Due Feb 11

1. Simple recurrence relations

Assume that you have a sequence of integers defined by an initial condition and a recursive relation. In each case find a simple expression that describes the elements of the sequence and then prove your answer using induction.

1. The sequence $a_1, a_2, \ldots$ satisfies $a_1 = 1$ and $a_{n+1} = 2 \times a_n$ for each $n \geq 1$.
2. The sequence $b_1, b_2, \ldots$ satisfies $b_1 = 3$ and $b_{n+1} = 2 \times b_n + 1$ for each $n \geq 1$.
3. The sequence $c_1, c_2, \ldots$ satisfies $c_1 = 2$ and $c_{n+1} = 5 \times c_n - 4$ for each $n \geq 1$.
4. The sequence $d_1, d_2, \ldots$ satisfies $d_1 = 2$ and $d_{n+1} = 3 \times d_n + 4$ for each $n \geq 1$.
5. The sequence $e_1, e_2, \ldots$ satisfies $e_1 = 100$ and $e_{n+1} = 23 \times e_n + 13$ for each $n \geq 1$.

2. Make it stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{2^n}$$

for every natural number n.

1. Suppose that we want to prove this statement using induction, can we let our induction hypothesis be simply $a_n \leq 3^{2^n}$? Show why this does not work.
2. Try to instead prove the statement $a_n \leq 3^{2^n - 1}$ using induction. Does this statement imply what you tried to prove in the previous part?

3. One side wins, the other loses

Mr. and Mrs. Matchmaker are at work in their matchmaking agency. There are n men and n women, each having strict preferences over people of the opposite gender as suitors for marriage. The Matchmakers want to produce a stable pairing (as defined in the lecture).

Minions who work for the matchmakers have produced two proposals each detailing one set of pairings. Mr. Matchmaker proposes the following scheme to produce the final pairings which are announced to the clients: for each man look at all of the women who are matched with him in at least one of the proposals (there could be one or two women). Then match this man with the best (according to his preferences) of these women.

1. Prove that Mr. Matchmaker's scheme actually results in a matching in which no two men are matched with the same woman.
2. Prove that the matching produced is stable.

3. Mrs. Matchmaker is very suspicious of this scheme and she thinks that it will do a terrible job for women. Help her by proving that Mr. Matchmaker's scheme results in a matching in which each woman is matched to her least favorite suitor among the two proposals produced by the minions.
4. In class you learned that the propose and reject algorithm results in a pairing which is optimal for men and pessimal for women. Give another proof using the things you learned in previous parts that such a pairing which is optimal for men and pessimal for women always exists.

4. I'm too good to marry

In the stable marriage problem, suppose that some men and women have standards and would not just settle for anyone. In other words, in addition to the orderings they have, they prefer being alone to being with some of the lower-ranked individuals (in their own preference list). A pairing would ultimately have to be partial, i.e. some individuals would remain single.

The notion of stability here should be adjusted a little bit: a pairing is stable if there is no rogue couple, and there is no paired individual who prefers being single over being with his/her pair. Note that a rogue couple here is a man and a woman such that: the man is either single and prefers to be with the woman over being single, or is paired and prefers being with the woman over being with his current paired woman. A similar condition should hold for the woman for the couple to be rogue.

1. Prove that we still have a stable pairing. You can approach this by introducing imaginary mates (one for each person) that people marry if they are single. How should you adjust the preference lists of people, including those of the newly introduced imaginary ones for this to work?
2. As you saw in the lecture, we may have different stable pairings. But interestingly, if a person remains single in one stable pairing, s/he must remain single in any other stable pairing as well (there really is no hope for some people!). Prove this fact using Mr. Matchmaker's scheme and its properties from the previous question (you don't need to have proved them to use the results here).

5. Modular decomposition of modular arithmetic

Complex systems are always broken down into simpler modules. In this problem you will learn how this might be done in modular arithmetic.

1. Write down the addition and multiplication table for modular-6 arithmetic (the rows and columns should be labeled $0,1,2,3,4,5$).
2. Each number $0,1,2,3,4,5$ has a remainder mod 2 and a remainder mod 3. For each number write down the pair (x,y) where x is its remainder mod 2 and y is its remainder mod 3. Obviously $0 \le x \le 1$ and $0 \le y \le 2$. Out of all possible pairs (x,y), where $0 \le x \le 1$ and $0 \le y \le 2$, how many times do you see each pair appear?
3. Again write down the addition and multiplication table you wrote in part 1, but this time replace each number with its corresponding pair (when a number appears as a row/column label and also when it appears somewhere in the table). Describe how one can add or multiply two pairs without looking at the original numbers.

6. Power in modular arithmetic

What are the two last digits of $3^{32} = 3^{2^5}$. Use as few multiplications as you can and show your work. All your work should be done by hand and no electronic devices (computers, calculators, etc.) should be used.