1. **Odd Degree Vertices**

   Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

   Prove the claim above using:

   (i) Induction on $m = |E|$ (number of edges)

   (ii) Induction on $n = |V|$ (number of vertices)

   (iii) Well-ordering principle

   (iv) Direct proof (e.g., counting the number of edges in G)

2. Directed Euler

Recall that in lecture we proved Euler's theorem for undirected graphs by giving a recursive algorithm whose correctness we proved by induction. One way to more fully understand the recursive algorithm is to unwind it into an iterative algorithm. In the case of Euler's theorem there is another reason to unwind the recursion — proving Euler's theorem for directed graphs via a recursive algorithm gets a little more tricky. In this exercise we will walk you through an iterative algorithm based proof of the *if* direction (the harder direction) of the directed Eulerian tour theorem. The theorem states that a directed graph G has an Eulerian tour that traverses each (directed) edge exactly once if and only if it is connected and for every vertex $v \in G$, $indegree(v) = outdegree(v)$. Here connected means that for any two edges in G it is possible to start from the first edge and walk along the directed graph edges to end up at the second edge.

(i) First prove that if you start from a vertex s and walk arbitrarily until you are stuck, the set of edges traversed forms a tour, T, (not necessarily Eulerian), and the remaining untraversed edges satisfy two properties:

  (a) For every vertex $v \in G$, $indegree(v) = outdegree(v)$.

  (b) There is some vertex v on T such that there is at least one untraversed edge (v, w) leaving v.

(ii) Fill in the details of an iterative procedure for finding the Eulerian tour, that in the next iteration starts from v and finds a tour T' among the untraversed edges. The iteration ends with an update to T by splicing T' into T.

(iii) Finally, prove by induction that your procedure results in an Eulerian tour.