EECS 70 Discrete Mathematics and Probability Theory
Spring 2013 Anant Sahai

HW 1

Due January 28

Administrative details of preparing and submitting homework submissions: In CS 70 this semester, you will be submitting all homework solutions online. (You might recognize the submission system from CS 61ABC.) To get ready for online submission, get an EECS instructional computer account if you don't have one already, and register with the grading system. See your GSI or Piazza for instructions on how to do this.

You can typeset your solution using the LaTeX document processing system. In that case, submit both the LaTeX source files (e.g., `hw1.prob1.tex`) as well as the resulting PDF document (e.g., `hw1.prob1.pdf`). You can produce a PDF file from your LaTeX input with the command

```
pdflatex hw1.prob1.tex
```

Alternatively, you can handwrite your solutions and scan them, or use any other software to generate your pdf. (If you use any other software, include your source document the same way that you would the LaTeX source files.

Before submitting, make sure you check carefully that the PDF comes out correctly and correct any errors. We suggest using

```
acroread hw1.prob1.pdf
```

Your submission needs to start with the following information:

Your full name
Your login name
The name of the homework assignment (e.g. hw1)
The number of the problem (e.g. 1)
Your section number
Your list of partners for this homework, or "none" if you had no partners

To submit your answers to this homework assignment, create a directory named `hw1`, copy your solution files to that directory, `cd` to that directory, and then give the command

```
submit hw1
```

You are welcome to form small groups (up to four people) to work through the homework, but you **must** write up all your solutions on your own.

1. **(2 pts.) Getting started**
What is Anant Sahai's second favorite number?

The answer is found on Piazza.

(Why are we having you do this? Piazza is your best source for recent announcements, clarifications on homeworks, and related matters, and we want you to be familiar with how to read the newsgroup.)

2. **(2 pts.) Tricksy Profs**
Prof. Sahai and his brother (who in real life is a CS Prof at UCLA) are guarding a pair of magical doors that appear to be identical. But one of the doors leads you to a room full of Smaug's treasure and the other to a room full of angry goats. You want to get to the room filled with treasure. Both profs know which door is which as well as knowing each other.

The problem is that one of them always lies and the other always tells the truth. But you don't know which is which. Can you ask a question whose answer will reveal to you which door to take? What question should this be? Argue why this works or why no question could possibly work.

3. **(2 pts.) Exclusive OR**
The "exclusive OR" connective (written as XOR or $\oplus$) is just what it sounds like: $P \oplus Q$ is true when exactly one of P, Q is true (but not both). Write down the truth table for $P \oplus Q$ and for $(P \wedge \neg Q) \vee (\neg P \wedge Q)$, and hence show that $P \oplus Q$ is logically equivalent to $(P \wedge \neg Q) \vee (\neg P \wedge Q)$.

4. **(8 pts.) Implications**
Which of the following implications is true?

   1. If 30 is divisible by 10 then 40 is divisible by 10.
   2. If 30 is divisible by 9 then 40 is divisible by 10.
   3. If 30 is divisible by 10 then 40 is divisible by 9.
   4. If 30 is divisible by 9 then 40 is divisible by 9.

5. **(2 pts.) Chasing chains of consequences**
Suppose that S is a set of integers with the following properties:

   1. For all x, if $x \in S$, then $-x \in S$.
   2. For all x, if $x \in S$, then $2x \in S$.

   Suppose that the numbers 10, 11, and 12 are not in S. What can you say about S? Tell us as much as you can.

6. **(10 pts.) Practice with quantifiers**
Which of the following propositions is true? In part 5, $Q(k)$ denotes the proposition "$1+2+\cdots+k = k(k+1)/2$".

   1. $(\forall x \in \mathbb{N} \, . \, x^2 < 5) \implies (\forall x \in \mathbb{N} \, . \, x^2 < 4)$.
   2. $(\forall x \in \mathbb{N} \, . \, x^2 < 4) \implies (\forall x \in \mathbb{N} \, . \, x^2 < 5)$.
   3. $\forall x \in \mathbb{N} \, . \, (x^2 < 5 \implies x^2 < 4)$.
   4. $\forall x \in \mathbb{N} \, . \, (x^2 < 4 \implies x^2 < 5)$.
   5. $\forall n \in \mathbb{N} \, . \, Q(n) \implies Q(n+1)$.

7. **(4 pts.) Working with quantifiers**
Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$ denotes the set of natural numbers.

1. For any natural number n, let $P(n)$ denote the proposition

$$P(n) = \forall i \in \mathbb{N} \,.\, (i < n \Longrightarrow (\forall j \in \mathbb{N} \,.\, (j = n \vee n \neq ij))).$$

   Concisely, for which numbers $n \in \mathbb{N}$ is $P(n)$ true?

2. Rewrite the following quantified proposition in an equivalent form with all negations ("$\neg$", "$\neq$") removed.

$$\neg\forall i \in \mathbb{N} \,.\, \neg\exists j \in \mathbb{N} \,.\, \exists k \in \mathbb{N} \,.\, \neg\forall \ell \in \mathbb{N} \,.\, f(i,j) \neq g(k,\ell).$$

8. (8 pts.) A few proofs

Prove or disprove each of the following statements. For each proof, state which of the proof types (as discussed in the Lecture Notes) you used.

1. For all natural numbers n, if n^2 is even then n^5 is even.
2. For all natural numbers n, $n^2 - n + 3$ is odd.
3. For all real numbers x, y, if $x + y \geq 20$ then $x \geq 10$ or $y \geq 10$.
4. For all real numbers r, if r is irrational then r^2 is irrational.

9. (2 pts.) You be the grader

Assign a grade of A (correct) or F (failure) to the following proof. If you give a F, please explain exactly everything that is wrong with the structure or the reasoning in the *proof*. Justify your answer (saying that the claim is false is *not* a justification).

Theorem 0.1: $\forall n \in \mathbb{N} \,.\, n^2 \leq n \Longrightarrow (n+1)^2 \leq n+1$.

Proof: Suppose that $n \in \mathbb{N}$ and $n^2 \leq n$. (Otherwise, there is nothing to prove.) We need to show that

$$(n+1)^2 \leq n+1.$$

Working backwards we see that:

$$\begin{aligned} (n+1)^2 &\leq n+1 \\ n^2+2n+1 &\leq n+1 \\ n^2+2n &\leq n \\ n^2 &\leq n \end{aligned}$$

So we get back to our original hypothesis which was assumed to be true. Hence, for every $n \in \mathbb{N}$ we know that if $n^2 \leq n$, then $(n+1)^2 \leq n+1$. $\square$