CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Walrand and Rao **Discussion 13a**

1. **Markov Chain Terminology**

   In this question, we will walk you through terms related to Markov chains. Keep in mind the following theorems and ideas as you work.

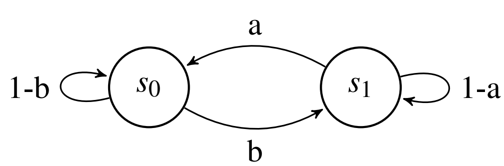
   (a) (Irreducibility) A Markov chain is irreducible if it can go from every state i to every other state j, possibly in multiple steps.

   (b) (Periodicity) $d(i) :=$ g.c.d$\{n > 0 \mid P^n(i,i) = Pr[X_n = i | X_0 = i] > 0\}, i \in \mathscr{X}$ where $d(i) = 1$ if and only if the Markov chain is aperiodic.

   (c) (Matrix Representation) Define the transition probability matrix P by filling entry i, j with probability $P(i, j)$, where $X_n = i, X_{n+1} = j$.

   (d) (Invariance) A distribution π is invariant for the transition probability matrix P if it satisfies the following balance equation: $\pi = \pi P$.

   Use the above theorems and the Markov chain below to answer the following questions.



   (a) For what values of a and b is the above Markov chain irreducible? Reducible?

   Solution: The markov chain is irreducible if both a and b are non-zero. It is reducible if at least one is 0.

   (b) For $a = 1, b = 1$, prove that the above Markov chain is periodic.

   Solution: We compute $d(0)$ to find that:

   $$d(0) = \text{g.c.d.}\{2, 4, 6...\} = 2$$

   Thus, the chain is periodic.

   (c) For $0 < a < 1, 0 < b < 1$, prove that the above Markov chain is aperiodic.

   Solution: We compute $d(0)$ to find that:

   $$d(0) = \text{g.c.d.}\{1, 2, 3...\} = 1$$

   Thus, the chain is aperiodic.

(d) Construct a transition probability matrix using the above Markov chain.

Solution:

$$\begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

(e) Write the balance equations for this Markov chain. If $a = 0, b = 0$, is this distribution invariant?

Solution:

$$\pi(0) = (1-b)\pi(0) + a\pi(1)$$
$$\pi(1) = b\pi(0) + (1-a)\pi(1)$$

Notice that we get this same system of equations by multiplying P in the previous part by $\begin{bmatrix} \pi(0) \\ \pi(1) \end{bmatrix}$.

There are two ways to see whether or not $a = 0, b = 0$ is invariant. We can plug in these values into the balance equation, for one.

$$\pi(0) = \pi(0)$$
$$\pi(1) = \pi(1)$$

Intuitively, we see that the transition matrix does not affect the distribution. However, note that if we plug in $a = 0, b = 0$ int the matrix, this becomes the identity matrix!

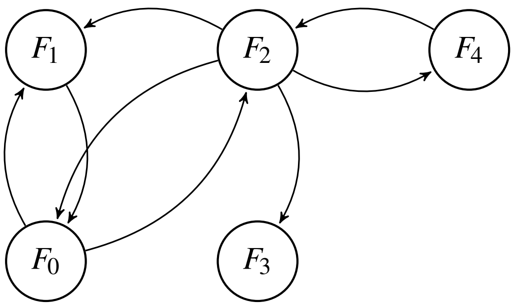
$$\begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This formally satisfies the necessary and sufficient condition for an invariant system, where $\pi = \pi P$.

2. **The Dwinelle Labyrinth**

You have decided to take a humanities class this semester, a French class to be specific. Instead of a final exam, your professor has issued a final paper. You must turn in this paper *before* noon to the professor's office on floor 3 in Dwinelle, and it's currently 11:48 a.m.

Let Dwinelle be modeled by the following Markov chain. Instead of rushing to turn it in, we will spend valuable time computing whether or not we *could have* made it. Suppose walking between floors takes 1 minute.



(a) Will you make it in time, if you choose a floor to transition to, uniformly at random? (i.e., If F_0 is the number of steps needed to get to F_3, is $E[F_0] < 12$?)

Solution: Write out all of our balance equations.

$E[F_0] = 1 + \frac{1}{2}E[F_1] + \frac{1}{2}E[F_2]$
$E[F_1] = 1 + E[F_0]$
$E[F_2] = 1 + \frac{1}{4}E[F_0] + \frac{1}{4}E[F_1] + \frac{1}{4}E[F_3] + \frac{1}{4}E[F_4]$
$E[F_3] = 0$
$E[F_4] = 1 + E[F_2]$

Let us rewrite these equations, before placing it in matrix form.

$-1 = -E[F_0] + \frac{1}{2}E[F_1] + \frac{1}{2}E[F_2]$
$-1 = -E[F_1] + E[F_0]$
$-1 = -E[F_2] + \frac{1}{4}E[F_0] + \frac{1}{4}E[F_1] + \frac{1}{4}E[F_3] + \frac{1}{4}E[F_4]$
$0 = E[F_3]$
$-1 = -E[F_4] + E[F_2]$

We can rewrite this in matrix form.

$$P = \begin{bmatrix} -1 & 1/2 & 1/2 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 1/4 & 1/4 & -1 & 1/4 & 1/4 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

We can now reduce the matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 15 \\ 0 & 1 & 0 & 0 & 0 & 16 \\ 0 & 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 13 \end{bmatrix}$$

We see that $E[F_0] = 15$, meaning it will take 15 minutes for us to get to floor 3. Unfortunately, we only have 12 minutes.

(b) Will you make it in time, if for every floor, you order all accessible floors and are twice as likely to take higher floors? (i.e., If you are considering 1, 2, or 3, you will take each with probabilities $\frac{1}{7}, \frac{2}{7}, \frac{4}{7}$, respectively.)

Solution: Write out all of our balance equations.
$E[F_0] = 1 + \frac{1}{3}E[F_1] + \frac{2}{3}E[F_2]$
$E[F_1] = 1 + E[F_0]$
$E[F_2] = 1 + \frac{1}{15}E[F_0] + \frac{2}{15}E[F_1] + \frac{4}{15}E[F_3] + \frac{8}{15}E[F_4]$
$E[F_3] = 0$
$E[F_4] = 1 + E[F_2]$
Let us rewrite these equations, before placing it in matrix form.
$-1 = -E[F_0] + \frac{1}{3}E[F_1] + \frac{2}{3}E[F_2]$
$-1 = -E[F_1] + E[F_0]$
$-1 = -E[F_2] + \frac{1}{15}E[F_0] + \frac{2}{15}E[F_1] + \frac{4}{15}E[F_3] + \frac{8}{15}E[F_4]$
$0 = E[F_3]$
$-1 = -E[F_4] + E[F_2]$
We can rewrite this in matrix form.

$$P = \begin{bmatrix} -1 & 1/3 & 2/3 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 1/15 & 2/15 & -1 & 4/15 & 8/15 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

We row reduce to get the following.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 11.5 \\ 0 & 1 & 0 & 0 & 0 & 12.5 \\ 0 & 0 & 1 & 0 & 0 & 8.5 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 9.5 \end{bmatrix}$$

We see that $E[F_0] = 11.5$, meaning it will take 11.5 minutes for us to get to floor 3. That's fewer than 12 minutes, so if you finished this computation in less than 30 seconds, you could make it!