EECS 70 Discrete Mathematics and Probability Theory
Spring 2013 Anant Sahai

HW 11

Due April 15

1. Introductions

It's the first discussion section and the GSI is trying to come up with a clever method to make students form groups and introduce themselves to each other. As there are always shy people in the class, the GSI cannot just ask them to do these things, so s/he comes up with this method: s/he prints the names of all n students, each name on a separate card, and then distributes these cards to all students in a completely random order. So by the end, each student is holding the name of another student (or possibly his/her own name). Assume that there are no absentees, so the number of cards is equal to the number of card-holders.

Then the GSI asks students to form groups this way: each person should find the person whose name s/he holds, and then be in the same group with that person. For example if A is holding B's name, B is holding C's name and C is holding A's name, then one of the groups will be $\{A, B, C\}$.

1. **You should have finished this in Homework 9.**
2. **You should have got "for each $1 \le k \le n$, the probability that a student ends up in a group of size k is $\frac{1}{n}$" in Homework 9.**
3. Now for a fixed $1 \le k \le n$, the GSI is interested in computing the expected number of students that end up in groups of size k. Compute this quantity using the results of the previous part. If you sum the results for all $1 \le k \le n$, what do you get? Give an intuitive explanation of why you get this number without referencing the calculations.
4. Now the GSI wants to know the number of groups of size k. Compute the expected number of such groups using the results of the previous part.
5. The GSI does not like very large groups. In particular s/he does not want a group to contain (strictly) more than half of the students. What are the chances that the GSI's scheme results in such a group? Simplify your answer by expressing it in terms of the Harmonic sequence of numbers which is defined like this:
$$H_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$
6. Now the GSI wants to know the approximate probability of this event, assuming that n is very large. Consider the following approximation for Harmonic numbers (you don't have to prove it):
$$H_n \simeq \ln n + \gamma$$
Here γ is just a constant which is independent of n.
Plug in this approximation into the formula you got for the last part, and find out the approximate probability when n is large. Hint: you should get a number which is independent of n.

2. Round the Clock

As your new year's resolution you decided to have three meals a day, with no exceptions. However you do not like putting names on things, and so you won't be calling your meals breakfast, lunch, and/or dinner. You simply love them equally and hate labeling them. Furthermore, towards your goal of being well-organized you decide that you will have each meal at the same time everyday.

For each one of your meals, you decide that you would have it at an exact but completely random hour during the whole day. i.e. for each meal you pick a time from the set of 24 exact hours $\{\text{midnight}, 1\text{am}, \ldots, 10\text{pm}, 11\text{pm}\}$ and then you have that meal at that exact time every day. Note that you might even have two meals at the same time (i.e. you sample with replacement)! But if you schedule two or more meals at the same time, you decide on a completely random ordering over the meals scheduled for that time, and then you'll have your meals according to that ordering every day. Furthermore, for simplicity assume that it takes no time for you to finish a meal.

1. For each one of your meals, what is the expected amount of time you have to wait after finishing it, until you get to have your next meal?
2. Now suppose that your friend calls you at an exact random hour during the day (24 different possibilities). S/he asks you to go to his/her place and have your next meal with him/her. You do not want to ruin your schedule, so you tell your friend that you'll be at his/her place in x hours to have your next meal, where $x \geq 0$ is the number of hours until your next meal is scheduled. **What is the expected value of x?** Assume that if your friend calls you at an exact hour when you have a meal scheduled, then the call is placed either before your meal or after your meal (or in between two of your meals if you have multiple meals scheduled for that time). Also the probability of the call happening in each of these places is uniformly random. This means that if you have one meal scheduled, then given that the call is placed at the same hour, it is placed before the meal with probability $1/2$ and after the meal with probability $1/2$. If you have two meals scheduled, then given that the call is placed at the same hour, it is placed before both meals with probability $1/3$, between the meals with probability $1/3$, and after both meals with probability $1/3$. When all the meals and the call are at the same hour, then the call is placed before all meals with probability $1/4$, after the first meal with probability $1/4$, after the second meal with probability $1/4$, and after all meals with probability $1/4$.

3. Expectation of Geometric Distribution

In this question, we will use another approach to calculate the expectation of the geometric distribution.

1. If $S = \sum_{i=1}^{\infty} ir^{i-1} = 1 + 2r + 3r^2 + \ldots$ where $-1 < r < 1$, prove

$$S = \frac{1}{(1-r)^2}. \tag{1}$$

(Hint: what is $S - rS$?)

2. Given a random variable X having the geometric distribution with parameter p where $0 < p \leq 1$, *i.e.*,

$$\Pr[X = i] = (1-p)^{i-1} p \qquad \text{for } i = 1, 2, 3, \ldots,$$

use Equation (**??**) to prove $\mathrm{E}(X) = \frac{1}{p}$.

4. Surviving Challenge

Is it a good choice to play the following game? You can use a computer (even a spreadsheet program) to help your calculation.

- Bet: 4 homework points.
- Rules:

- Initially, there are 52 cards with 4 suits (♠, ♡, ♢, ♣) and 13 values (A,2,3,4,5,6,7,8,9,10,J,Q,K).
- In each round, you draw a card and put it in front of you. If two cards in front of you have the same value, the game is over; otherwise, you survive this round and earn 1 homework point.

5. Manufacturing Failures

Products in a factory are manufactured in a two step process. Let's call the steps A and B. Step A has a probability of p_a of failing and step B has a probability of p_b of failing. The failure of the two steps are independent of each other.

A product is considered defective if at least one of the two steps fails. Let X denote the number of products until and including the first defective one. Compute the distribution of X.

What is its expectation?

Hint: You have seen this distribution on another question on this homework set.

6. James Bond

James Bond is imprisoned in a cell from which there are three possible ways to escape: an air-conditioning duct, a sewer pipe and the door (which is unlocked). The air-conditioning duct leads him on a two-hour trip whereupon he falls through a trap door onto his head, much to the amusement of his captors. The sewer pipe is similar but takes five hours to traverse. Each fall produces temporary amnesia and he is returned to the cell immediately after each fall. Assume that he always immediately chooses one of the three exits from the cell with probability $\frac{1}{3}$. On the average, how long does it take before he realizes that the door is unlocked and escapes?

7. Hug Collecting

The class of CS70 has 250 sad people out of 450 people. They are sad because they had an unfairly difficult midterm and did not do so well. So the Professor sends friendly volunteers into the class who each choose one person at random (independent of whether they are sad or not) and gives them a hug.

1. He keeps sending volunteers until every sad person has received at least one hug. What is the expected number of volunteers that he has to send into the room?
2. Suppose that he instead has to choose the number of volunteers to send in advance. He would like there to be a less than a 1/3 probability that any sad people remain without a hug. About how many volunteers should he send in?
3. Use a computer to simulate and estimate what the answer to part (2) above really should be.

8. Your Own Problem

Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student should submit a unique problem. What is the problem? How to formulate it? How to solve it? What is the solution?