Introduction to Sets

A **set** is a well defined collection of objects considered as a whole. These objects are called **elements** or **members** of a set, and they can be anything, including numbers, letters, people, cities, and even other sets. By convention, sets are usually denoted by capital letters and can be described or defined by listing its elements and surrounding the list by curly braces. For example, we can describe the set A to be the set whose members are the first five prime numbers, or we can explicitly write: $A = \{2, 3, 5, 7, 11\}$. If x is an element of A, we write $x \in A$. Similarly, if y is not an element of A, then we write $y \notin A$. Two sets A and B are said to be equal, written as $A = B$, if they have the same elements. The order and repetition of elements do not matter, so {red, white, blue} = {blue, white, red} = {red, white, white, blue}. Sometimes, more complicated sets can be defined by using a different notation. For example, the set of all rational numbers denoted by $\mathbb{Q}$ can be written as: $\{\frac{a}{b} \mid a, b \text{ are integers}, b \neq 0\}$. In English, this is read as "the set of all fractions such that the numerator is an integer and the denominator is a non-zero integer."

Cardinality

We can also talk about the size of a set, or its **cardinality**. If $A = \{1,2,3,4\}$, then the cardinality of A, denoted by $|A|$, is 4. It is possible for the cardinality of a set to be 0. This set is called the **empty set**, denoted by the symbol $\emptyset$. A set can also have an infinite number of elements, such as the set of all integers, prime numbers, or odd numbers.

Subsets and Proper Subsets

If every element of a set A is also in a set B, then we say that A is a **subset** of B, written $A \subseteq B$, or A is contained in B. We can also write $B \supseteq A$, meaning that B is a superset of A, or B contains A. A **proper subset** is a set A that is strictly contained in B, written as $A \subset B$, meaning that A excludes at least one element of B. For example, consider the set $B = \{1,2,3,4,5\}$. Then $\{1,2,3\}$ is both a subset and a proper subset of B, while $\{1,2,3,4,5\}$ is a subset but not a proper subset of B. Here are a few basic properties regarding subsets:

- The empty set is a proper subset of any nonempty set A: $\emptyset \subset A$.
- The empty set is a subset of every set B: $\emptyset \subseteq B$.
- Every set A is a subset of itself: $A \subseteq A$.

Intersections and Unions

The **intersection** of a set A with a set B, written as $A \cap B$, is a set of all elements which are members of both A and B. Two sets are said to be **disjoint** if $A \cap B = \emptyset$. The **union** of a set A with a set B, written as $A \cup B$, is a set of all elements which are either members of A or B. For example, if A is the set of all positive even numbers, and B is the set of all positive odd numbers, then $A \cap B = \emptyset$, and $A \cup B = \mathbb{Z}^+$, or the set of all positive integers. Here are a few properties of intersections and unions:

- $A \cup B = B \cup A$

- $A \cup \emptyset = A$
- $A \cap B = B \cap A$
- $A \cap \emptyset = \emptyset$

Complements

If A and B are two sets, then the **relative complement** of A in B, written as $B - A$ or $B \backslash A$, is the set of elements in B, but not in A: $B \backslash A = \{x \in B \mid x \notin A\}$. For example, if $B = \{1,2,3\}$ and $A = \{3,4,5\}$, then $B \backslash A = \{1,2\}$. For another example, if $\mathbb{R}$ is the set of real numbers and $\mathbb{Q}$ is the set of rational numbers, then $\mathbb{R} \backslash \mathbb{Q}$ is the set of irrational numbers. Here are some important properties of complements:

- $A \backslash A = \emptyset$
- $A \backslash \emptyset = A$
- $\emptyset \backslash A = \emptyset$

Significant Sets

In mathematics, some sets are referred to so commonly that they are denoted by special symbols. Some of these numerical sets include:

- $\mathbb{P}$ denotes the set of all prime numbers: $\{2,3,5,7,11,\ldots\}$.
- $\mathbb{N}$ denotes the set of all natural numbers: $\{0,1,2,3,...\}$.
- $\mathbb{Z}$ denotes the set of all integer numbers: $\{\ldots,-2,-1,0,1,2,\ldots\}$.
- $\mathbb{Q}$ denotes the set of all rational numbers: $\{\frac{a}{b} \mid a,b \in \mathbb{Z},\ b \neq 0\}$.
- $\mathbb{R}$ denotes the set of all real numbers.
- $\mathbb{C}$ denotes the set of all complex numbers.

In addition, the **Cartesian product** (also called the **cross product**) of two sets A and B, written as $A \times B$, is the set of all pairs whose first component is an element of A and whose second component is an element of B. In set notation, $A \times B = \{(a,b) \mid a \in A, b \in B\}$. For example, if $A = \{1,2,3\}$ and $B = \{u,v\}$, then $A \times B = \{(1,u),(1,v),(2,u),(2,v),(3,u),(3,v)\}$. Given a set S, another significant set is the **power set** of S, denoted by $\mathscr{P}(S)$, is the set of all subsets of S: $\{T \mid T \subseteq S\}$. For example, if $S = \{1,2,3\}$, then the power set of S is: $\mathscr{P}(S) = \{\{\},\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$. It is interesting to note that, if $|S| = k$, then $|\mathscr{P}(S)| = 2^k$.