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CS61C : Machine Structures

Lecture #11: Floating Point

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Quote of the day

"95% of the folks out there are completely clueless about floating-point."

James Gosling
Sun Fellow
Java Inventor
1998-02-28

Review of Numbers

- Computers are made to deal with numbers
- What can we represent in N bits?
  - Unsigned integers:
    - 0 to $2^N - 1$
  - Signed Integers (Two's Complement)
    - $-2^{(N-1)}$ to $2^{(N-1)} - 1$

Other Numbers

- What about other numbers?
  - Very large numbers? (seconds/century)
    - $3{,}155{,}760{,}000_{10}$ ($3.15576_{10} \times 10^9$)
  - Very small numbers? (atomic diameter)
    - 0.00000001_{10} ($1.0_{10} \times 10^{-8}$)
  - Rationals (repeating pattern)
    - 2/3 (0.666666666. . .)
  - Irrationals
    - $2^{1/2}$ (1.414213562373. . .)
  - Transcendentals
    - *e* (2.718...), π (3.141...)
- All represented in scientific notation

Scientific Notation (in Decimal)

mantissa → $6.02_{10} \times 10^{23}$ ← exponent

decimal point   radix (base)

- Normalized form: no leadings 0s (exactly one digit to left of decimal point)
- Alternatives to representing 1/1,000,000,000
  - Normalized: 1.0×10^{-9}
  - Not normalized: 0.1×10^{-8}, 10.0×10^{-10}

Scientific Notation (in Binary)

mantissa → $1.0_{two} \times 2^{-1}$ ← exponent

"binary point"   radix (base)

- Normalized mantissa always has exactly one "1" before the point.
- Computer arithmetic that supports it called floating point, because it represents numbers where binary point is not fixed, as it is for integers
- Declare such variable in C as `float`

Floating Point Representation (1/2)

- Normal format: +1.xxxxxxxxxx$_{two}$*2^{yyyy}$_{two}$
- Multiple of Word Size (32 bits):

31	30 … 23	22 … 0
S	Exponent	Significand
1 bit	8 bits	23 bits

- S represents Sign
- Exponent represents y's
- Significand represents x's

Represent numbers as small as 2.0×10^{-38} to as large as 2.0×10^{38}

Floating Point Representation (2/2)

- What if result too large? (> 2.0×10^{38})
  - Overflow!
  - Overflow ⇒ Exponent larger than represented in 8-bit Exponent field
- What if result too small? (>0, < 2.0×10^{-38})
  - Underflow!
  - Underflow ⇒ Negative exponent larger than represented in 8-bit Exponent field
- How to reduce chances of overflow or underflow?

Double Precision Fl. Pt. Representation

- Next Multiple of Word Size (64 bits)

31	30 … 20	19 … 0
S	Exponent	Significand
1 bit	11 bits	20 bits
Significand (cont'd)		
32 bits		

- Double Precision (vs. Single Precision)
  - C variable declared as `double`
  - Represent numbers almost as small as 2.0×10^{-308} to almost as large as 2.0×10^{308}
  - But primary advantage is greater accuracy due to larger significand

QUAD Precision Fl. Pt. Representation

- Next Multiple of Word Size (128 bits)
- Unbelievable range of numbers
- Unbelievable precision (accuracy)
- This is currently being worked on
- The version in progress has 15 bits for the exponent and 112 bits for the significand

IEEE 754 Floating Point Standard (1/4)

- Single Precision, DP similar
- Sign bit: 1 means negative
  0 means positive
- Significand:
  - To pack more bits, leading 1 implicit for normalized numbers
  - 1 + 23 bits single, 1 + 52 bits double
- Note: 0 has no leading 1, so reserve exponent value 0 just for number 0

IEEE 754 Floating Point Standard (2/4)

- Kahan wanted FP numbers to be used even if no FP hardware; e.g., sort records with FP numbers using integer compares
- Could break FP number into 3 parts: compare signs, then compare exponents, then compare significands
- Wanted it to be faster, single compare if possible, especially if positive numbers
- Then want order:
  - Highest order bit is sign (negative < positive)
  - Exponent next, so big exponent => bigger #
  - Significand last: exponents same => bigger #

IEEE 754 Floating Point Standard (3/4)

- Negative Exponent?
  - 2's comp? 1.0 x 2^{-1} v. 1.0 x2^{+1} (1/2 v. 2)

	S	Exponent	Significand
1/2	0	1111 1111	000 0000 0000 0000 0000 0000
2	0	0000 0001	000 0000 0000 0000 0000 0000

  - This notation using integer compare of 1/2 v. 2 makes 1/2 > 2!
- Instead, pick notation 0000 0001 is most negative, and 1111 1111 is most positive
  - 1.0 x 2^{-1} v. 1.0 x2^{+1} (1/2 v. 2)

	S	Exponent	Significand
1/2	0	0111 1110	000 0000 0000 0000 0000 0000
2	0	1000 0000	000 0000 0000 0000 0000 0000

IEEE 754 Floating Point Standard (4/4)

- Called **Biased Notation**, where bias is number subtracted to get real number
  - IEEE 754 uses bias of 127 for single prec.
  - Subtract 127 from Exponent field to get actual value for exponent
- Summary (single precision):

31	30 … 23	22 … 0
S	Exponent	Significand
1 bit	8 bits	23 bits

- $(-1)^S$ x (1 + Significand) x $2^{(\text{Exponent}-127)}$
  - Double precision identical, except with exponent bias of 1023

?

S	Exponent	Significand
0	0111 1101	0000 0000 0000 0000 0000 000

Is this floating point number:

> 0?

= 0?

< 0?

Understanding the Significand (1/2)

- Method 1 (Fractions):
  - In decimal: 0.340_{10} => $340_{10}/1000_{10}$ => $34_{10}/100_{10}$
  - In binary: 0.110_2 => $110_2/1000_2 = 6_{10}/8_{10}$ => $11_2/100_2 = 3_{10}/4_{10}$
  - Advantage: less purely numerical, more thought oriented; this method usually helps people understand the meaning of the significand better

Understanding the Significand (2/2)

- Method 2 (Place Values):
  - Convert from scientific notation
  - In decimal: 1.6732 = $(1\text{x}10^0)$ + $(6\text{x}10^{-1})$ + $(7\text{x}10^{-2})$ + $(3\text{x}10^{-3})$ + $(2\text{x}10^{-4})$
  - In binary: 1.1001 = $(1\text{x}2^0)$ + $(1\text{x}2^{-1})$ + $(0\text{x}2^{-2})$ + $(0\text{x}2^{-3})$ + $(1\text{x}2^{-4})$
  - Interpretation of value in each position extends beyond the decimal/binary point
  - Advantage: good for quickly calculating significand value; use this method for translating FP numbers

Example: Converting Binary FP to Decimal

S	Exponent	Significand
0	0110 1000	101 0101 0100 0011 0100 0010

- Sign: 0 => positive
- Exponent:
  - 0110 1000_{two} = 104_{ten}
  - Bias adjustment: 104 - 127 = -23
- Significand:
  - 1 + $1\text{x}2^{-1}$+ $0\text{x}2^{-2}$ + $1\text{x}2^{-3}$ + $0\text{x}2^{-4}$ + $1\text{x}2^{-5}$ +... $=1+2^{-1}+2^{-3}+2^{-5}+2^{-7}+2^{-9}+2^{-14}+2^{-15}+2^{-17}+2^{-22}$ = 1.0_{ten} + 0.666115_{ten}
- Represents: $1.666115_{ten}*2^{-23}$ ~ $1.986*10^{-7}$ (about 2/10,000,000)

Peer Instruction #1

What is the decimal equivalent of this floating point number?

1	1000 0001	111 0000 0000 0000 0000 0000

Answer

What is the decimal equivalent of:

S	Exponent	Significand
1	1000 0001	111 0000 0000 0000 0000 0000

$(-1)^S \times (1 + \text{Significand}) \times 2^{(\text{Exponent}-127)}$

$(-1)^1 \times (1 + .111) \times 2^{(129-127)}$

$-1 \times (1.111) \times 2^{(2)}$

-111.1

-7.5

```
1: -1.75
2: -3.5
3: -3.75
4: -7
5: -7.5
6: -15
7: -7 * 2^129
8: -129 * 2^7
```


Converting Decimal to FP (1/3)

- Simple Case: If denominator is an exponent of 2 (2, 4, 8, 16, etc.), then it's easy.
- Show MIPS representation of -0.75
  - -0.75 = -3/4
  - $-11_{two}/100_{two} = -0.11_{two}$
  - Normalized to $-1.1_{two} \times 2^{-1}$
  - $(-1)^S \times (1 + \text{Significand}) \times 2^{(\text{Exponent}-127)}$
  - $(-1)^1 \times (1 + .100\ 0000\ ...\ 0000) \times 2^{(126-127)}$

1	0111 1110	100 0000 0000 0000 0000 0000

Converting Decimal to FP (2/3)

- Not So Simple Case: If denominator is not an exponent of 2.
  - Then we can't represent number precisely, but that's why we have so many bits in significand: for precision
  - Once we have significand, normalizing a number to get the exponent is easy.
  - So how do we get the significand of a never-ending number?

Converting Decimal to FP (3/3)

- Fact: All rational numbers have a repeating pattern when written out in decimal.
- Fact: This still applies in binary.
- To finish conversion:
  - Write out binary number with repeating pattern.
  - Cut it off after correct number of bits (different for single v. double precision).
  - Derive Sign, Exponent and Significand fields.

Example: Representing 1/3 in MIPS

- 1/3
  - $= 0.33333..._{10}$
  - = 0.25 + 0.0625 + 0.015625 + 0.00390625 + …
  - = 1/4 + 1/16 + 1/64 + 1/256 + …
  - $= 2^{-2} + 2^{-4} + 2^{-6} + 2^{-8} + \ldots$
  - $= 0.0101010101\ldots_2 * 2^0$
  - $= 1.0101010101\ldots_2 * 2^{-2}$
  - Sign: 0
  - Exponent = -2 + 127 = 125 = 01111101
  - Significand = 0101010101…

0	0111 1101	0101 0101 0101 0101 0101 010

Administrivia

- Project 1 Due Tonight
- HW4 will be out soon
- Midterm Results (out of 45 possible):
  - Average: 31.25
  - Standard Deviation: 8
  - Median: 30.875
  - Max: 44
  - Will be handed back in discussion

"Father" of the Floating point standard

IEEE Standard 754 for Binary Floating-Point Arithmetic.

1989 ACM Turing Award Winner!

Prof. Kahan

`www.cs.berkeley.edu/~wkahan/`
`…/ieee754status/754story.html`

Representation for ± ∞

- In FP, divide by 0 should produce ± ∞, not overflow.
- Why?
  - OK to do further computations with ∞ E.g., X/0 > Y may be a valid comparison
  - Ask math majors
- IEEE 754 represents ± ∞
  - Most positive exponent reserved for ∞
  - Significands all zeroes

Representation for 0

- Represent 0?
  - exponent all zeroes
  - significand all zeroes
  - What about sign?
    - `+0: 0 00000000 00000000000000000000000`
    - `-0: 1 00000000 00000000000000000000000`
- Why two zeroes?
  - Helps in some limit comparisons
  - Ask math majors

Special Numbers

- What have we defined so far? (Single Precision)

Exponent	Significand	Object
0	0	0
0	nonzero	???
1-254	anything	+/- fl. pt. #
255	0	+/- ∞
255	nonzero	???

- Professor Kahan had clever ideas; "Waste not, want not"
  - Exp=0,255 & Sig!=0 …

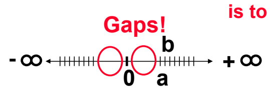
Representation for Not a Number

- What is `sqrt(-4.0)` or `0/0`?
  - If ∞ not an error, these shouldn't be either.
  - Called Not a Number (NaN)
  - Exponent = 255, Significand nonzero
- Why is this useful?
  - Hope NaNs help with debugging?
  - They contaminate: op(NaN,X) = NaN

Representation for Denorms (1/2)

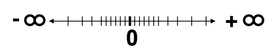
- Problem: There's a gap among representable FP numbers around 0
  - Smallest representable pos num:
    $a = 1.0\ldots_2 * 2^{-126} = 2^{-126}$
  - Second smallest representable pos num:
    $b = 1.000\ldots\ldots1_2 * 2^{-126} = 2^{-126} + 2^{-149}$
  - $a - 0 = 2^{-126}$
  - $b - a = 2^{-149}$

Normalization and implicit 1 is to blame!



Representation for Denorms (2/2)

- Solution:
  - We still haven't used Exponent = 0, Significand nonzero
  - Denormalized number: no leading 1, implicit exponent = -126.
  - Smallest representable pos num:
    $a = 2^{-149}$
  - Second smallest representable pos num:
    $b = 2^{-148}$



Peer Instruction 2

1. Converting `float` -> `int` -> `float` produces same `float` number
2. Converting `int` -> `float` -> `int` produces same `int` number
3. FP add is associative: `(x+y)+z = x+(y+z)`

"And in conclusion…"

- Floating Point numbers approximate values that we want to use.
- IEEE 754 Floating Point Standard is most widely accepted attempt to standardize interpretation of such numbers
  - Every desktop or server computer sold since ~1997 follows these conventions
- Summary (single precision):

31	30 … 23	22 … 0
S	Exponent	Significand
1 bit	8 bits	23 bits

- $(-1)^S \times (1 + \text{Significand}) \times 2^{(\text{Exponent}-127)}$
  - Double precision identical, bias of 1023