\cdot[\mathrm{Sin}(x),\mathrm{Cos}(y)] = -\mathrm{Cos}(x)+\mathrm{Sin}(y)$$

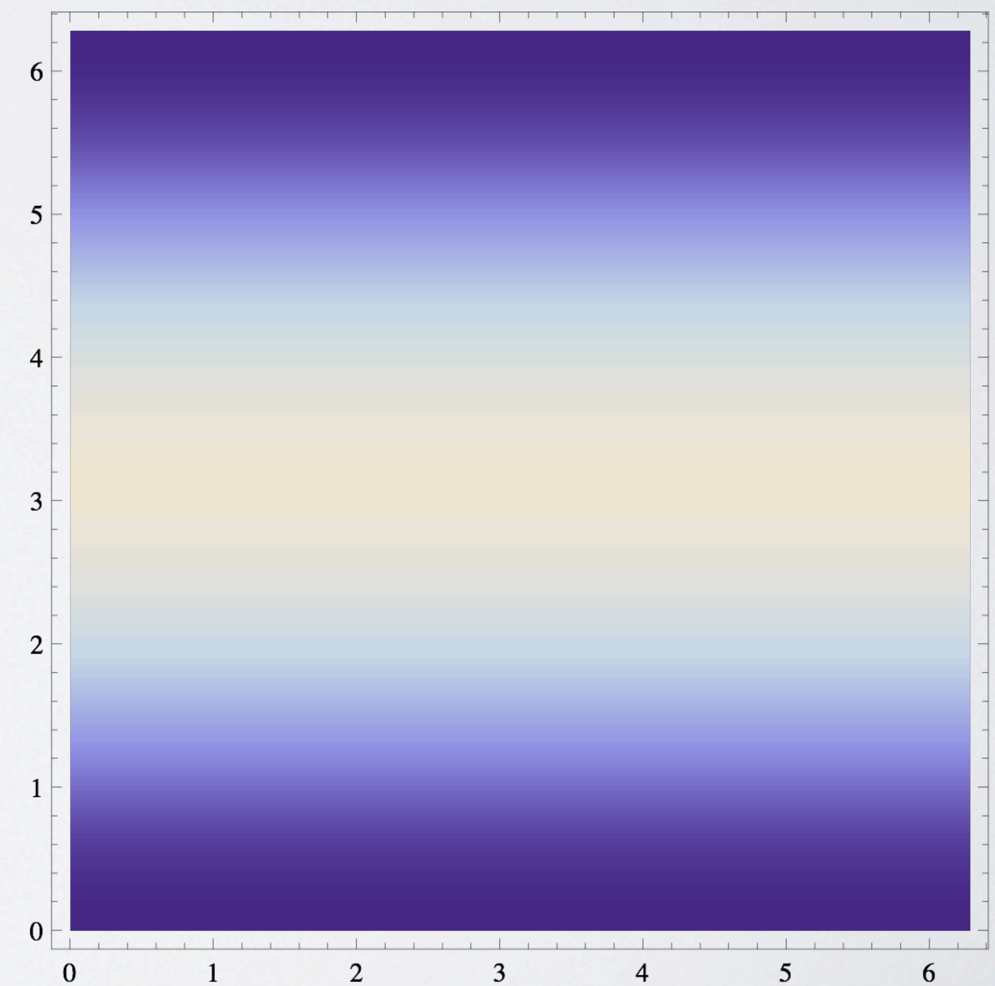
Curl

- For a vector field it describes the net "rotation"
- Cross product of derivatives with the field
  - Scaler in 2D, vector in 3D



$$\mathrm{curl}\,\mathbf{v}(\mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{v}(\mathbf{x})$$



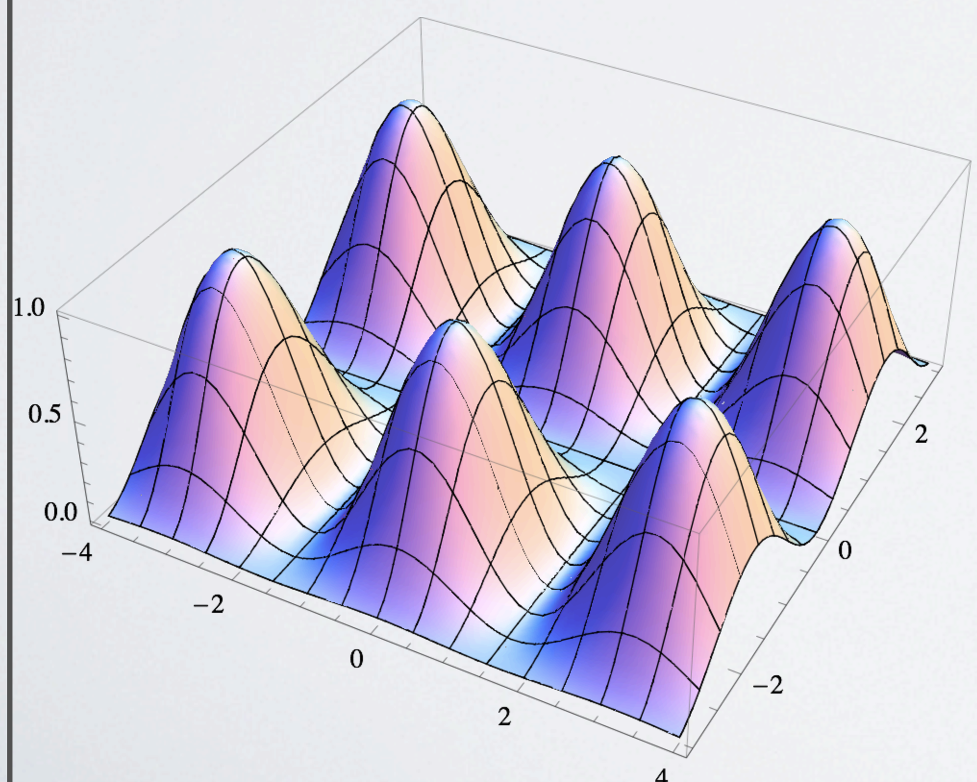


$$\boldsymbol{\nabla} \times [\mathrm{Cos}(y), 0] = -\mathrm{Sin}(y)$$

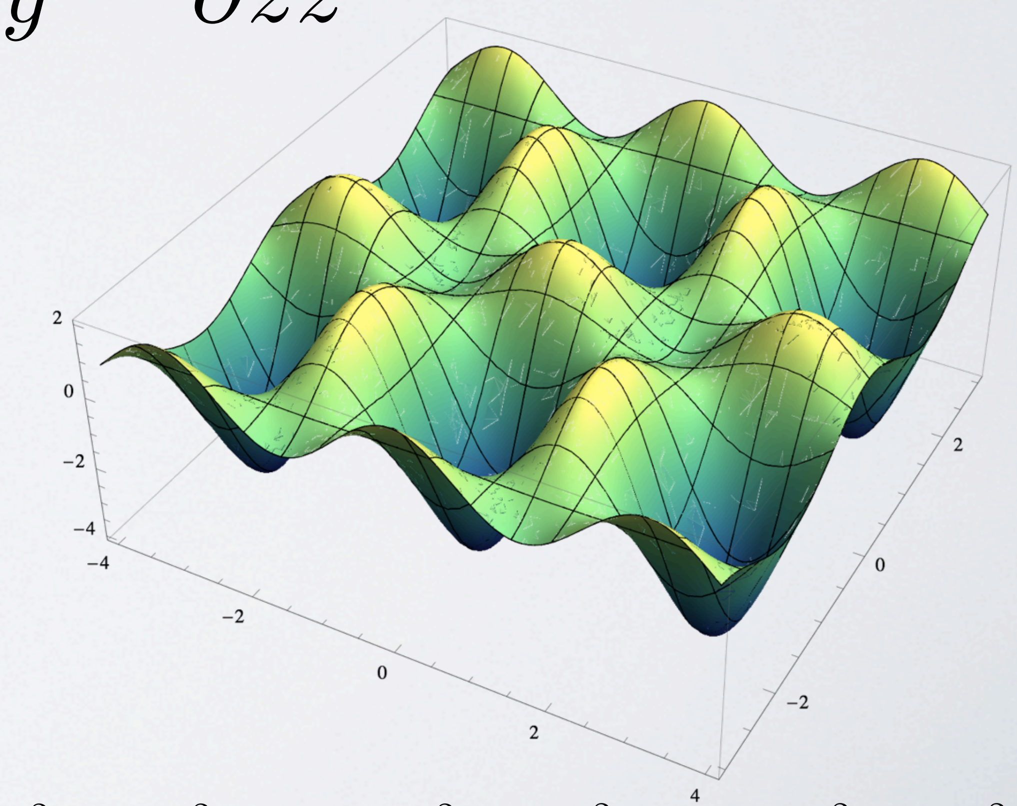
Laplacian

- Divergence of Gradient
  - Scalar second derivative operator
  - Difference between a point and its surround
    - Often used for smoothing of some sort

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial xx} + \frac{\partial^2}{\partial yy} + \frac{\partial^2}{\partial zz}$$



$\cos^2(x)\sin^2(y)$



$2\cos^2(x)\cos^2(y) - 4\cos^2(x)\sin^2(y) + 2\sin^2(x)\sin^2(y)$

Notation Examples

$$\mathbf{v}(\mathbf{x}) = \boldsymbol{\nabla} f(\mathbf{x}) \quad \longrightarrow \quad v_i = \partial_i f$$

$$s(\mathbf{x}) = \boldsymbol{\nabla} \cdot \mathbf{v}(\mathbf{x}) \quad \longrightarrow \quad s = \partial_i v_i$$

$$\mathbf{c}(\mathbf{x}) = \boldsymbol{\nabla} \times \mathbf{v}(\mathbf{x}) \quad \longrightarrow \quad c_i = \varepsilon_{ijk} \partial_j v_k$$

$$\mathbf{a}(\mathbf{x}) = (\mathbf{v}(\mathbf{x}) \cdot \boldsymbol{\nabla}) \mathbf{b}(\mathbf{x}) \quad \longrightarrow \quad a_i = v_j \partial_j b_i$$

Fun Facts

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0$$

$$\nabla \times (\nabla s) = 0$$

Both are obvious in tensor notation

- Helmholtz-Hodge decomposition
  - Smooth, differentiable vector field

Scalar and vector potentials

$$\mathbf{a} = \nabla s + \nabla \times \mathbf{v} + \mathbf{h}$$

∇s irrotational or curl-free part

$\nabla \times \mathbf{v}$ solenoidal or divergence-free part

$\mathbf{h}$ harmonic part

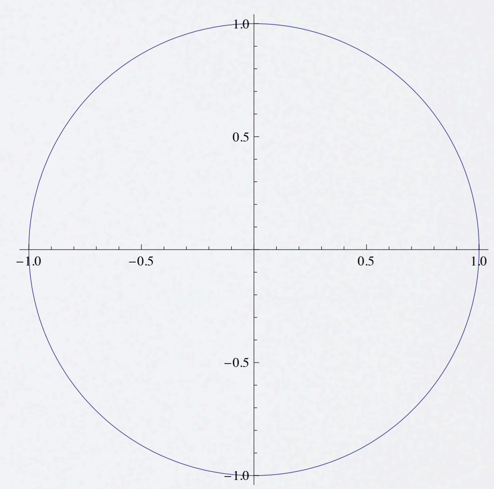
Directional Derivative

$$\frac{\mathrm{d}f}{\mathrm{d}\mathbf{x}} = \mathbf{x} \cdot \boldsymbol{\nabla} f$$

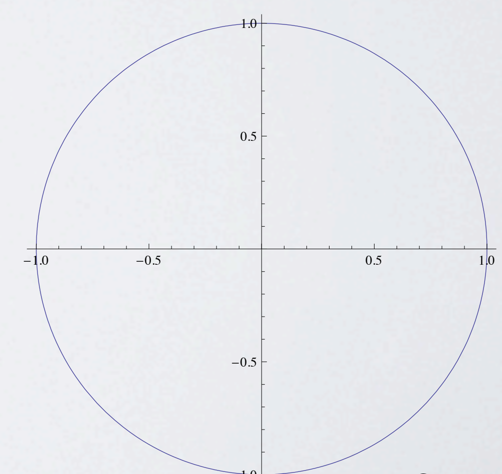
Add a picture or something...

Parametric Curves

- Curve is a geometric entity
  - Set of points in space
  - In neighborhood of any point it is isomorphic to a line
- Generator function: $\mathbf{x} = \mathbf{x}(t)$
  - A vector valued function (careful with "vector")
  - A scalar function for each dimension of embedding space
- A particular parameterization is arbitrary and not unique
  - Parameterization is not intrinsic



$$[\cos(\theta), \sin(\theta)]$$



$$\left[\frac{2u}{u^2+1}, \frac{1-u^2}{u^2+1}\right]$$

Derivatives

- Given function for curve we can take derivatives w.r.t. the parameter:

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$$

- The derivatives have names based on physical analogs
  - Velocity
  - Acceleration
  - Jerk
  - Snap, Crackle, and Pop
- Speed is the magnitude of velocity $s = ||\dot{\mathbf{x}}||$
- All are dependent on parameterization and not intrinsic
- Note that, e.g., velocity is a vector field on t

Arclength

- Let $s = A(t) = \int_0^t ||\mathbf{x}(\tau)|| \, d\tau$
- $A(t)$ is the *arclength* of the curve
- The arclength reparameterization of the curve is

  $$\hat{\mathbf{x}}(s) = \mathbf{x}(A^{-1}(s))$$

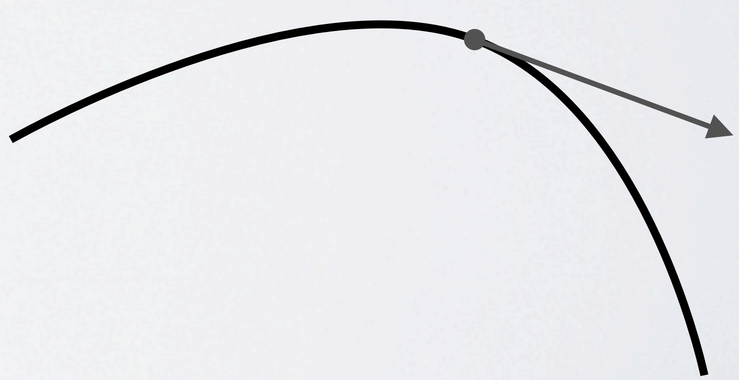
- The arclength parameterization is unique up to sign change and translation
- $\frac{d\hat{\mathbf{x}}(s)}{ds} = \frac{d\mathbf{x}(t)}{dt} \left|\left| \frac{d\mathbf{x}(t)}{dt} \right|\right|^{-1}$ and $\left|\left| \frac{d\hat{\mathbf{x}}(s)}{ds} \right|\right| = 1$

Closed form arclength parameterization may be hard to find.

Tangent Vector

- Tangent vector is a geometric property of the curve
  - Does not depend on parameterization
  - Tangent may exist where velocity is zero or may be undefined

$$\mathbf{T} = \frac{d\hat{\mathbf{x}}(s)}{ds}$$

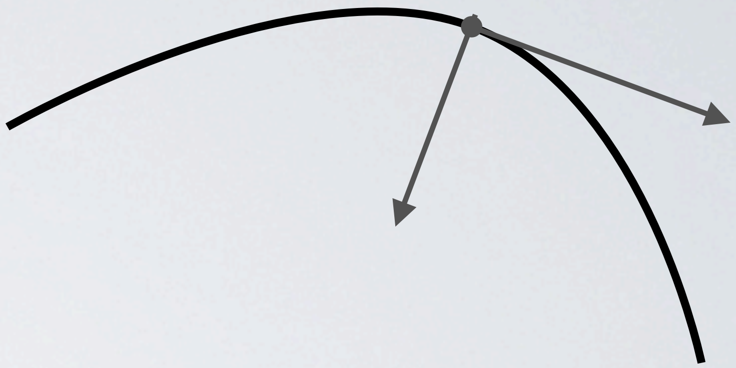


Curvature and Normal

Note: $\mathbf{T} \cdot \mathbf{T} = 1$

$$(\mathbf{T} \cdot \mathbf{T})' = (1)'$$

$$\mathbf{T} \cdot \mathbf{T}' = 0$$



Therefore: $\mathbf{T} \perp \mathbf{T}'$

We can write: $\mathbf{T}' = \kappa \mathbf{N}$

Curvature of the curve at this point (κ)

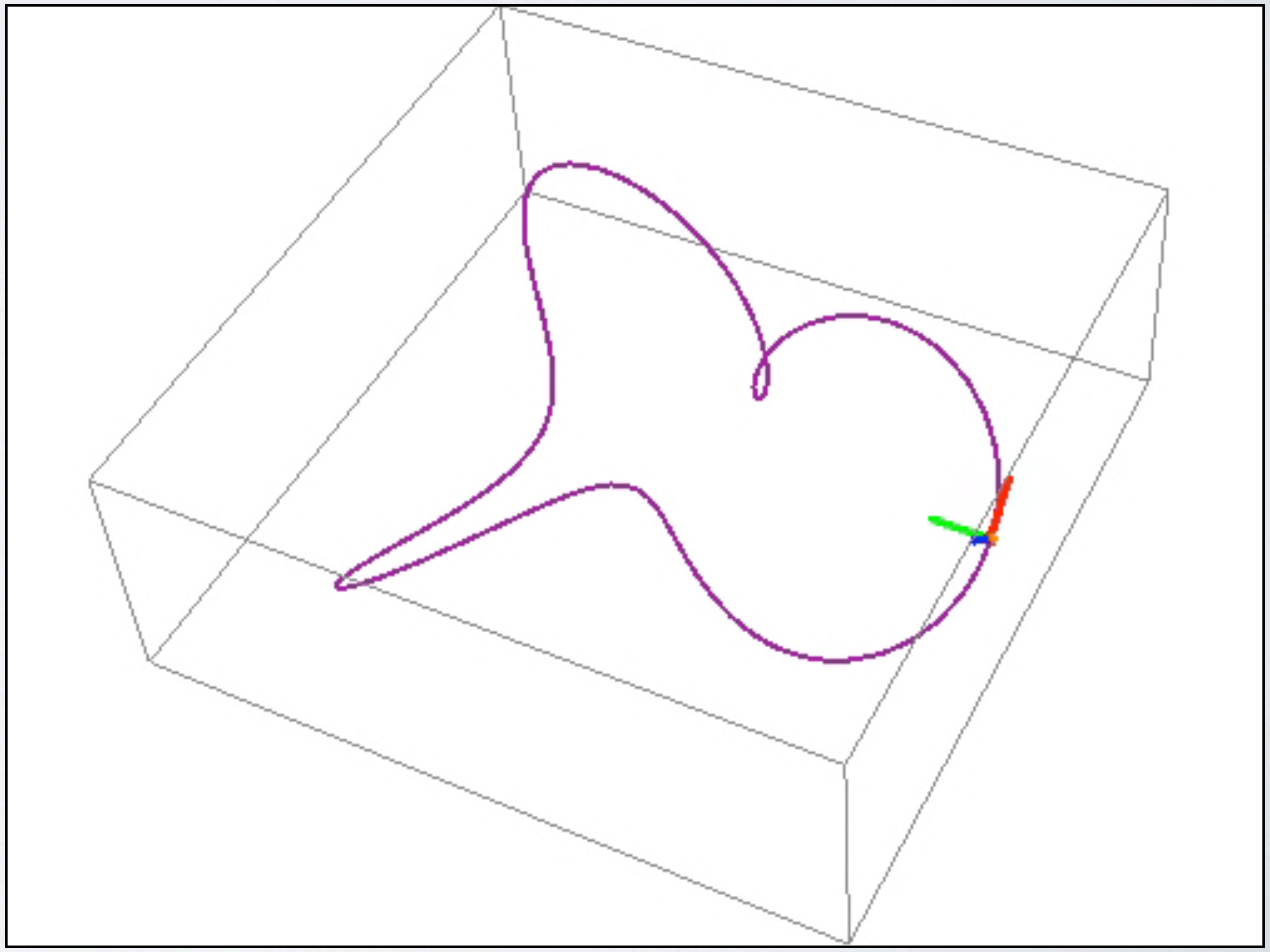
Normal of the curve at this point ($\mathbf{N}$)

Taylor expansion implies that if curvature is zero curve must be locally a straight line.

Frenet Frame

- Define *binormal* by $\mathbf{B} = \mathbf{T} \times \mathbf{N}$
- Gives us orthonormal coordinate frame: Frenet Frame
  - Moves along curve
  - Give local frame of reference

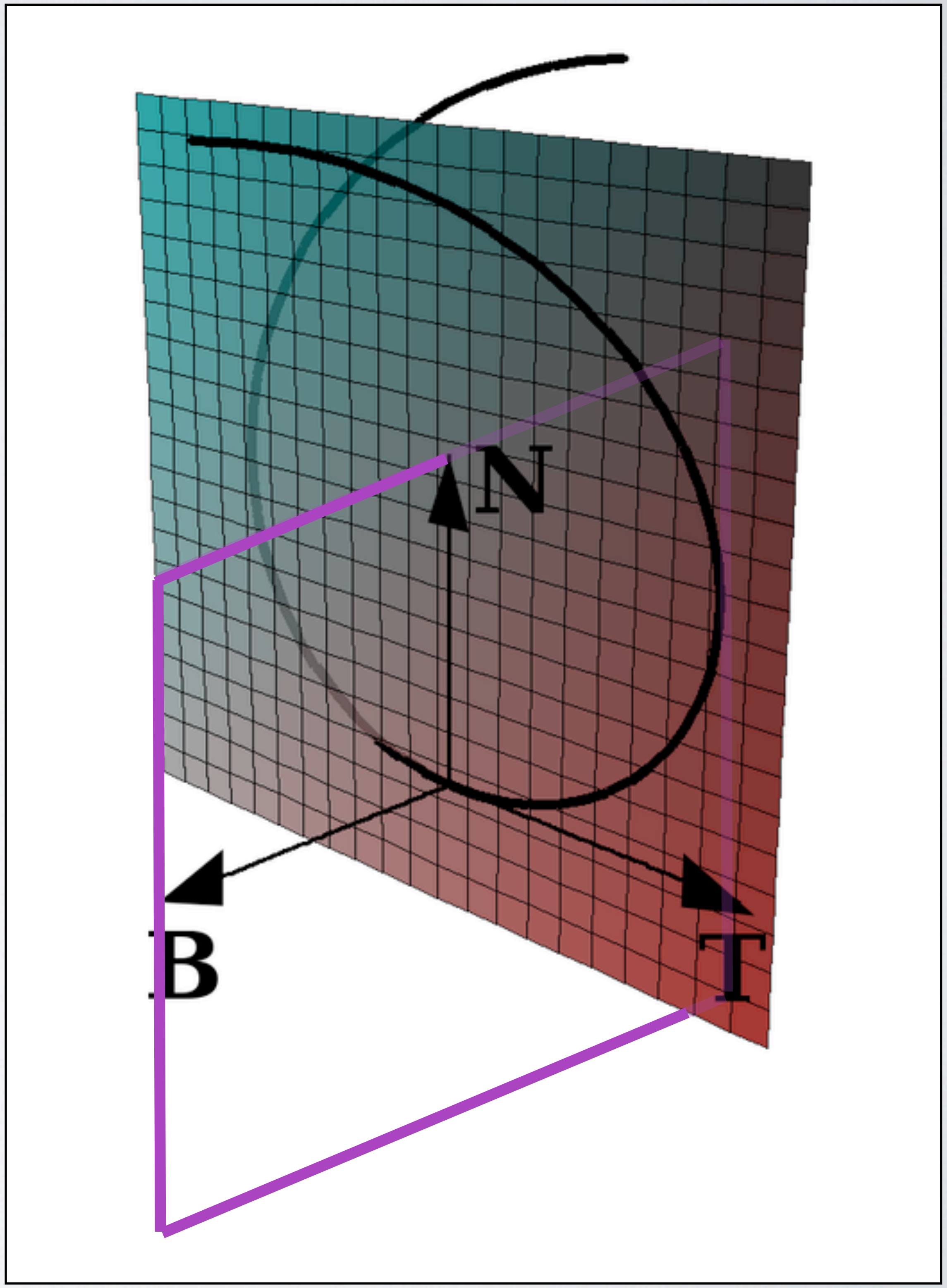
$\mathbf{T}$
$\mathbf{N}$
$\mathbf{B}$



Not defined at inflection points where there is no curvature...

Frenet Frame

Image from Wikipedia



- Osculating Plane
  - Defined by N and T
  - Locally contains the curve
- Normal Plane
  - Defined by N and B
  - Locally perpendicular to the curve

Torsion

$$\mathbf{B}\cdot\mathbf{B}=1 \quad \rightarrow \quad \mathbf{B}\cdot\mathbf{B}'=0$$

$$\mathbf{B}\cdot\mathbf{T}=0 \quad \rightarrow \quad \mathbf{B}'\cdot\mathbf{T}+\mathbf{B}\cdot\mathbf{T}'=0$$

$$\rightarrow \quad \mathbf{B}'\cdot\mathbf{T}=-\mathbf{B}\cdot\mathbf{T}'=-\mathbf{B}.\kappa\mathbf{N}=0$$

$$\mathbf{B}'\perp\mathbf{B} \quad \text{and} \quad \mathbf{B}'\perp\mathbf{T}$$

Change in binormal is then $\mathbf{B}'=-\tau\mathbf{N}$

Torsion

If torsion is zero, we have a planar curve.

The minus sign is to make positive torsion CCW w.r.t tangent.

Evolution of Frenet Frame

$$\mathbf{N}' \perp \mathbf{N} \rightarrow \mathbf{N}' = \alpha\mathbf{T} + \beta\mathbf{B}$$

$$\begin{aligned} \alpha &= \mathbf{N}' \cdot \mathbf{T} \\ \beta &= \mathbf{N}' \cdot \mathbf{B} \end{aligned}$$

Recall it's an orthonormal basis.

Differentiate $\mathbf{N} \cdot \mathbf{T} = 0$ and $\mathbf{N} \cdot \mathbf{B} = 0$

Yields $\mathbf{N}' \cdot \mathbf{T} = -\mathbf{N} \cdot \kappa\mathbf{N} = -\kappa$

$\mathbf{N}' \cdot \mathbf{B} = -\mathbf{N} \cdot (-\tau)\mathbf{N} = \tau$

Therefore $\mathbf{N}' = -\kappa\mathbf{T} + \tau\mathbf{B}$

We know $\mathbf{T}' = \kappa\mathbf{N}$ and $\mathbf{B}' = -\tau\mathbf{B}$

Evolution of Frenet Frame

$$\mathbf{T}' = \kappa \mathbf{N}$$

$$\mathbf{N}' = -\kappa \mathbf{T} + \tau \mathbf{B}$$

$$\mathbf{B}' = -\tau \mathbf{N}$$

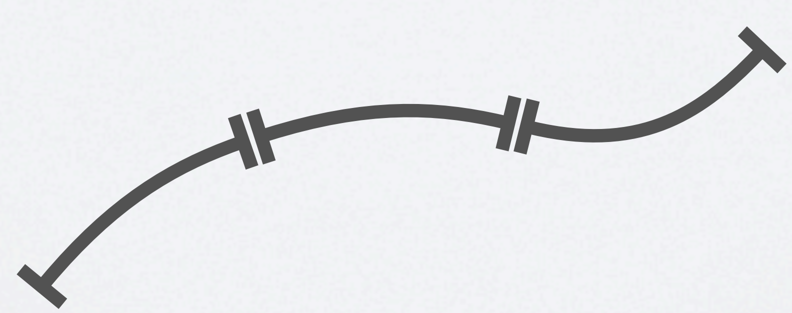
$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

ODE for evolution of Frenet Frame

Given starting point, if you know curvature and torsion, then you can build curve.

(Need "speed" also if not arclength parameterized.)

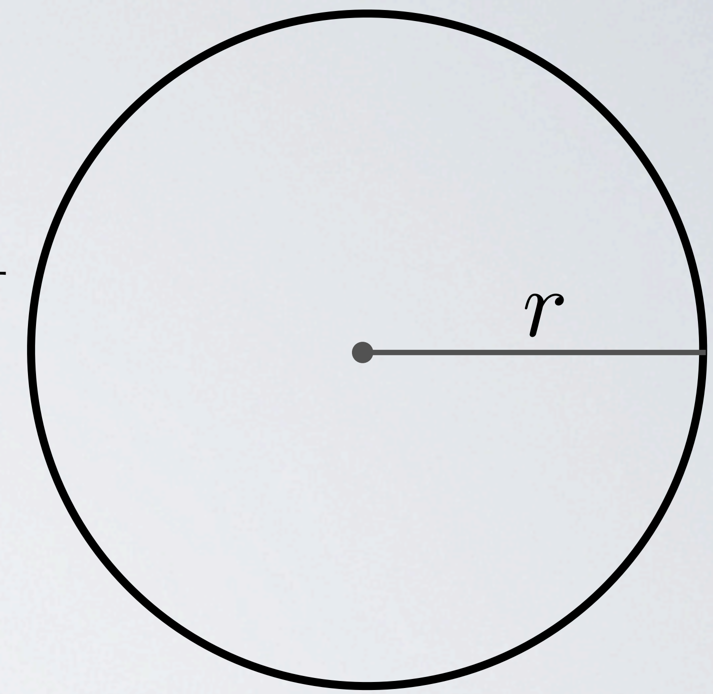
Discrete analogy: stacking up macaroni



Radius of Curvature

$$\hat{\mathbf{x}}(s) = [r\cos\left(\frac{s}{r}\right), r\sin\left(\frac{s}{r}\right)]$$

Note that $||\hat{\mathbf{x}}'|| = 1$



$$\mathbf{T} = [-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right)]$$

$$\mathbf{T}' = [-\frac{1}{r}\cos\left(\frac{s}{r}\right), -\frac{1}{r}\sin\left(\frac{s}{r}\right)]$$

$$\kappa = ||\mathbf{T}'|| = \frac{1}{r}$$

Curvature is inverse of *radius of curvature.*

Some Formulae

- For arclength parameterized curve

$$\kappa = ||\hat{\mathbf{x}}(s)''||$$

$$\tau = \frac{\hat{\mathbf{x}}' \cdot (\hat{\mathbf{x}}'' \times \hat{\mathbf{x}}''')}{||\hat{\mathbf{x}}''||^2}$$

- For arbitrarily parameterized curve

$$\kappa = \frac{||\mathbf{x}'(t) \times \mathbf{x}''(t)||}{||\mathbf{x}'(t)||^3}$$

$$\tau = \frac{\mathbf{x}'(t) \times \mathbf{x}''(t) \cdot \mathbf{x}'''(t)}{||\mathbf{x}'(t) \times \mathbf{x}''(t)||^2}$$

Field Evaluated Along a Curve

- Curve defined in some space
  - $\mathbf{x}(t)$
- Function on embedding space of curve
  - $f(\mathbf{x})$
- Composition function
  - $f(\mathbf{x}(t))$
  - $\frac{\mathrm{d}f}{\mathrm{d}t} = \boldsymbol{\nabla} f \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$

Parametric Surfaces

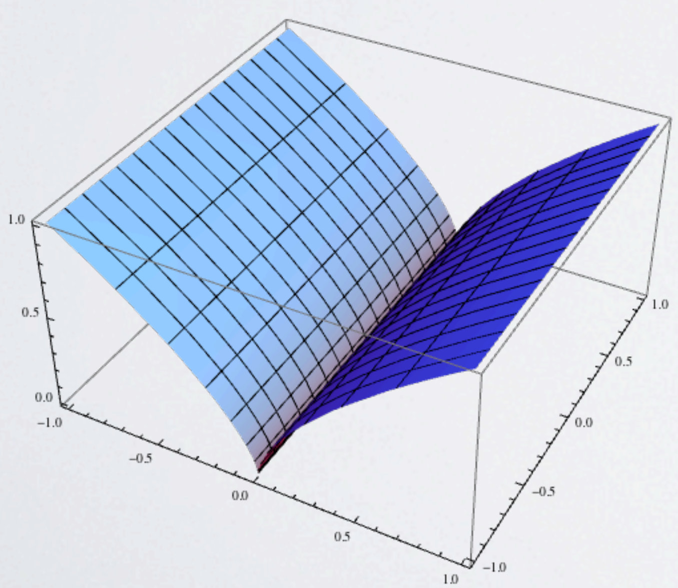
- Surface is a geometric entity
  - Set of points in space
  - In neighborhood of any point it is isomorphic to a plane
- Generator function: $\mathbf{x}(\mathbf{u})$
  - A vector valued function (careful with "vector")
  - A scalar function for each dimension of embedding space
  - Dimension of parameter is two
- A particular parameterization is arbitrary and not unique
  - Parameterization is not intrinsic

Derivatives

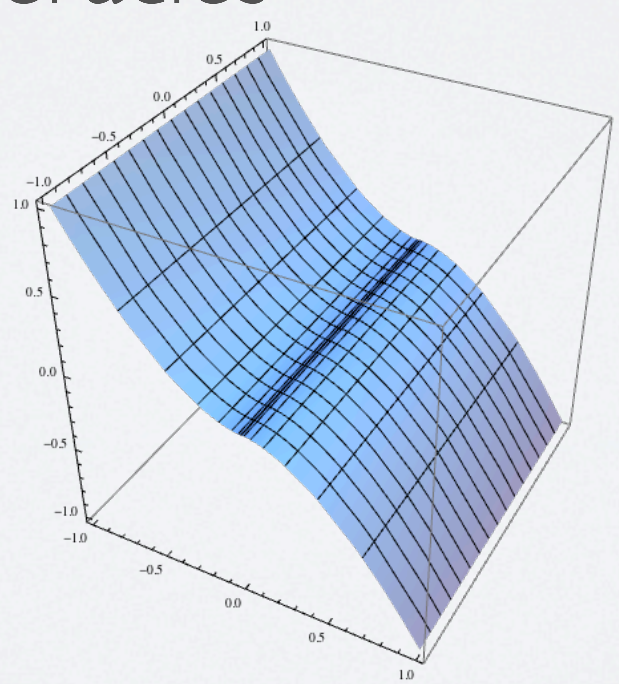
- Given function for curve we can take derivatives w.r.t. the parameter:

$$\frac{\partial \mathbf{x}(\mathbf{u})}{\partial u} \qquad \frac{\partial \mathbf{x}(\mathbf{u})}{\partial v}$$

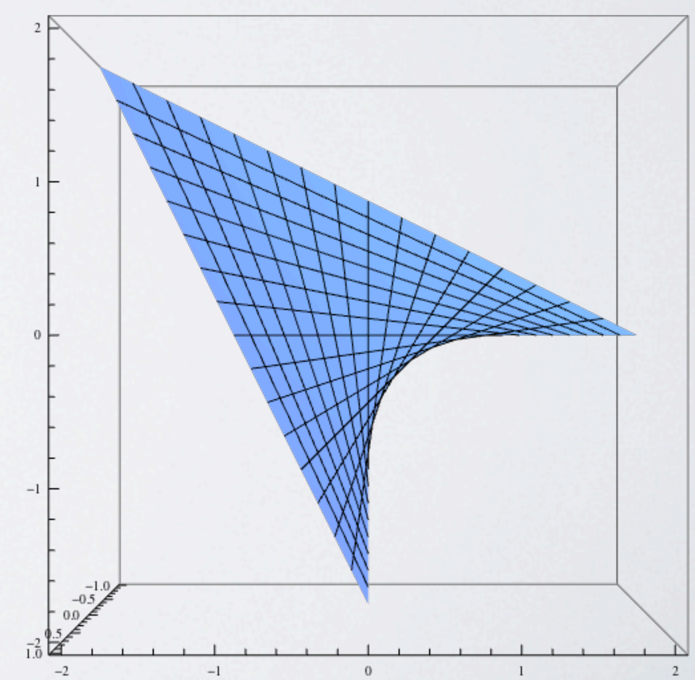
- All are dependent on parameterization and not intrinsic
- Note that each one is a vector field on $\mathbf{u}$
- Examples of degeneracies



$[v^3, u, v^2]$



$[v^3, u, -v^5]$



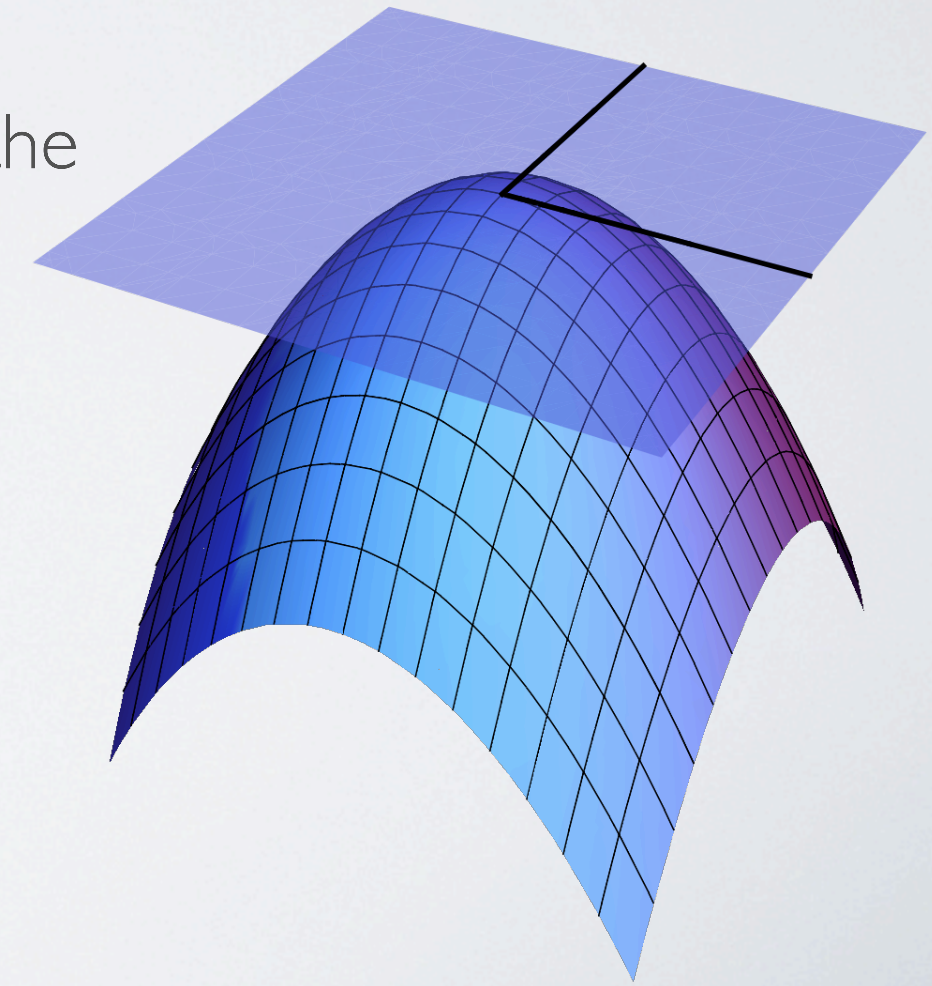
$[v(u+1), u(1-v), 0]$

Tangent Space

- The *tangent space* at a point on a surface is the vector space spanned by

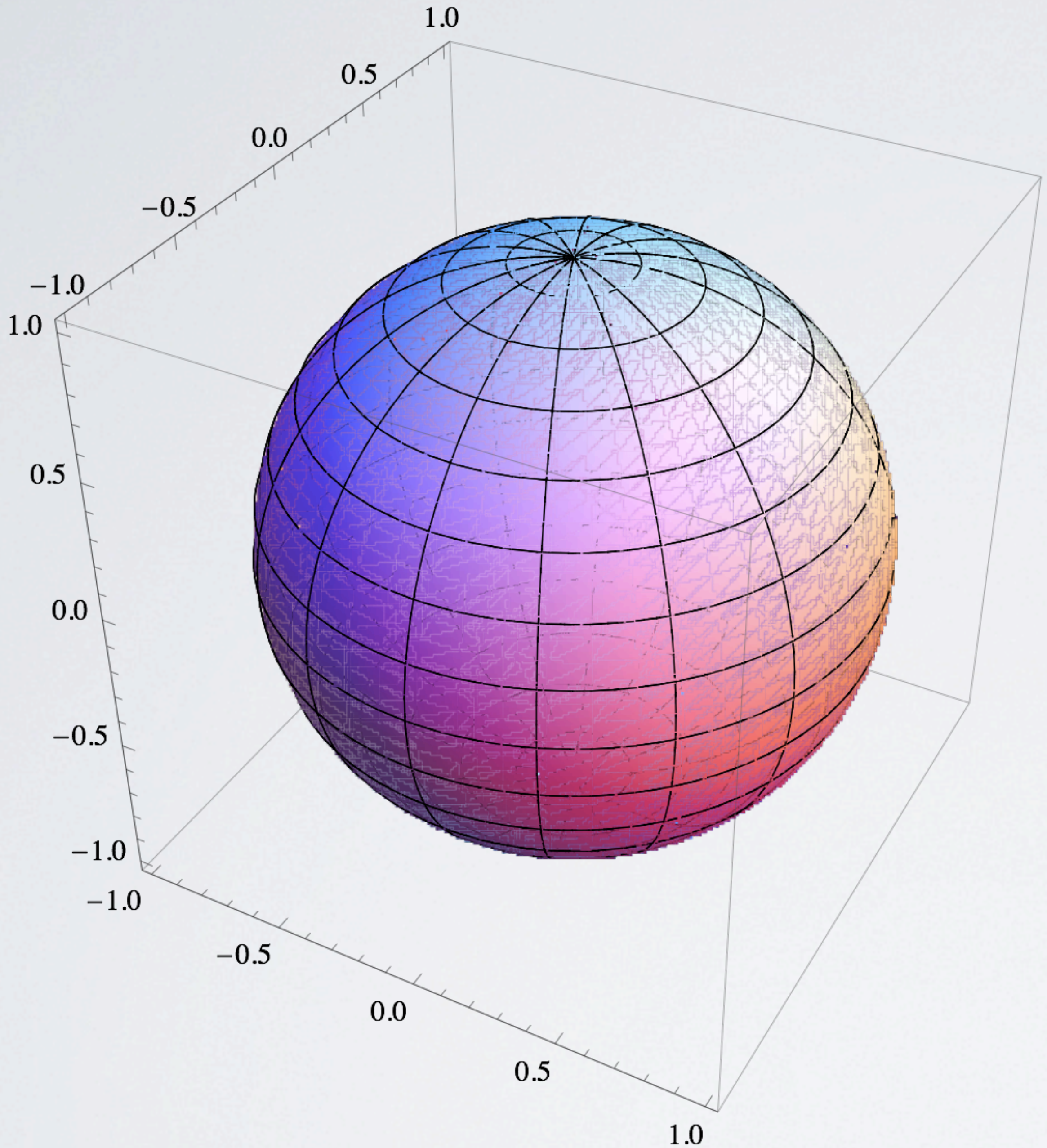
$$\frac{\partial \mathbf{x}(\mathbf{u})}{\partial u} \qquad \frac{\partial \mathbf{x}(\mathbf{u})}{\partial v}$$

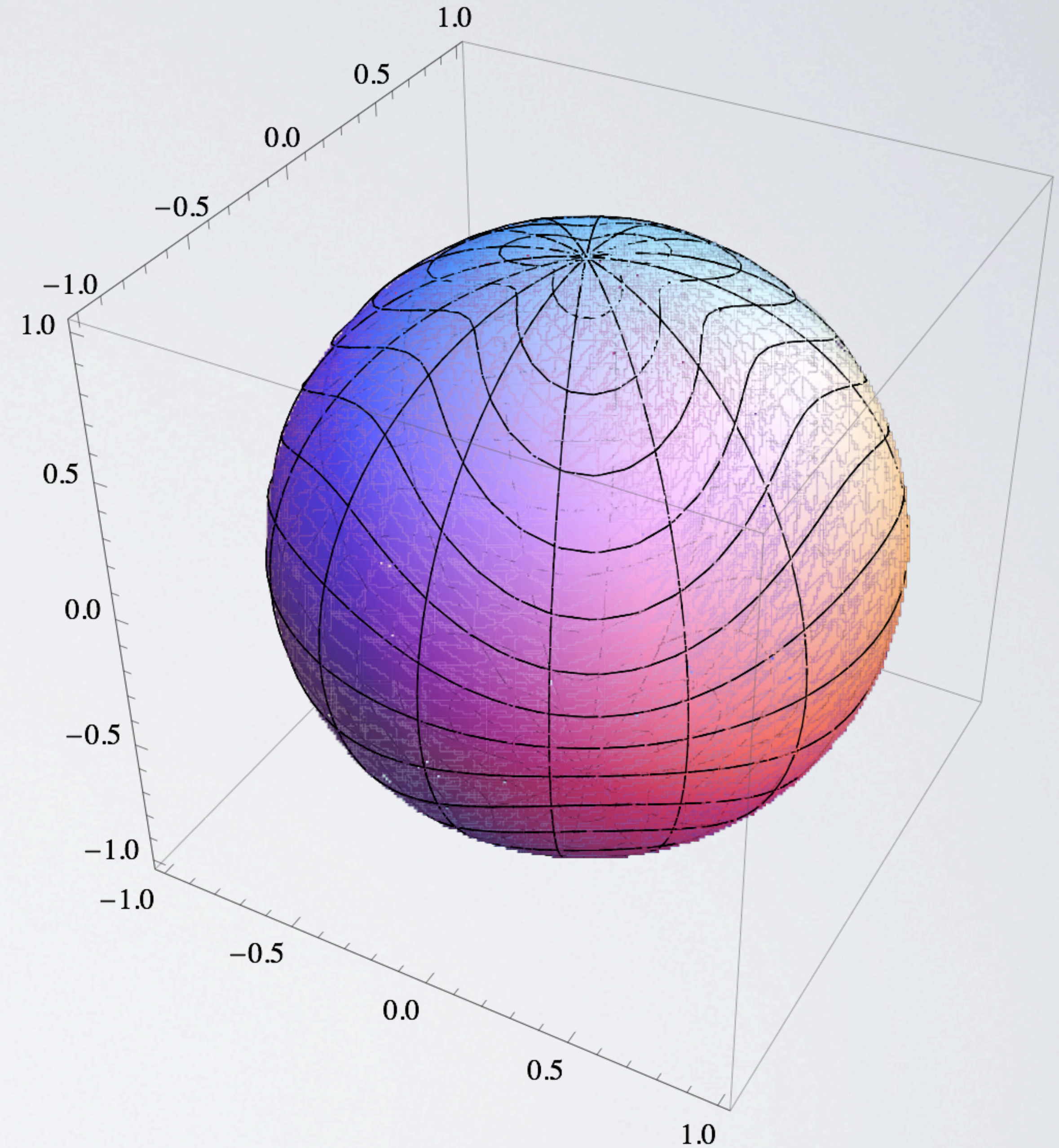
  - Definition assumes that these directional derivatives are linearly independent.
  - Tangent space of surface may exist even if the parameterization is bad



- For surface the space is a plane
  - Generalized to higher dimension manifolds

Non Orthogonal Tangents





$$\begin{bmatrix} \cos(\theta 2\pi)\cos(\phi pi/2) \\ \sin(\theta 2\pi)\cos(\phi\pi/2) \\ \sin(\phi\pi/2) \end{bmatrix}$$

$$\begin{bmatrix} \cos(2\pi\theta)\cos\left(\frac{1}{2}\pi\left(\frac{1}{2}(1-|\phi|)\cos(6\pi\theta)\phi+\phi\right)\right) \\ \cos\left(\frac{1}{2}\pi\left(\frac{1}{2}(1-|\phi|)\cos(6\pi\theta)\phi+\phi\right)\right)\sin(2\pi\theta) \\ \sin\left(\frac{1}{2}\pi\left(\frac{1}{2}(1-|\phi|)\cos(6\pi\theta)\phi+\phi\right)\right) \end{bmatrix}$$

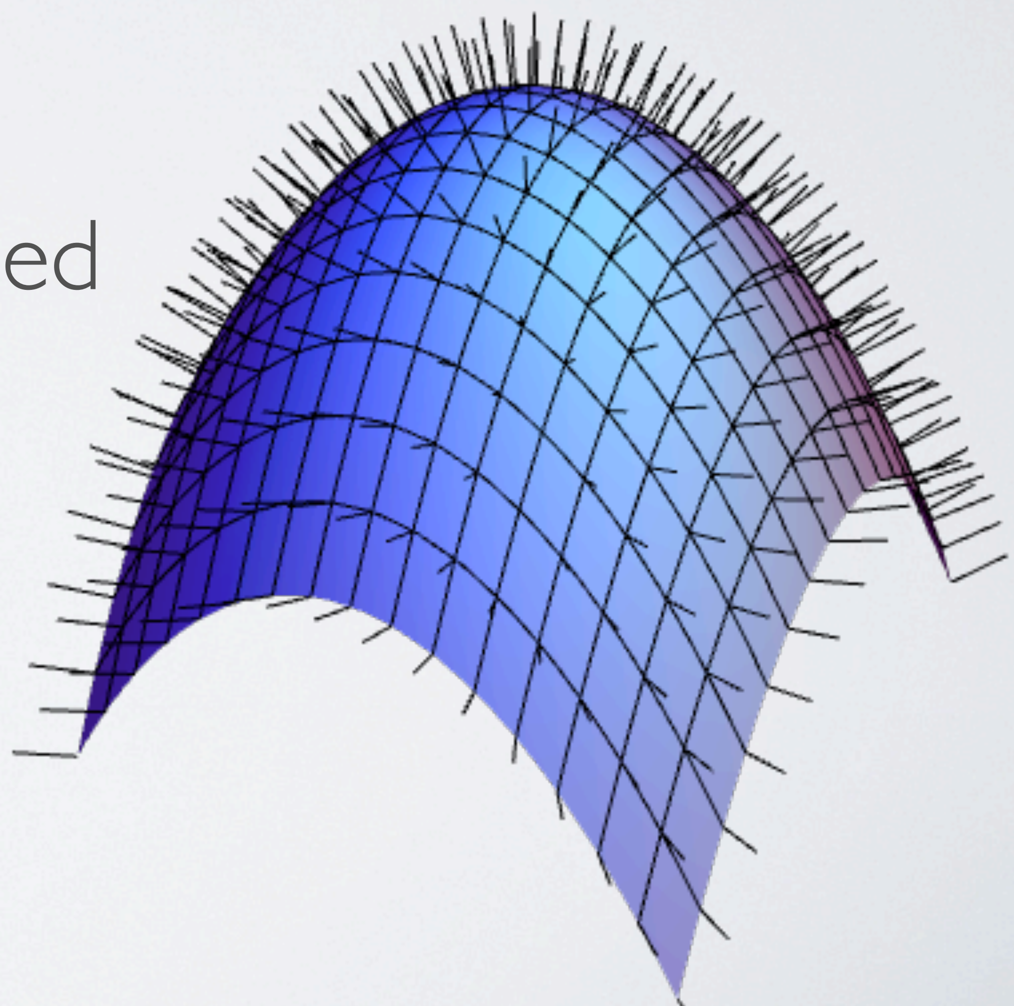
$\theta \in [0..1] \quad \phi \in [-1..1]$

Normals

- The normal at a point is the unit vector perpendicular to the tangent space

- $\mathbf{N} = \frac{\partial_u \mathbf{x} \times \partial_v \mathbf{x}}{||\partial_u \mathbf{x} \times \partial_v \mathbf{x}||}$

- The normal direction is determined
  - Up to a sign change
  - Relative to surface



First Fundamental

Pick a direction in parametric space: $\mathbf{du} = [\mathrm{d}u, \mathrm{d}v]$

Corresponding direction in the tangent plane:

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{x}}{\partial v}\mathrm{d}v$$

$$\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{x}(\mathbf{u})$$

For unit speed in parametric space, the sped in the embedding space is

$$s^2 = \mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{u}^{\mathsf{T}} \cdot (\boldsymbol{\nabla}\mathbf{x}) \cdot (\boldsymbol{\nabla}\mathbf{x})^{\mathsf{T}} \cdot \mathrm{d}\mathbf{u}$$

$$\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{u}^{\mathsf{T}} \cdot \mathbf{I} \cdot \mathrm{d}\mathbf{u}$$

$$\mathbf{I} = \begin{bmatrix} \partial_u \mathbf{x} \cdot \partial_u \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_u \mathbf{x} \\ \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_v \mathbf{x} \end{bmatrix} \qquad I_{ij} = (\partial_i x_k)(\partial_j x_k)$$

First Fundamental

$$\mathbf{I} = \begin{bmatrix} \partial_u \mathbf{x} \cdot \partial_u \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_u \mathbf{x} \\ \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_v \mathbf{x} \end{bmatrix} \qquad I_{ij} = (\partial_i x_k)(\partial_j x_k)$$

- Encodes distance metric on the surface
- If tangents are orthonormal it reduces to identity
- Used as metric by Green's Strain
- Invariant w.r.t. translations and rotations of surface:

e.g.

$$x'_i = R_{ij} x_j$$

$$\begin{aligned} (\partial_i x'_k)(\partial_j x'_k) &= (\partial_i R_{kp} x_p)(\partial_j R_{kq} x_q) \\ &= R_{kp} R_{kq} (\partial_i x_p)(\partial_j x_q) \\ &= \delta_{pq} (\partial_i x_p)(\partial_j x_q) \\ &= (\partial_i x_p)(\partial_j x_p) \end{aligned}$$

First Fundamental

$$\mathbf{I} = \left[\begin{array}{cc} \partial_u \mathbf{x} \cdot \partial_u \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_u \mathbf{x} \\ \partial_u \mathbf{x} \cdot \partial_v \mathbf{x} & \partial_v \mathbf{x} \cdot \partial_v \mathbf{x} \end{array} \right] \qquad I_{ij} = (\partial_i x_k)(\partial_j x_k)$$

- Transforms like a tensor in parameter space:

$$u'_i = R_{ij} u_j \longrightarrow u_i = R_{ji} u'_j$$

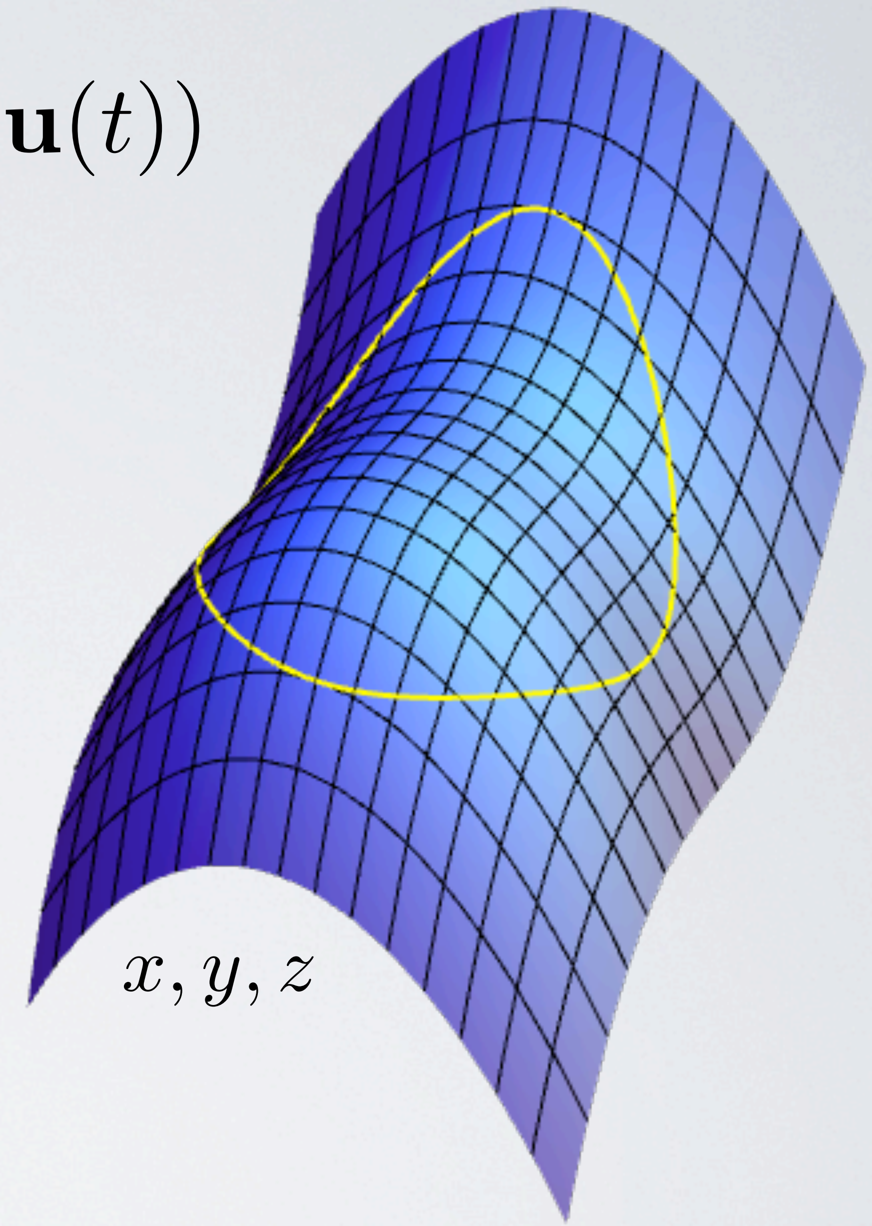
Assume orthonormal transform...

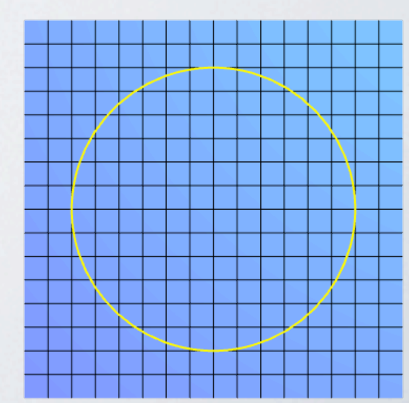
$$\begin{aligned} \frac{\partial x_k}{\partial u'_i} \frac{\partial x_k}{\partial u'_j} &= \frac{\partial x_k}{\partial u_p} \frac{\partial u_p}{\partial u'_i} \; \frac{\partial x_k}{\partial u_q} \frac{\partial u_q}{\partial u'_j} \\ &= R_{ip} \frac{\partial x_k}{\partial u_p} \; \frac{\partial x_k}{\partial u_q} R_{jq} \\ I'_{ij} &= R_{ip} \, I_{pq} \, R_{jq} \end{aligned}$$

Arclength Over Surface

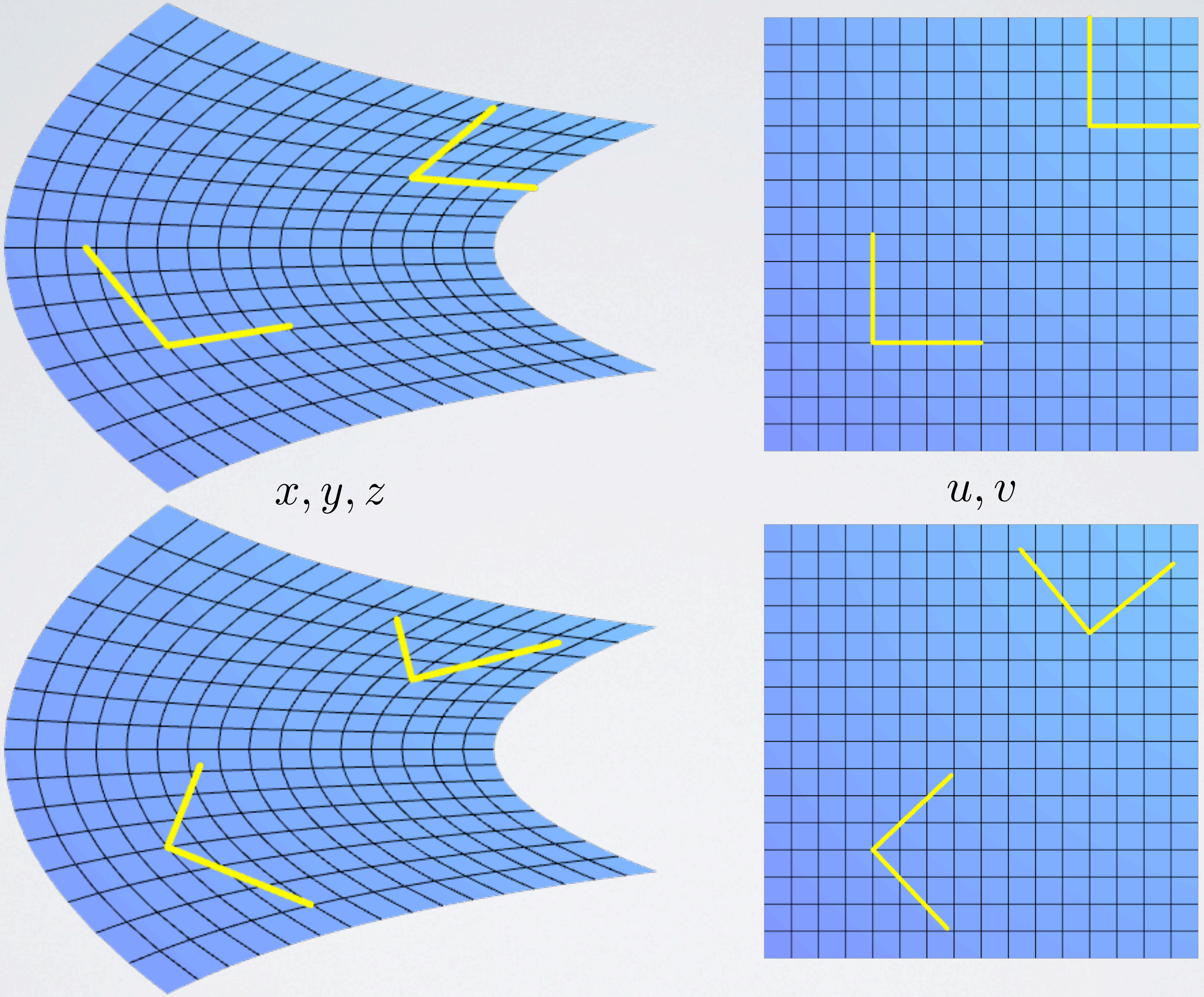
$$\mathbf{c}(t) = \mathbf{x}(\mathbf{u}(t))$$

$$\begin{aligned} l &= \int_a^b \left|\left|\frac{\mathrm{d}\mathbf{c}(t)}{\mathrm{d}t}\right|\right| \mathrm{d}t \\ &= \int_a^b \sqrt{||\mathrm{d}\mathbf{x}||^2 \mathrm{d}t} \\ &= \int_a^b \sqrt{\mathrm{d}\mathbf{x} \cdot \mathrm{d}\mathbf{x}}\, \mathrm{d}t \\ &= \int_a^b \sqrt{\mathrm{d}\mathbf{u}^{\mathsf{T}} \cdot \mathbf{I} \cdot \mathrm{d}\mathbf{u}}\, \mathrm{d}t \end{aligned}$$





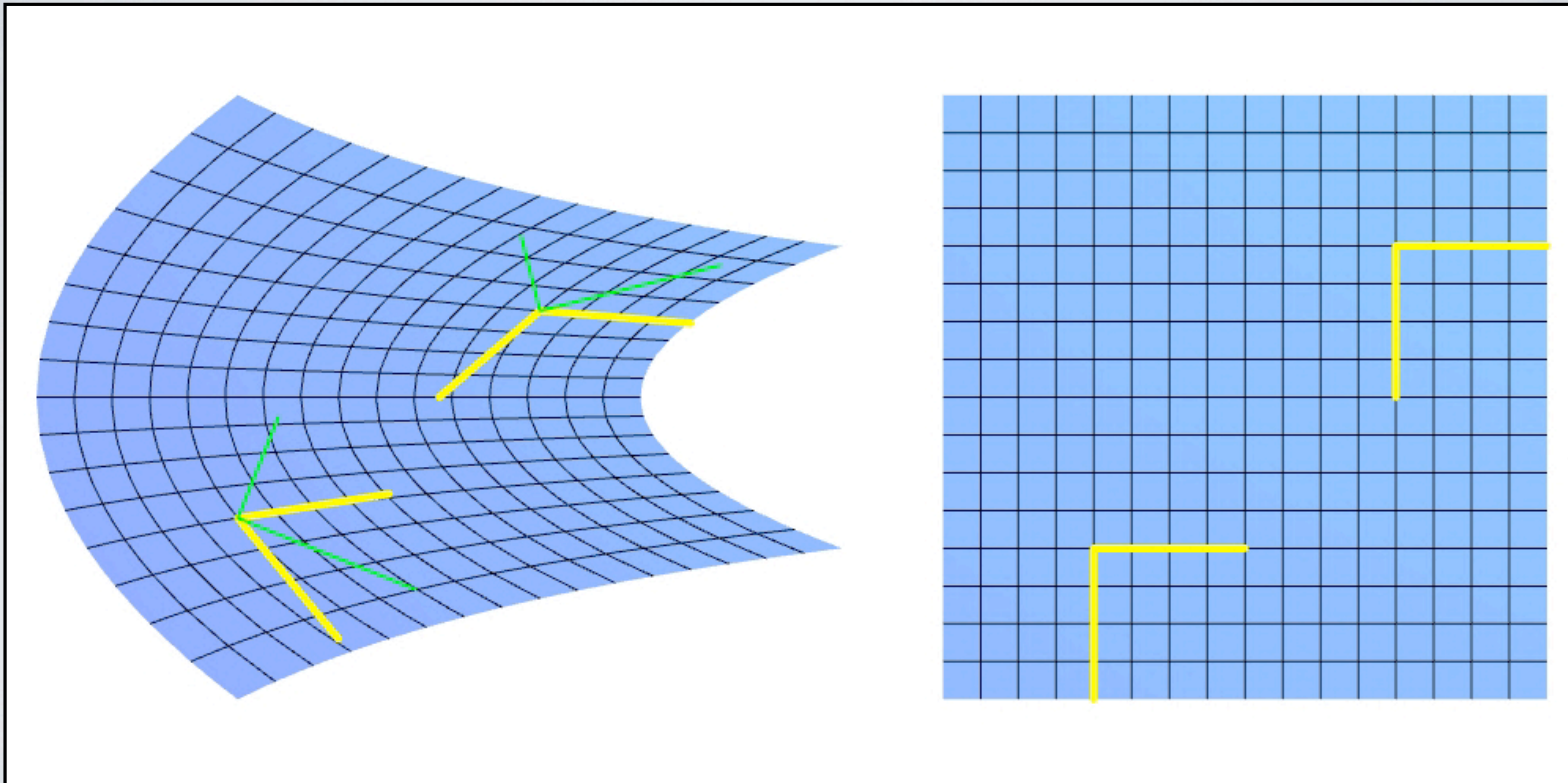
Principle Tangents



Bottom row is eigenvectors of $\mathbf{I}$

Not intrinsic features of the surface!

Principle Tangents



Orthonormal Parameterization

Eigen decomposition of Fist Fundamental

$$\mathbf{I} = \mathbf{R}\mathbf{S}^2\mathbf{R}^\mathsf{T} = \mathbf{A}\mathbf{A}^\mathsf{T}$$

Define coordinate transform by

$$\mathrm{d}\mathbf{u}' = \mathbf{S}\mathbf{R}^\mathsf{T}\mathrm{d}\mathbf{u} = \mathbf{A}^\mathsf{T}\mathrm{d}\mathbf{u}$$

$$\mathrm{d}\mathbf{u} = \mathbf{R}(1/\mathbf{S})\mathrm{d}\mathbf{u}' = \mathbf{A}^{-\mathsf{T}}\mathrm{d}\mathbf{u}'$$

In transformed parameterization $\mathbf{I}$ is the identity.

$$\begin{aligned} \mathrm{d}\mathbf{u}'^\mathsf{T} \cdot \mathbf{I}' \cdot \mathrm{d}\mathbf{u}' &= \mathrm{d}\mathbf{u}'^\mathsf{T} \cdot (1/\mathbf{S}) \cdot \mathbf{R}^\mathsf{T} \cdot \left(\mathbf{R} \cdot \mathbf{S}^2 \cdot \mathbf{R}^\mathsf{T}\right) \cdot \mathbf{R} \cdot (1/\mathbf{S}) \cdot \mathrm{d}\mathbf{u}' \\ &= \mathrm{d}\mathbf{u}'^\mathsf{T} \cdot \left((1/\mathbf{S}) \cdot \mathbf{R}^\mathsf{T} \cdot \mathbf{R} \cdot \mathbf{S}^2 \cdot \mathbf{R}^\mathsf{T} \cdot \mathbf{R} \cdot (1/\mathbf{S}) \cdot\right) \mathrm{d}\mathbf{u}' \end{aligned}$$

Similar to definition of arclength reparameterization.

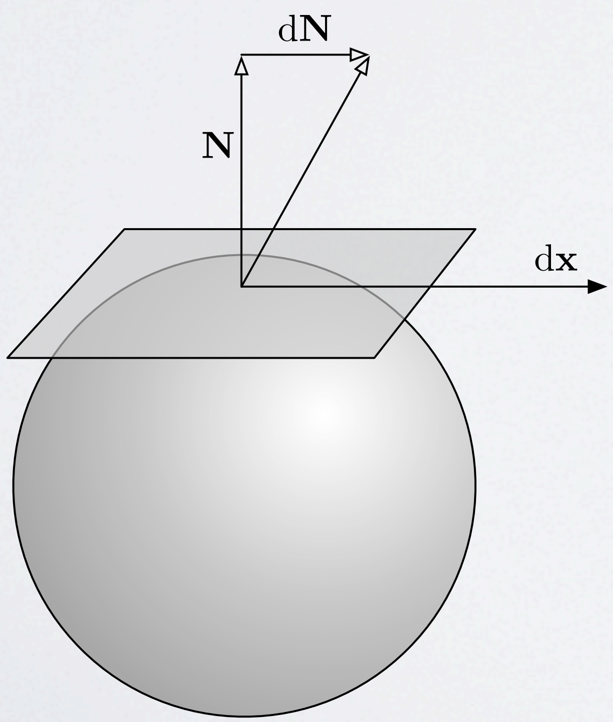
Second Fundamental

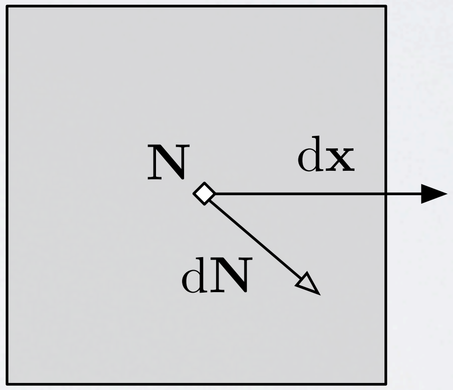
Let $\mathbf{dx}$ be some tangent direction $\quad \mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{x}(\mathbf{u})$

The directional derivative of the normal is

$$\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{N} = \frac{\partial \mathbf{N}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{N}}{\partial v}\mathrm{d}v$$

The normal is unit length so it is perpendicular to its derivative.





As shown in top-down view, the three vectors may not be co-planar.
Surface may tilt to side as point moves.

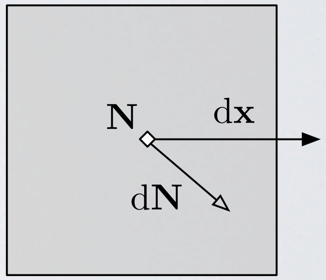
Second Fundamental

Let $\mathrm{d}\mathbf{x}$ be some tangent direction $\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{x}(\mathbf{u})$

The directional derivative of the normal is

$$\boldsymbol{\nabla}_{\mathbf{u}}\mathbf{N} = \frac{\partial \mathbf{N}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{N}}{\partial v}\mathrm{d}v$$

The change in normal restricted to the plane containing the tangent and normal is given by



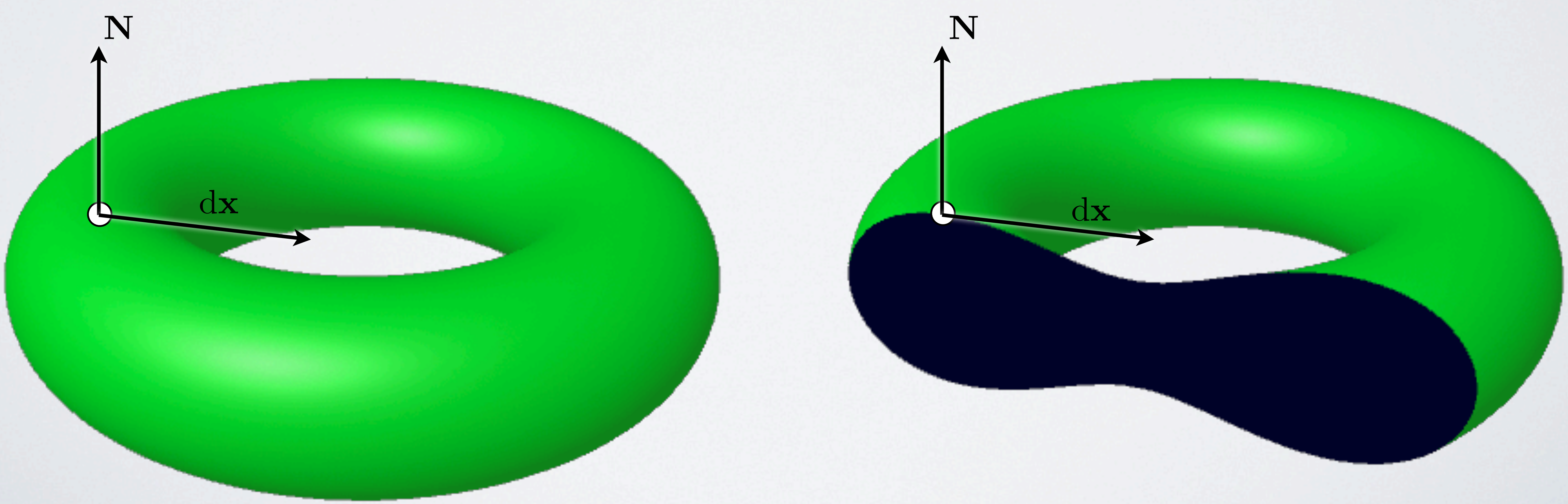
$$\begin{aligned} -\mathbf{T} \cdot \mathbf{N}_T &= -\mathrm{d}\mathbf{x} \cdot \boldsymbol{\nabla}_{\mathbf{u}}\,\mathbf{N} \\ &= -(\mathrm{d}\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{x}) \cdot (\mathrm{d}\mathbf{u} \cdot \boldsymbol{\nabla}\mathbf{N}) \\ &= \mathrm{d}\mathbf{u}^{\mathsf{T}} \begin{bmatrix} -\partial_u \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_u \mathbf{x} \cdot \partial_v \mathbf{N} \\ -\partial_v \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_v \mathbf{x} \cdot \partial_v \mathbf{N} \end{bmatrix} \mathrm{d}\mathbf{u} \end{aligned}$$

Second Fundamental

$$-\mathbf{T} \cdot \mathbf{N}_T = \mathrm{d}\mathbf{u}^{\mathsf{T}} \begin{bmatrix} -\partial_u \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_u \mathbf{x} \cdot \partial_v \mathbf{N} \\ -\partial_v \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_v \mathbf{x} \cdot \partial_v \mathbf{N} \end{bmatrix} \mathrm{d}\mathbf{u}$$

$$= \mathrm{d}\mathbf{u}^{\mathsf{T}} \, \mathbf{II} \, \mathrm{d}\mathbf{u}$$

Matches definition of curvature for curve defined by cutting surface with the normal-tangent plane, but scaled by the surface metric.



Second Fundamental

$$\mathbf{II} = \begin{bmatrix} -\partial_u \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_u \mathbf{x} \cdot \partial_v \mathbf{N} \\ -\partial_v \mathbf{x} \cdot \partial_u \mathbf{N} & -\partial_v \mathbf{x} \cdot \partial_v \mathbf{N} \end{bmatrix}$$

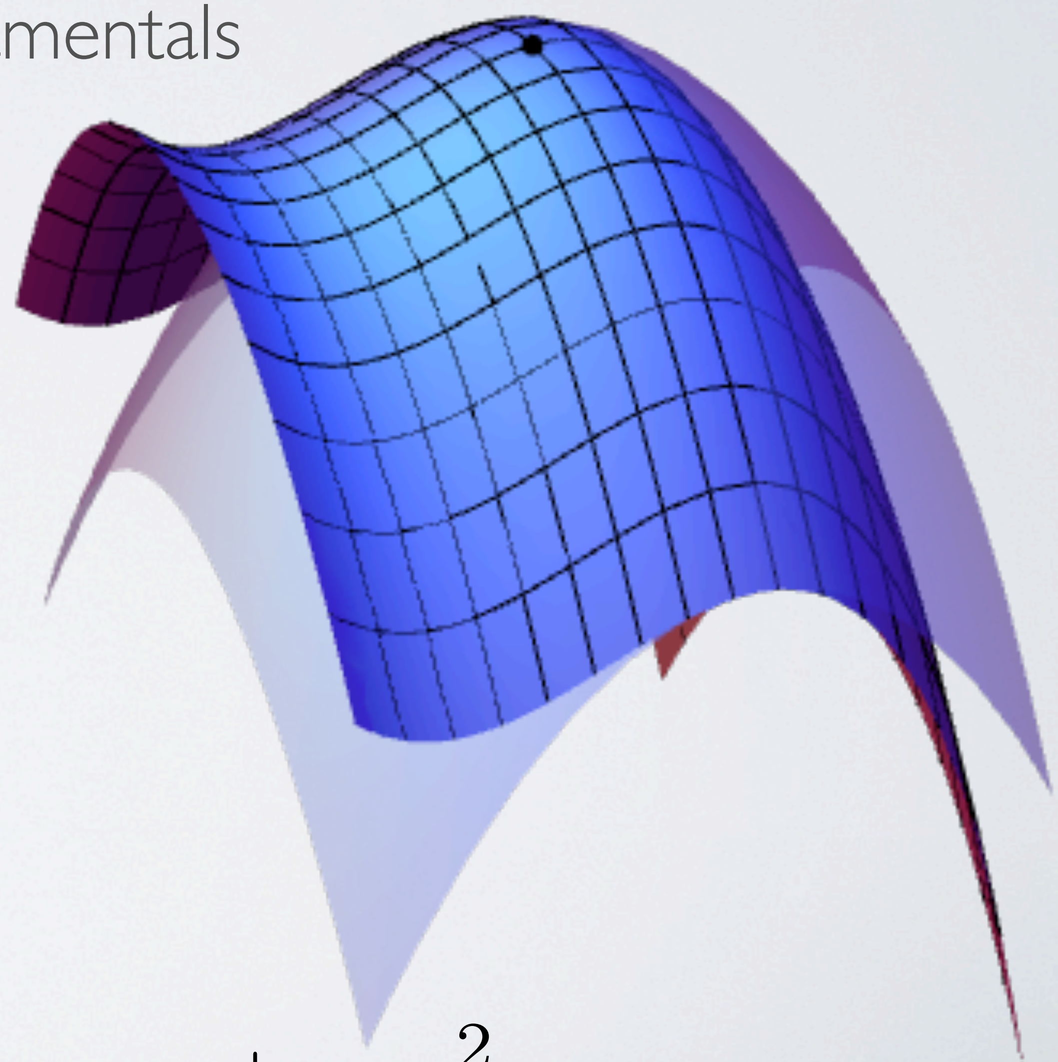
$$= \begin{bmatrix} \partial_{uu} \mathbf{x} \cdot \mathbf{N} & \partial_{uv} \mathbf{x} \cdot \mathbf{N} \\ \partial_{vu} \mathbf{x} \cdot \mathbf{N} & \partial_{vv} \mathbf{x} \cdot \mathbf{N} \end{bmatrix}$$

Symmetry

- Easy to show second version by expanding normal
  - Box product with repeat is zero
  - Any change in normal length will be perpendicular to surface
  - Permutation of box product does not change results

Osculating Paraboloid

- Tangent plane is linear approximation to surface at a point
- Osculating paraboloid is quadratic approximation to surface at a point
  - Matches surface's First and Second Fundamentals at the point

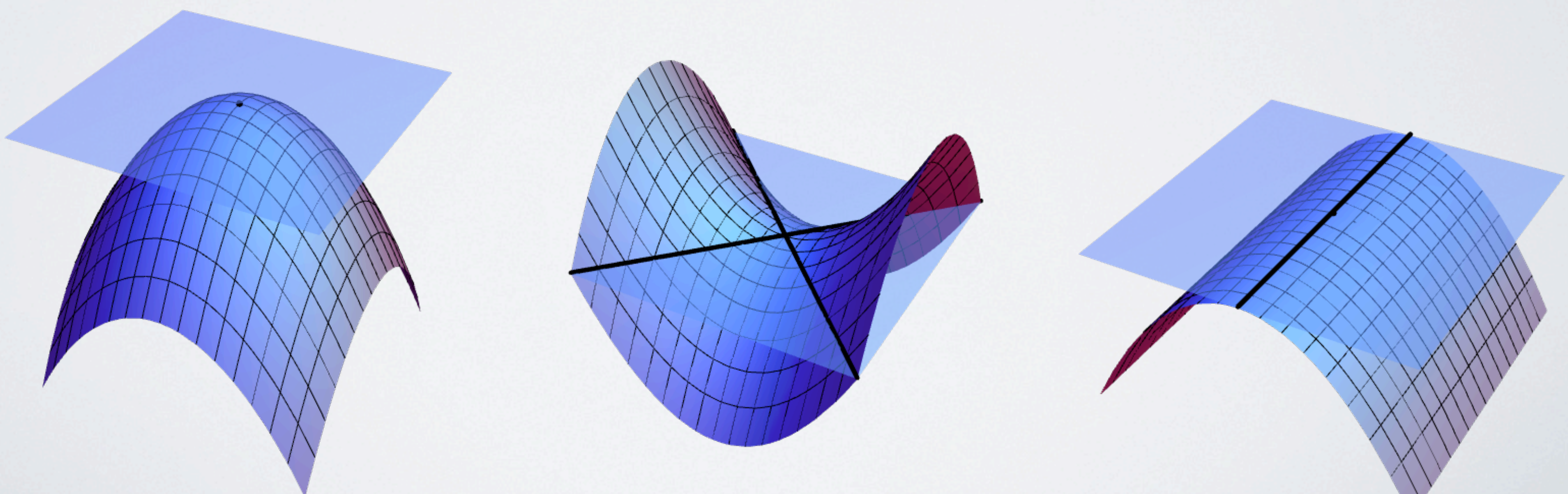


$$\mathbf{P}(\mathbf{u}) = \mathbf{c}_0 + \mathbf{c}_1 u + \mathbf{c}_2 v + \mathbf{c}_3 u^2 + \mathbf{c}_4 uv + \mathbf{c}_5 v^2$$

Nature of Surface

$$\mathbf{P}(\mathbf{u}) = \mathbf{c}_0 + \mathbf{c}_1 u + \mathbf{c}_2 v + \mathbf{c}_3 u^2 + \mathbf{c}_4 uv + \mathbf{c}_5 v^2$$

Elliptic	Hyperbolic	Parabolic
$\mathbf{c}_3\mathbf{c}_4 - (\mathbf{c}_5/2)^2 > 0$	$\mathbf{c}_3\mathbf{c}_4 - (\mathbf{c}_5/2)^2 < 0$	$\mathbf{c}_3\mathbf{c}_4 - (\mathbf{c}_5/2)^2 = 0$
		Includes planar case



Normal Curvature

- Curvature adjusted for surface metric and for velocity in parameter space:

$$\kappa = \frac{\mathrm{d}\mathbf{u}^{\mathsf{T}} \cdot \mathbf{II} \cdot \mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{u}^{\mathsf{T}} \cdot \mathbf{I} \cdot \mathrm{d}\mathbf{u}}$$

Normal Curvature

$$\kappa = \frac{\mathrm{d}\mathbf{u}^{\mathsf{T}} \cdot \mathbf{II} \cdot \mathrm{d}\mathbf{u}}{\mathrm{d}\mathbf{u}^{\mathsf{T}} \cdot \mathbf{I} \cdot \mathrm{d}\mathbf{u}}$$

Recall

$$\mathbf{I} = \mathbf{R}\mathbf{S}^2\mathbf{R}^{\mathsf{T}} = \mathbf{A}\mathbf{A}^{\mathsf{T}}$$

$$\mathrm{d}\mathbf{u} = \mathbf{R}(1/\mathbf{S})\mathrm{d}\mathbf{u}' = \mathbf{A}^{-\mathsf{T}}\mathrm{d}\mathbf{u}'$$

$$\kappa\, \mathrm{d}\mathbf{u}^{\mathsf{T}} \cdot \mathbf{I} \cdot \mathrm{d}\mathbf{u} = \mathrm{d}\mathbf{u}^{\mathsf{T}} \cdot \mathbf{II} \cdot \mathrm{d}\mathbf{u}$$

$$\kappa\, \mathrm{d}\mathbf{u}'^{\mathsf{T}} \cdot \mathbf{A}^{-1} \cdot \mathbf{I} \cdot \mathbf{A}^{-\mathsf{T}} \cdot \mathrm{d}\mathbf{u}' = \mathrm{d}\mathbf{u}'^{\mathsf{T}} \cdot \mathbf{A}^{-1} \cdot \mathbf{II} \cdot \mathbf{A}^{-\mathsf{T}} \cdot \mathrm{d}\mathbf{u}'$$

$$\kappa\, \mathrm{d}\mathbf{u}'^{\mathsf{T}} \cdot \mathrm{d}\mathbf{u}' = \mathrm{d}\mathbf{u}'^{\mathsf{T}} \cdot \mathbf{A}^{-1} \cdot \mathbf{II} \cdot \mathbf{A}^{-\mathsf{T}} \cdot \mathrm{d}\mathbf{u}'$$

$$\kappa = \frac{\mathrm{d}\mathbf{u}'^{\mathsf{T}} \cdot \mathbf{A}^{-1} \cdot \mathbf{II} \cdot \mathbf{A}^{-\mathsf{T}} \cdot \mathrm{d}\mathbf{u}'}{||\mathrm{d}\mathbf{u}'||}$$

Principal Curvatures

$$\kappa = \frac{\mathrm{d}\mathbf{u}'^{\mathsf{T}} \cdot \mathbf{II}' \cdot \mathrm{d}\mathbf{u}'}{||\mathrm{d}\mathbf{u}'||} \qquad \mathbf{II}' = \mathbf{A}^{-1} \cdot \mathbf{II} \cdot \mathbf{A}^{-\mathsf{T}}$$

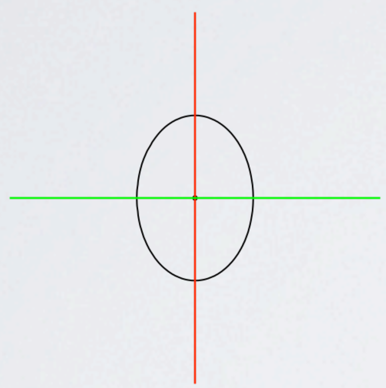
- Dot product projects away "twisting" curvature
- Eigenvectors are where there is nothing to project away
  - Notice that it's a real and symmetric matrix

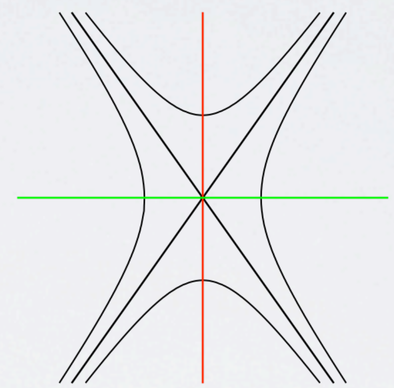
$$\mathbf{II}' \cdot \mathbf{v} = \kappa \, \mathbf{v}$$

Principal Curvatures

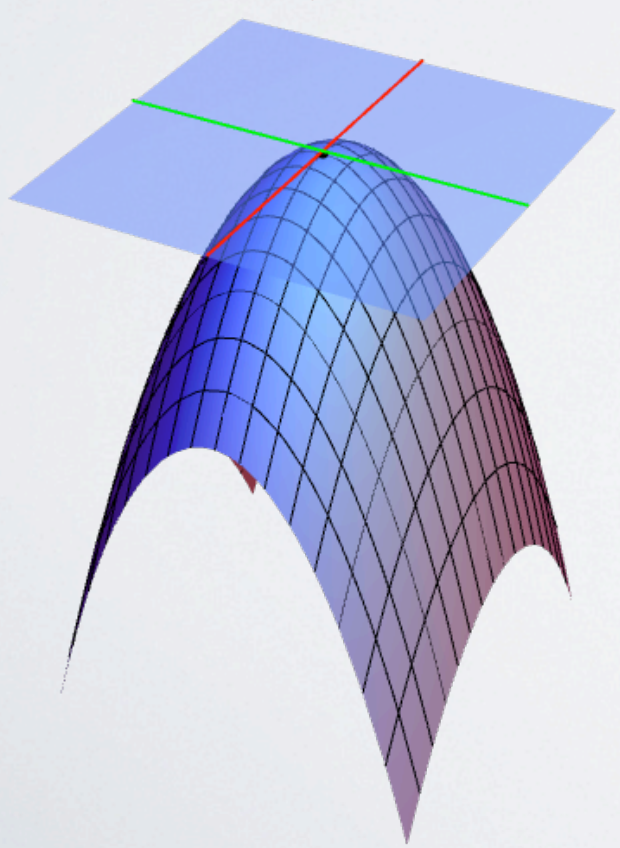
$$\mathbf{II}' \cdot \mathbf{v} = \kappa \, \mathbf{v}$$

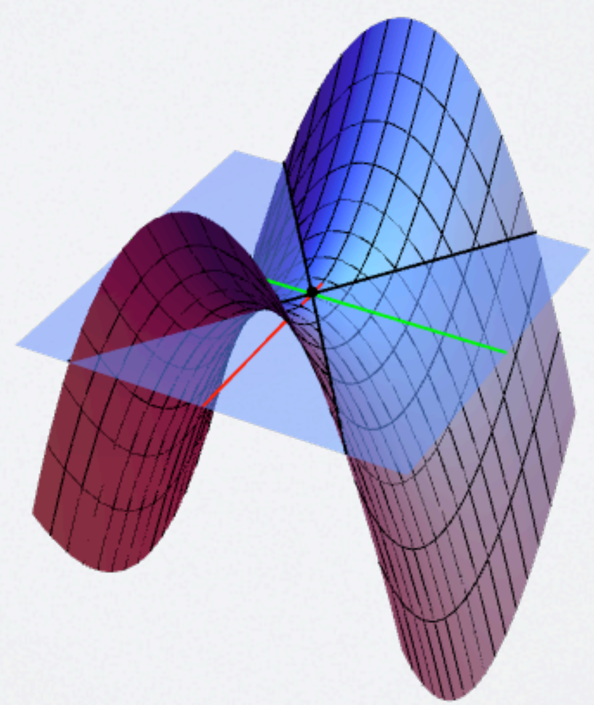
Elliptic	Hyperbolic	Parabolic
$\kappa_1 \kappa_2 > 0$	$\kappa_1 \kappa_2 < 0$	$\kappa_1 \kappa_2 = 0$

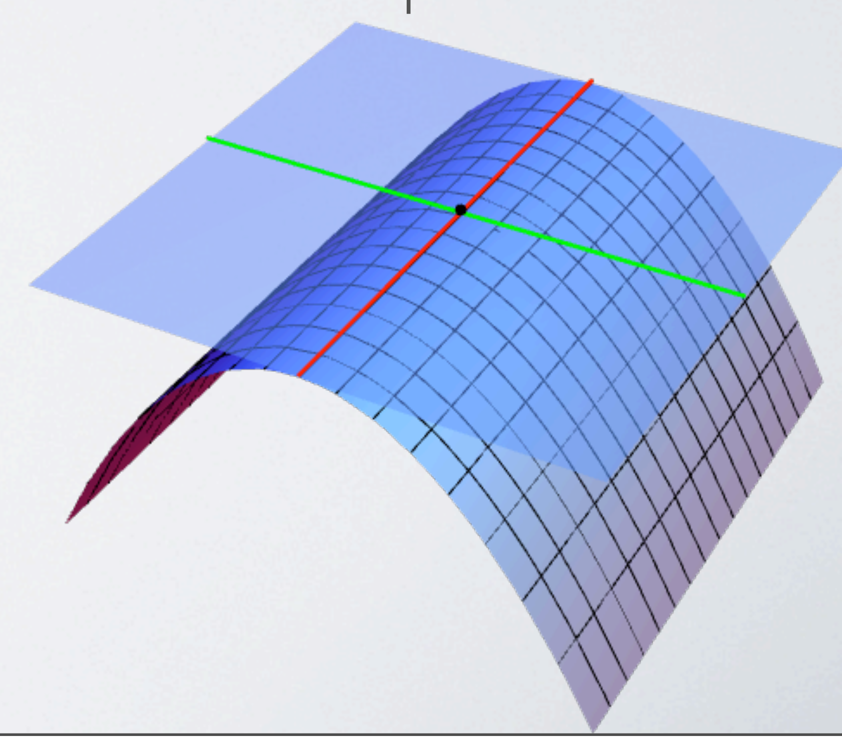




Includes planar case







Weingarten Operator

$$\begin{aligned}
\mathbf{W} &= \mathbf{I}^{-1} \cdot \mathbf{II} \\
&= \mathbf{A}^{-\mathsf{T}} \cdot \mathbf{A}^{-1} \cdot \mathbf{II} \\
&= \mathbf{A}^{-\mathsf{T}} \cdot \mathbf{A}^{-1} \cdot \mathbf{A} \cdot \mathbf{II}' \cdot \mathbf{A}^{\mathsf{T}} \\
&= \mathbf{A}^{-\mathsf{T}} \cdot \mathbf{II}' \cdot \mathbf{A}^{\mathsf{T}}
\end{aligned}$$

If κ and $\mathbf{u}'$ are an eigenvalue/vector pair of $\mathbf{II}'$

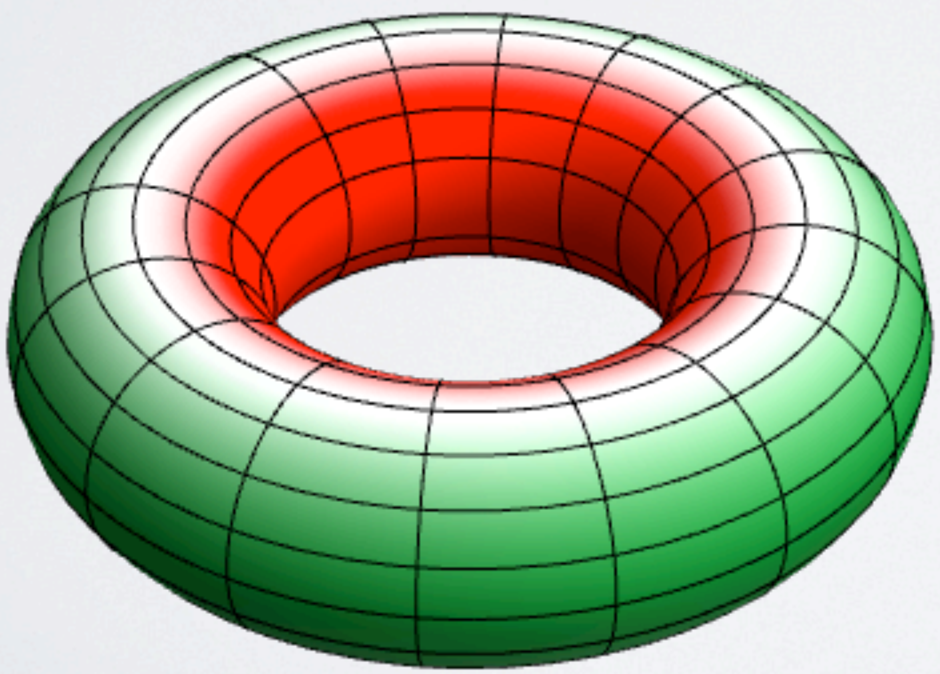
Then $\mathbf{u} = \mathbf{A}^{-\mathsf{T}}\mathbf{u}'$ is an eigenvector of $\mathbf{W}$ with the eigenvalue κ

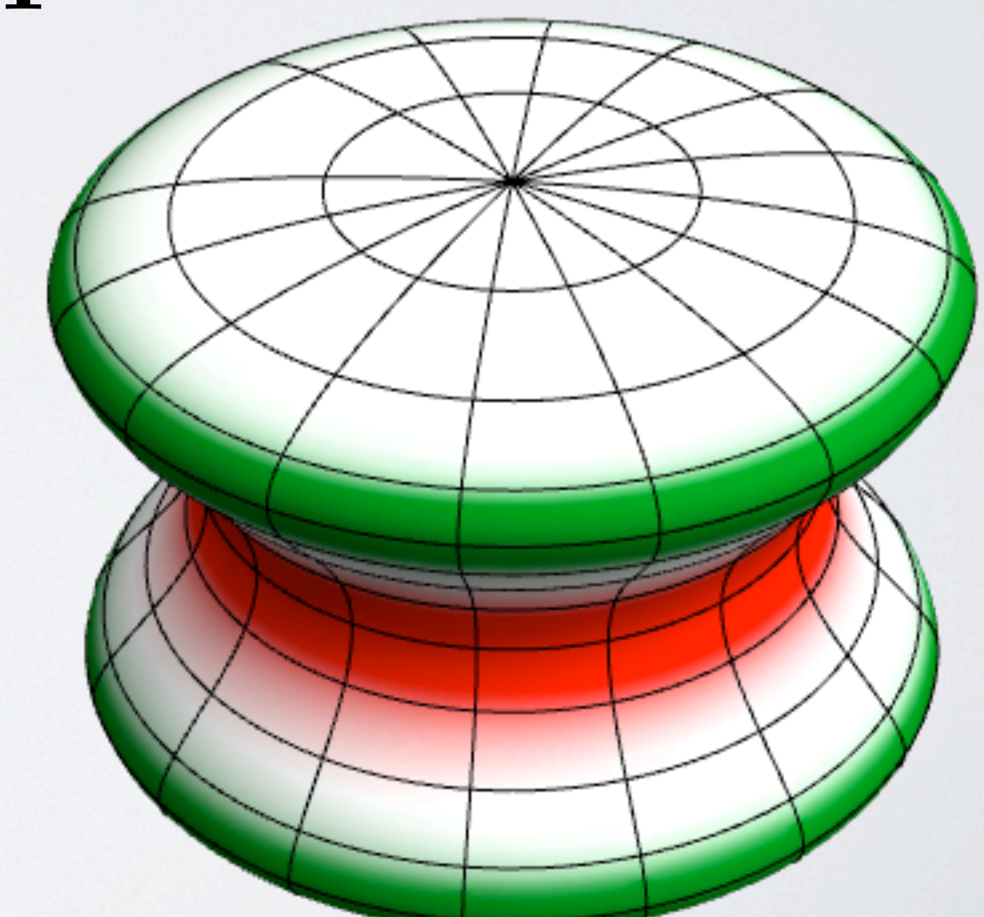
The eigenvectors are expressed in the original parameterization

Gaussian curvature

- Measure of intrinsic flatness of the surface
  - Imagine flat-landers computing π on the surface

$$K = \kappa_1 \, \kappa_2 = \det \mathbf{W} = \frac{\det \mathbf{II}}{\det \mathbf{I}}$$

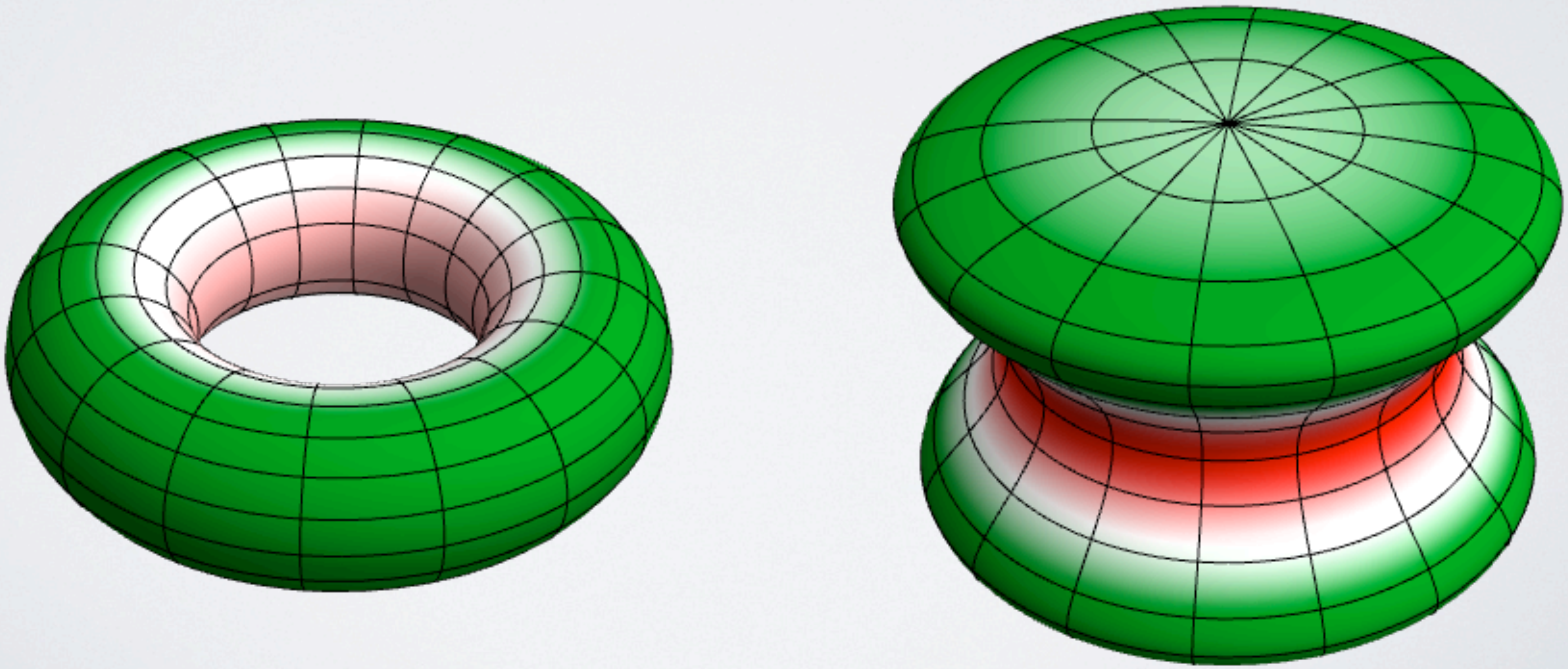




Mean curvature

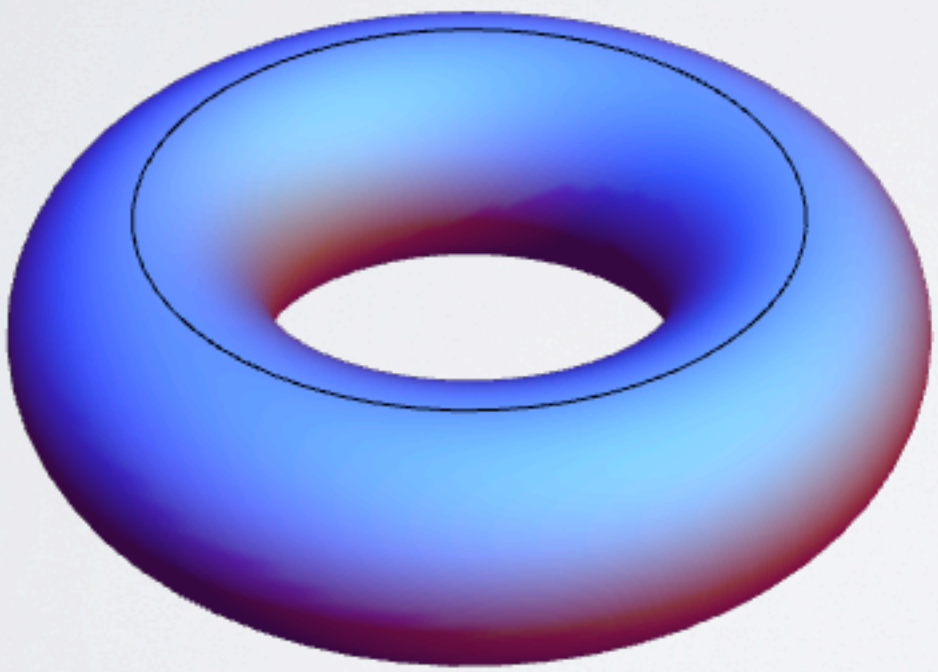
- Average curvature of the surface
  - Will be zero for minimal surfaces

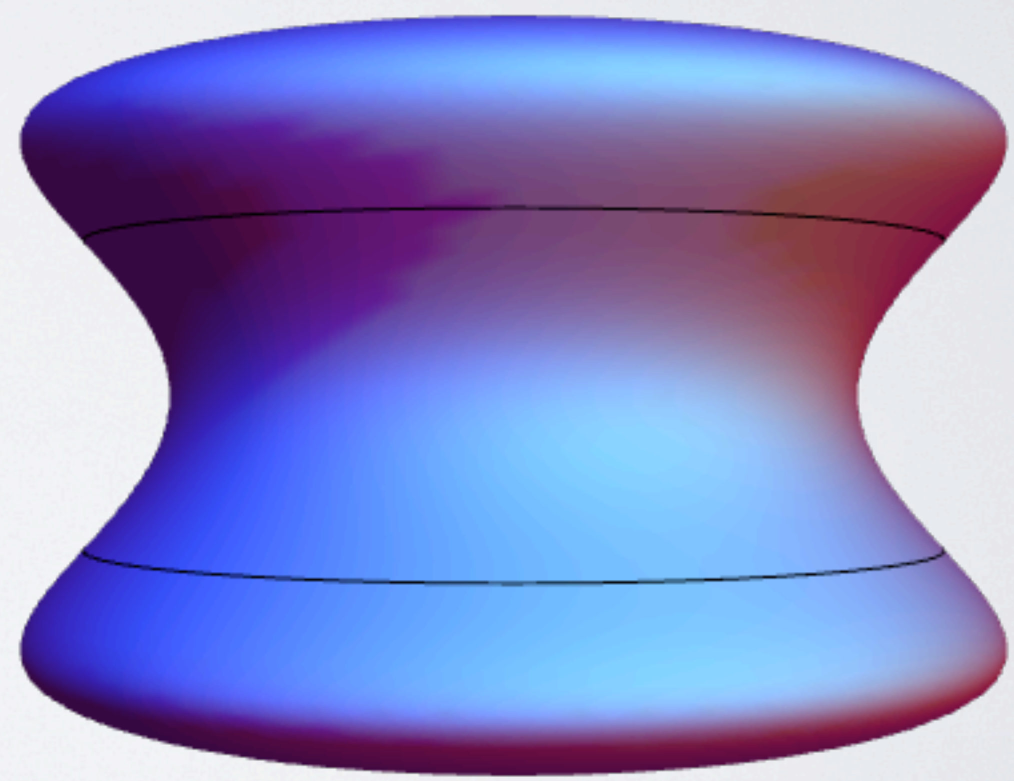
$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{\mathrm{Tr}\left(\mathbf{I} \cdot \mathbf{II}^*\right)}{2 \det \mathbf{I}}$$



Parabolic Lines

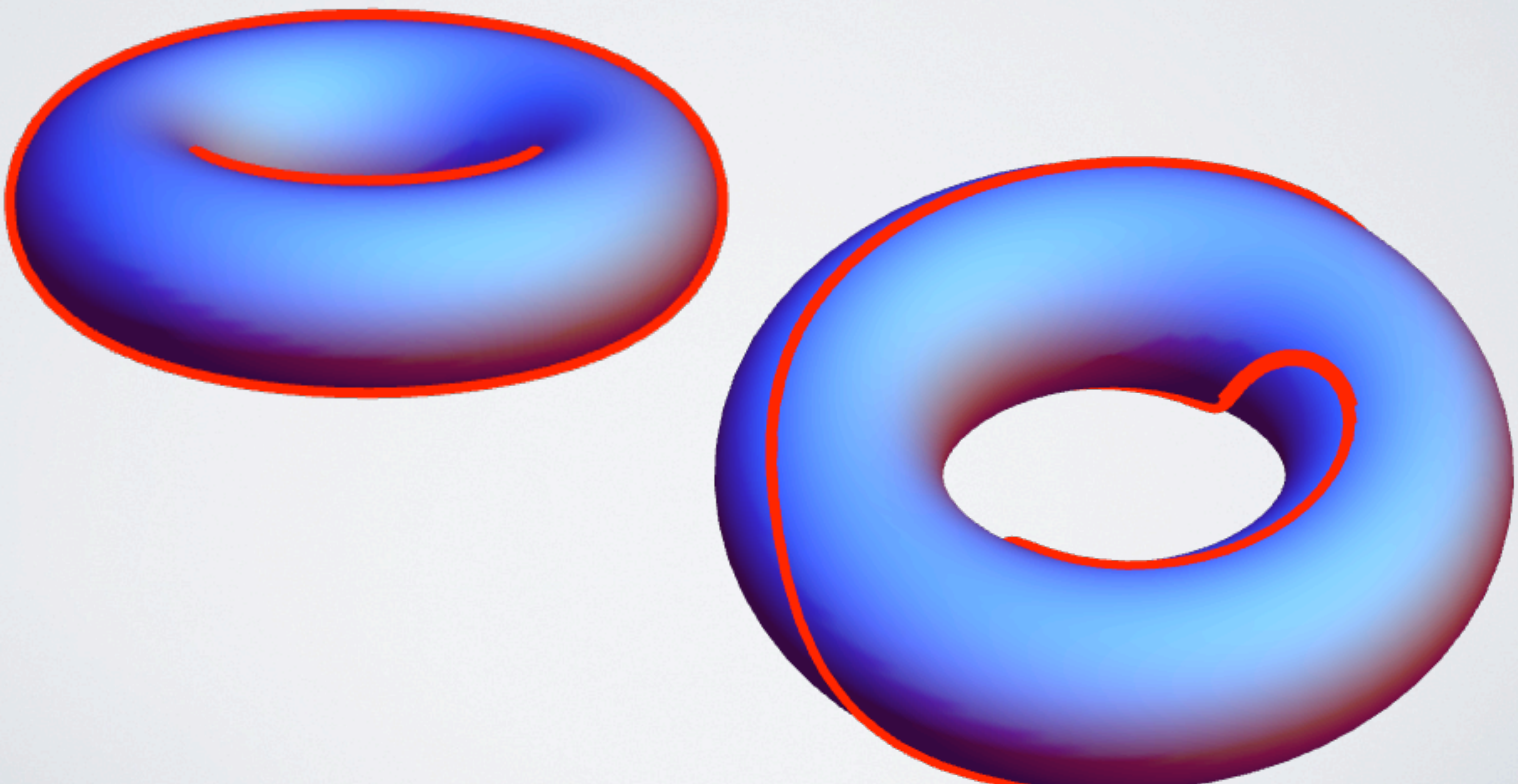
- Curves on surface where Gaussian curvature is zero





Contours

- Surface normal perpendicular to view direction
  - Generator curve: $f(u,v) = (\partial_u \mathbf{S}(u,v) \times \partial_v \mathbf{S}(u,v)).\mathbf{v} = 0$



Contours

- Surface normal perpendicular to view direction
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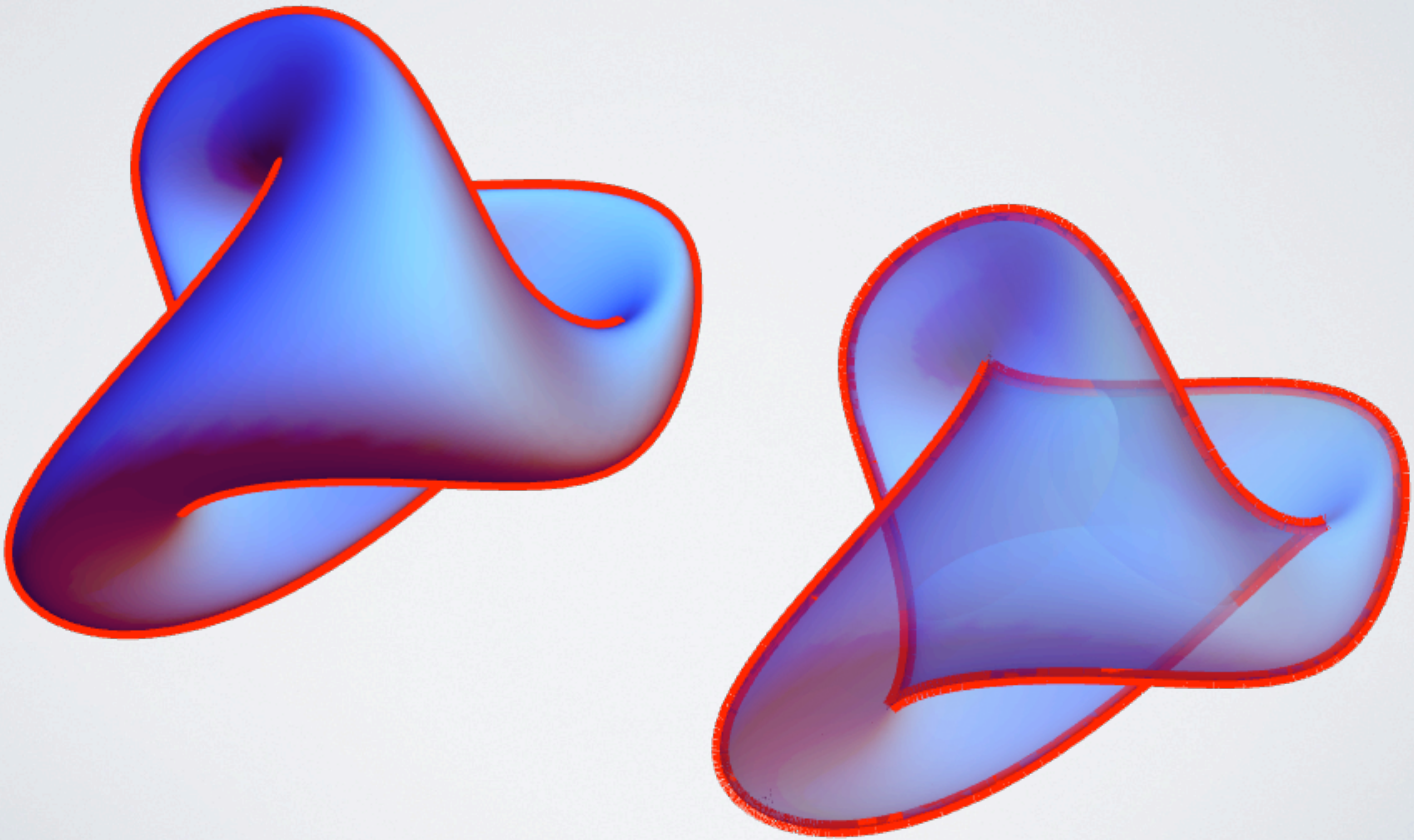


Geodesic Curves

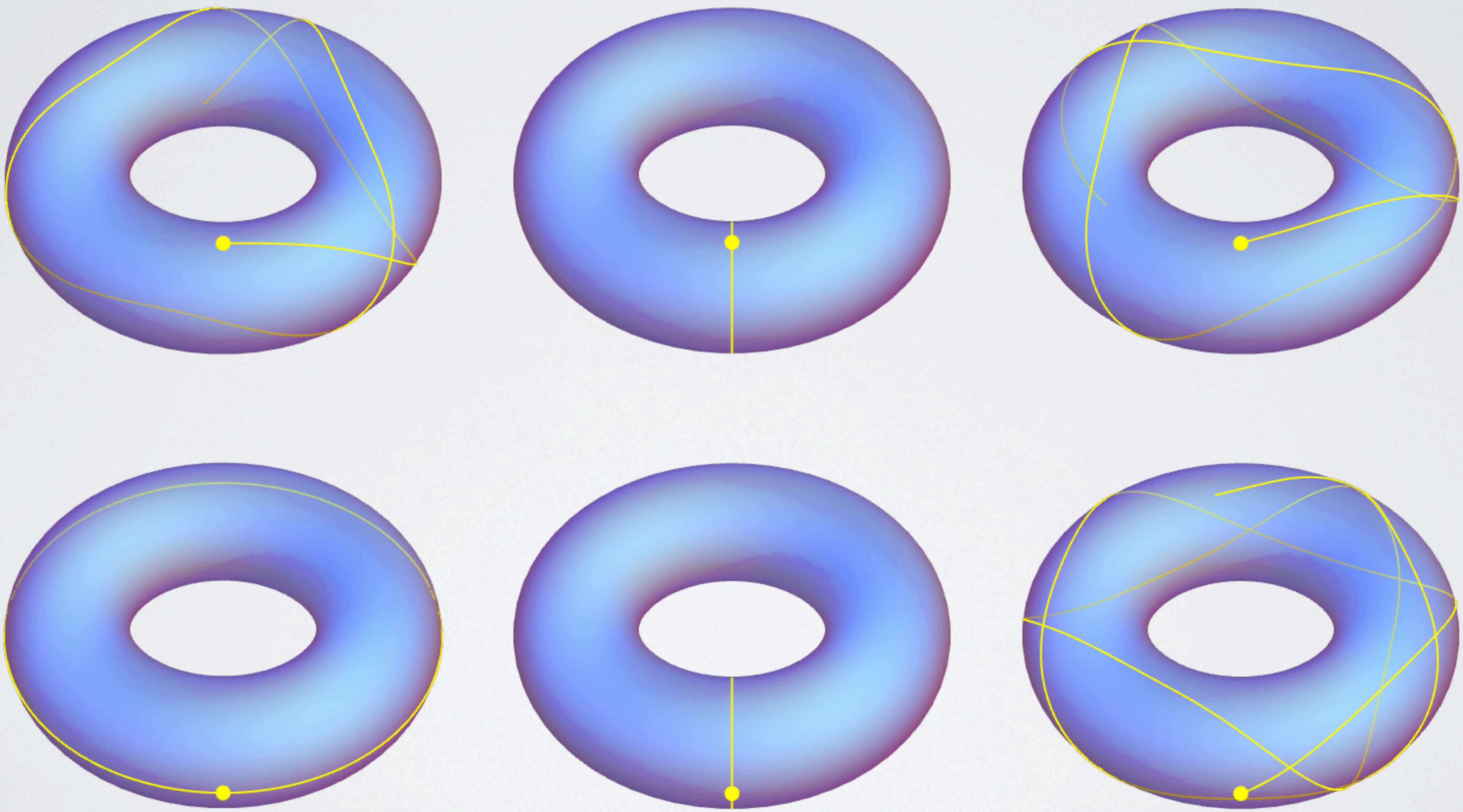
- Given a curve, $\mathbf{C}$, on a surface, $\mathbf{S}$
  - $\mathbf{C}(t) = \mathbf{S}(u(t), v(t))$
- The *geodesic curvature* is
  - $\kappa^2 = \kappa_g^2 + \kappa_n^2$
  - $\kappa_n = \kappa(\hat{\mathbf{N}}_s \cdot \hat{\mathbf{N}}_c)$
- Separates curvature into
  - What's necessary to stay on surface
  - What's wiggling in tangent plane
- *Geodesics* are curves with $\kappa_g = 0$
  - Generalize straight lines
  - <u>Locally</u> shortest path between points
  - On a circle they are great arcs

$$\frac{\mathrm{d}^2\mathbf{C}}{\mathrm{d}t^2} \cdot \frac{\partial \mathbf{S}}{\partial u_i} = 0 \;\; \forall i$$

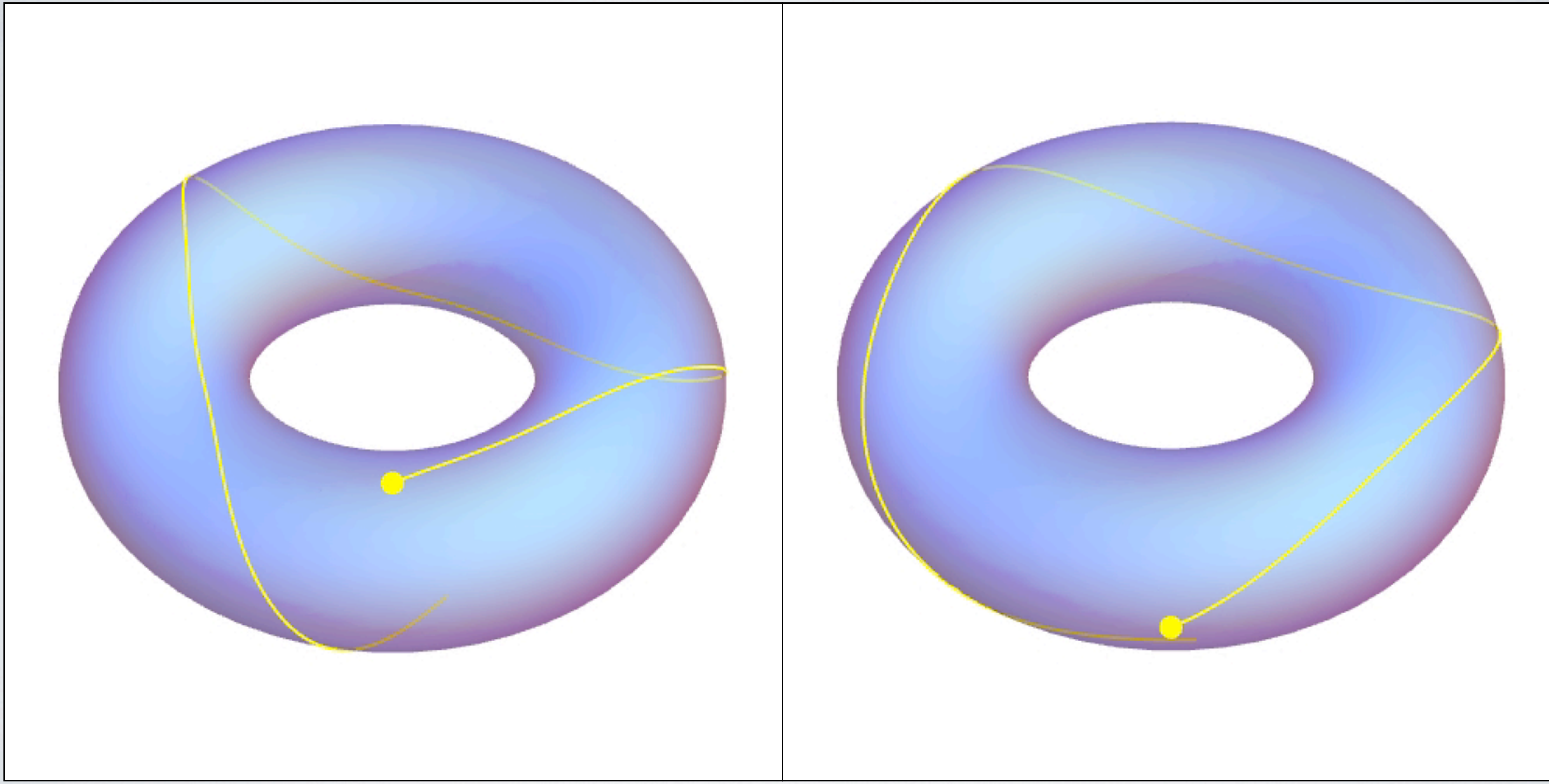
ODE for curve

$$\ddot{u}_q = (\mathbf{I}^{-1})_{qp} \frac{\partial S_k}{\partial u_p} \frac{\partial^2 S_k}{\partial u_i \partial u_j} \dot{u}_i \dot{u}_j$$

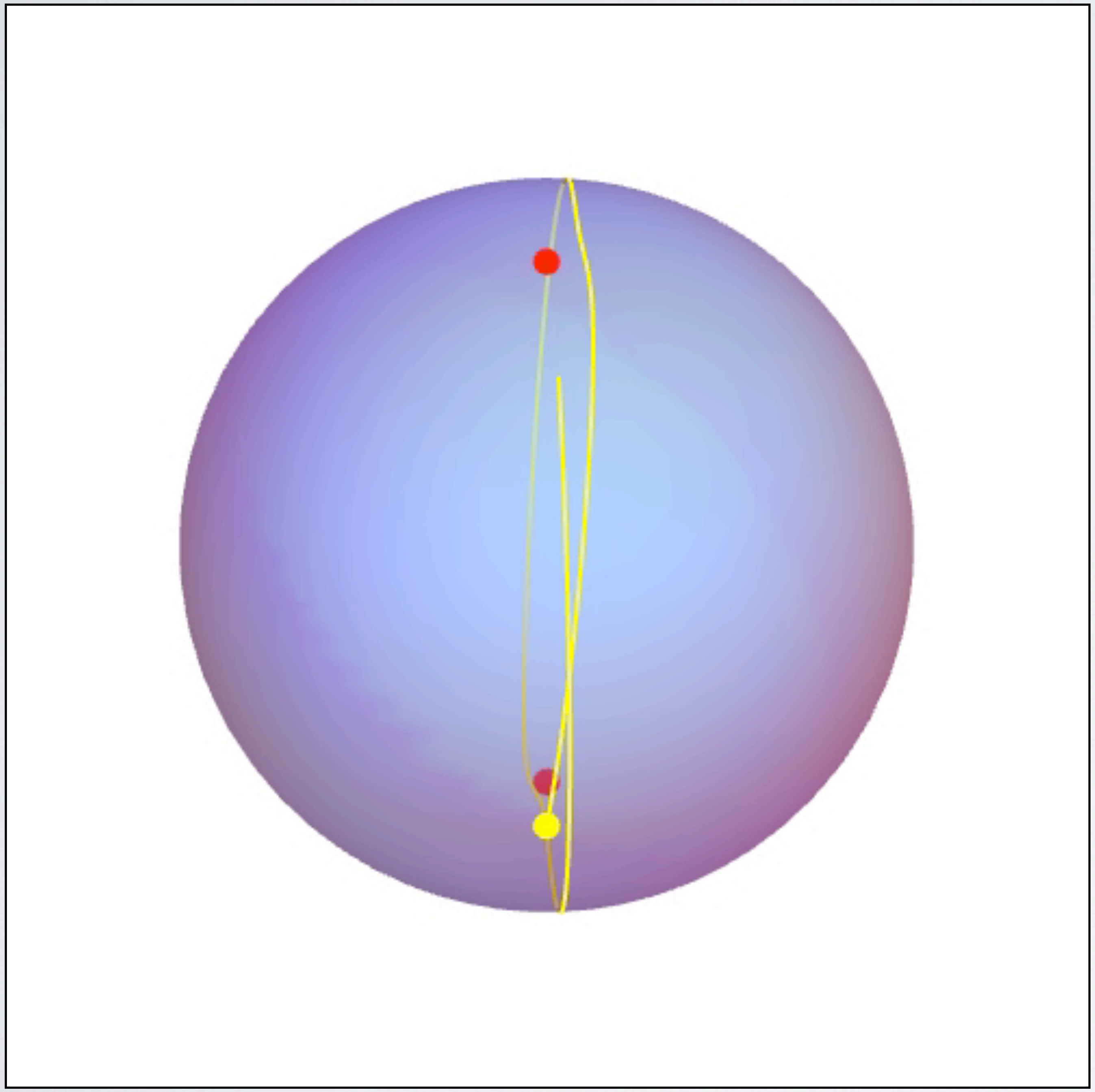
Geodesic Curves



Geodesic Curves

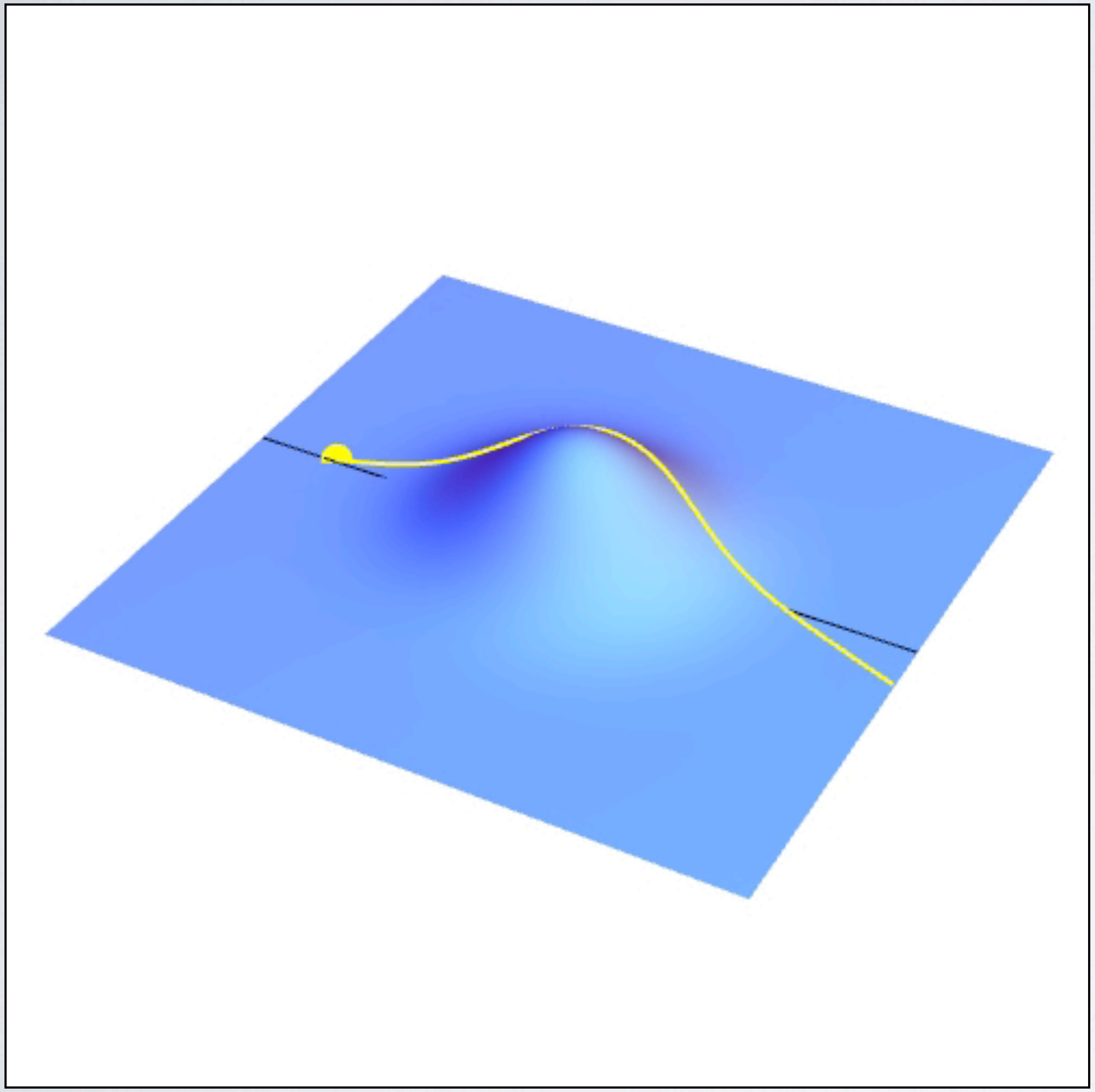


Geodesic Curves

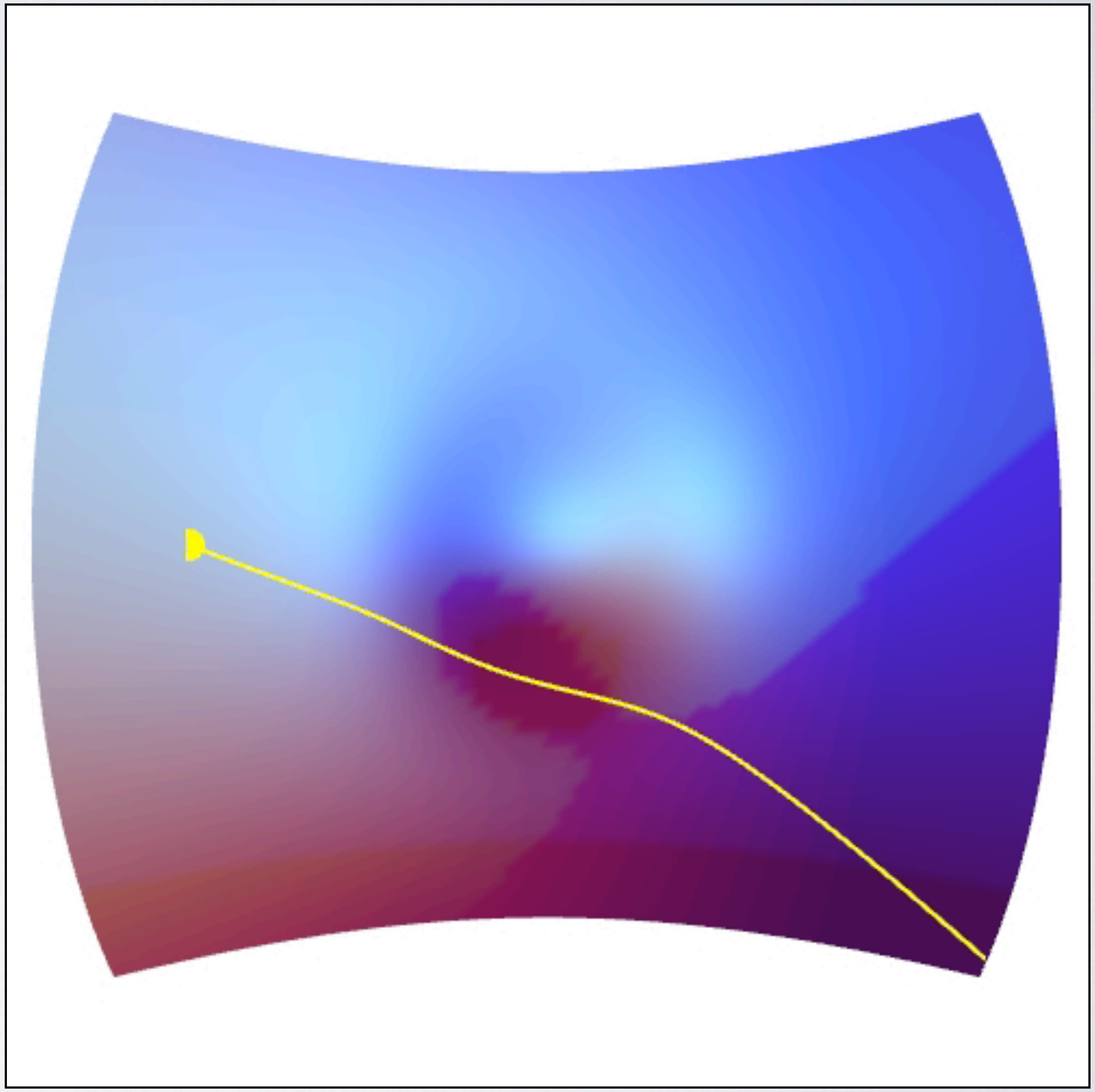


Note integration errors when passing near poles.

Geodesic Curves



Flat w/ bump



Hyperbolic w/ bump

Lines of Curvature

- A line of curvature on a surface is tangent everywhere to one of the principal curvatures
    - Except at umbilic points where the two principal curvatures are equal

Need to check: lines of curvature geodesic?

Implicit Surfaces

See 2005 paper by Ron Goldman

$$\{\mathbf{x} | f(\mathbf{x}) = 0\}$$

$$\mathbf{N}(\mathbf{x}) = \frac{\boldsymbol{\nabla} f}{||\boldsymbol{\nabla} f||}$$

$$K_G = \frac{\boldsymbol{\nabla} f \cdot (\boldsymbol{\nabla}\boldsymbol{\nabla}^{\mathsf{T}} f)^* \cdot \boldsymbol{\nabla} f}{||\boldsymbol{\nabla} f||^4}$$

$$K_M = \frac{\boldsymbol{\nabla} f \cdot (\boldsymbol{\nabla}\boldsymbol{\nabla}^{\mathsf{T}} f) \cdot \boldsymbol{\nabla} f - ||\boldsymbol{\nabla} f||^2 \mathrm{Tr}(\boldsymbol{\nabla}\boldsymbol{\nabla}^{\mathsf{T}} f)}{2||\boldsymbol{\nabla} f||^3}$$

$$\kappa_{1|2} = K_M \pm \sqrt{K_M^2 - K_G}$$