EECS 70 Discrete Mathematics and Probability Theory
Spring 2013 Anant Sahai HW 8

Due March 18

1. Deja Vu

Gandalf has this habit of pacing back and forth when he is in deep thought. Feeling a bit concerned, Bilbo claims that this behavior will eventually get Gandalf lost. However, Gandalf always ends up back where he started at some point. More formally, let Gandalf be initially at the position $x_0 = 0$. Every second in thought, Gandalf moves either forward one pace or backwards one pace, *i.e.*, if he is at position x_t at time t, he will be either at $x_{t+1} = x_t - 1$ or $x_{t+1} = x_t + 1$ at time $t+1$. We want to compute the number of paths that Gandalf returns 0 at time t and it is his first return, *i.e.*, $x_t = 0$ and $x_s \neq 0$ for all s where $0 < s < t$.

1. How many paths can Gandalf take if he returns 0 at $t = 6$ and it is his first return?
2. How many paths can Gandalf take if he returns 0 at $t = 7$ and it is his first return?
3. How many paths can Gandalf take if he returns 0 at $t = 8$ and it is his first return?
4. How many paths can Gandalf take if he returns 0 at $t = 2n+1$ for $n \in \mathbb{N}$ and it is his first return?
5. How many paths can Gandalf take if he returns 0 at $t = 2n+2$ for $n \in \mathbb{N}$ and it is his first return? (Hint: read http://en.wikipedia.org/wiki/Catalan_number and use any result there if you need.)

2. The Monty Hall Problem

1. Read Note 11 first and Explain the Monty Hall problem in your own words. Pretend you are teaching yourself as you were at the beginning of this term.
2. What is the sample space of this problem? Draw it as a tree structure (as suggested on page 5 of Note 11).
3. What is the probability of each sample point?
4. What is the event of interest?
5. If the contestant is using the "sticking strategy," which sample points are in this event? What is the probability of this event?
6. If the contestant is using the "switching strategy," which sample points are in this event? What is the probability of this event?

3. Error Correction Codes

Alice wishes to send $n = 5$ characters to Bob. However, there is a 0.1 probability for each character to be corrupted during transmission due to channel noise. Therefore, Alice wants to transmit m additional characters to correct potential errors, and Bob uses the Reed-Solomon approach (in the lecture note) to correct errors.

1. If $m = 2$, what it the probability that Bob gets Alice's message successfully?
2. If $m = 3$, what it the probability that Bob gets Alice's message successfully?

3. By the above results, is adding more characters always helpful in this case?
4. To guarantee that the probability that Bob gets Alice's message successfully is at least 0.9, how many additional characters does Alice need to transmit?

Now, we are considering a general case where $n \geq 0$, $m \geq 0$, and $0 \leq p \leq 1$, which is the probability for each character to be corrupted.

5. True or False (prove it or provide a counterexample): for any n, m, p, transmitting $m+2$ additional characters is always at least as good as (never worse than) transmitting m additional characters, in terms of the probability that Bob gets Alice's message successfully.

4. Best-of-Five Series

A best-of-five series is the competition between two teams where a team wins the series by winning three games in the series. After a team wins three games, the following games will not be played. An outcome of a best-of-five series is defined by the result of each game, *i.e.*, WWW, WWLW, WLWW, and WLWLW are all different outcomes for a team. Now, two basketball teams, California Golden Bears and Stanford Cardinal, are going to play a best-of-five series.

1. How many possible outcomes for Golden Bears to win the series with 3 wins and 2 losses?
2. How many possible outcomes for Golden Bears to win the series?
3. Prove the following statement (where $n \geq 0$) with explicit calculations.

$$\sum_{i=0}^{n} \binom{n+i}{i} = \binom{2n+1}{n+1}.$$

4. If it is a best-of-$(2n+1)$ series, use the above equality to prove that there are $\binom{2n+1}{n+1}$ outcomes for Golden Bears to win the series.
5. If it is a best-of-$(2n+1)$ series, explain "there are $\binom{2n+1}{n+1}$ outcomes for Golden Bears to win the series" without any explicit calculation.
6. If it is a best-of-(n_1+n_2-1) series where Golden Bears win the series by winning n_1 games in the series, and Cardinal wins the series by winning n_2 games in the series, explain "there are $\binom{n_1+n_2-1}{n_1}$ outcomes for Golden Bears to win the series" without any explicit calculation.

It is believed that Golden Bears have a 0.7 probability to win each game, if the game is played.

7. What is the probability for Golden Bears to win the series with 3 wins and 2 losses?
8. What is the probability for Golden Bears to win the series.
9. The caption of Cardinal proposes to change the series from best-of-five to best-of-three. Is this change advantageous to Golden Bears or Cardinal?

Now, let's consider some conditional probabilities. We still assume that Golden Bears have a 0.7 probability to win each game, if the game is played.

10. Pr[(Golden Bears win the series) | (Golden Bears win the first game)]=?
11. Pr[(Golden Bears win the first game) | (Golden Bears win the series)]=?
12. What is Pr[Golden Bears win the fifth game]? Explain your answer.

13. Pr[(Golden Bears win the series) | (Golden Bears win the fifth game)]=?
14. Pr[(Golden Bears win the fifth game) | (Golden Bears win the series)]=?

5. Stirling's Approximation

In this question, suppose $n \in \mathbb{Z}^+$, we want to find approximations for $n!$. If you are asked to plot a function, you can use any software (*e.g.*, MATLAB) or online tool (*e.g.*, go to http://www.wolframalpha.com/ and type "plot $\ln x$").

1. Plot the function $f(x) = \ln x$.

For the following three questions, please note that $\ln x$ is strictly increasing and concave because, when $x > 0$, its first and second derivatives are positive and negative, respectively. Concavity means that all line segments from two points on the function are below the function.

2. Suppose $n \in \mathbb{Z}^+$, use the plot to explain why

$$\ln 1 + \ln 2 + \ldots + \ln n \geq \int_1^n \ln x dx. \tag{1}$$

3. Suppose $n \in \mathbb{Z}^+$, use the plot to explain why

$$\ln 1 + \ln 2 + \ldots + \ln n < \int_1^{n+1} \ln x dx. \tag{2}$$

4. Suppose $a \in \mathbb{Z}^+$, use the plot to explain why

$$\left(\frac{\ln a + \ln(a+1)}{2}\right) < \int_a^{a+1} \ln x dx. \tag{3}$$

5. Use Equation (1) to prove $n! \geq e\left(\frac{n}{e}\right)^n$.
6. Use Equation (2) to prove $n! \leq en\left(\frac{n}{e}\right)^n$.
7. Use Equation (3) to prove $n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n$, which is a tighter upper bound.
8. The Stirling's approximation is usually written as $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ or a simpler version $n! \approx \left(\frac{n}{e}\right)^n$. Plot the function $f(n) = \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{n!}$. What do you observe?
9. Suppose $m = \frac{k}{n}$, use m, n and apply the simpler version of the Stirling's approximation to rewrite $\binom{n}{k}$.
10. Now, suppose $m_1 = \frac{k_1}{n} = 0.25$, $m_2 = \frac{k_2}{n} = 0.5$, and $m_3 = \frac{k_3}{n} = 0.75$, plot $\binom{n}{k_1}$, $\binom{n}{k_2}$, and $\binom{n}{k_3}$ as functions of n on a log-scaled $f(n)$ axis. What do you observe?
11. What are the slopes of the functions in the plot.

6. Your Own Problem

Write your own problem related to this week's material and solve it. You may still work in groups to brainstorm problems, but each student should submit a unique problem. An example is the birthday paradox (of course, do not use this problem again) in Note 11. What is the problem? How to formulate it? How to solve it? What is the solution?