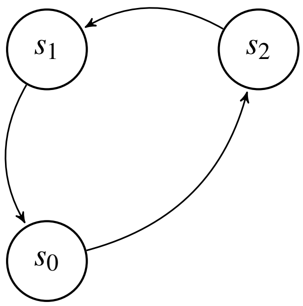
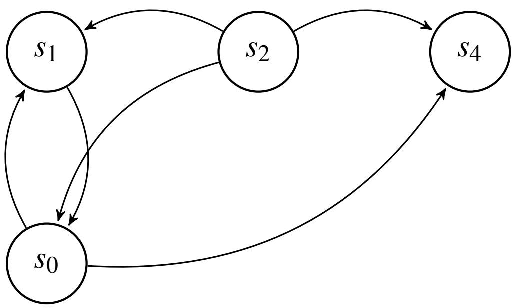
CS 70 Spring 2016 — Discrete Mathematics and Probability Theory — Walrand and Rao — Discussion 13b

1. **Markov Chain Terminology Test**

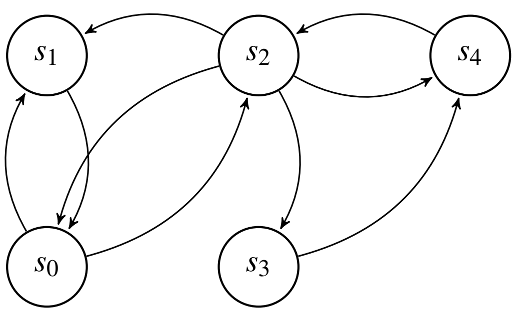
   For each of the following Markov chains, determine if it is (a) periodic and/or (b) reducible.

   (a) 

   Solution: This chain is (a) periodic and (b) irreducible.

   (b) 

   Solution: This chain is (a) aperiodic and (b) reducible.

   (c) 

   Solution: This chain is (a) aperiodic and (b) irreducible.

2. **Number of Ones**

In this problem, we will revisit dice-rolling, except with conditional expectation.

(a) If we roll a die until we see a 6, how many ones should we expect to see?

Solution: Let Y be the number of ones we see. Let X be the number of rolls we take until we get a 6. We are, then, solving for $E[Y|X]$.

Let us first compute $E[Y|X]$. We know that since in each of our $k-1$ rolls before the kth, we necessarily roll a number in $\{1, 2, 3, 4, 5\}$. Thus, we have a $\frac{1}{5}$ chance of getting a one, meaning $E[Y|X=k] = \frac{1}{5}(k-1)$, so $E[Y|X] = \frac{1}{5}(X-1)$.

If this is confusing, write Y as a sum of indicator variables.

$$Y = Y_1 + Y_2 + \ldots Y_k$$

where Y_i is 1 if we see a one on the ith roll. This means

$$E[Y|X=k] = E[Y_1|X=k] + E[Y_2|X=k] + \ldots E[Y_k|X=k]$$

We know for a fact that on the kth roll, we roll a 6, thus $E[Y_k] = 0$. Thus, we actually consider

$$\begin{aligned} E[Y_1|X=k] + E[Y_2|X=k] + \ldots E[Y_{k-1}|X=k] \\ = (k-1)E[Y_1|X=k] \\ = (k-1)Pr[Y_1|X=k] \\ = (k-1)\frac{1}{5} \end{aligned}$$

Using the Law of Total Expectation, we know that

$$\begin{aligned} E[E[Y|X]] = E[Y] = E[\frac{1}{5}(X-1)] \\ = \frac{1}{5}E[X-1] \\ = \frac{1}{5}(E[X]-1) \end{aligned}$$

Since, $XGeom(\frac{1}{6})$, the expected number of rolls until we roll a 6 is $E[X] = 6$.

$$\begin{aligned} = \frac{1}{5}(6-1) \\ = 1 \end{aligned}$$

(b) If we roll a die until we see a number greater than 3, how many ones should we expect to see?

Solution:

The first change from the previous part is the probability of rolling a 1, given we have made k rolls to get to our first roll that satisfies the condition. This makes

$$E[E[Y|X]] = E[Y] = E[\frac{1}{3}(X-1)]$$
$$= \frac{1}{3}(E[X]-1)$$

Since $XGeom(\frac{1}{2})$, we know that the expected number of rolls until we roll a number greater than 3 is $E[X] = 2$.
this makes $E[Y] = \frac{1}{3}$.

(c) We add r red marbles, b blue marbles and g green marbles to the same bag. If we sample balls, with replacement, until we get 3 red marbles (not necessarily consecutively), how many blue marbles should we expect to see?

Solution:

Let Y be the number of blue marbles we see. Let X be the samples we take until we get 3 red marbles. We are, then, solving for $E[Y|X]$.
Let us first compute $E[Y|X]$. Let Y_i be 1 if we see a blue marble on the ith roll and $Y = \sum_i^k Y_i$. This means

$$E[Y|X=k] = E[\sum_i^k Y_i|X=k]$$
$$= \sum_i^k E[Y_i|X=k]$$

However, We also three Y_i have $E[Y_i] = 0$, since there are necessarily 3. This means the other $k-3$ marbles are necessarily blue or green.

$$= \sum_{i \neq a,b,c}^k Pr[Y_i|X=k]$$
$$= \sum_{i \neq a,b,c}^k \frac{b}{b+g}$$
$$= (k-3)\frac{b}{b+g}$$

This means $E[Y|X] = (X-3)\frac{b}{b+g}$.
Using the Law of Total Expectation, we know that

$$E[E[Y|X]] = E[Y] = E[\frac{b}{b+g}(X-3)]$$
$$= \frac{b}{b+g}E[X-3]$$
$$= \frac{b}{b+g}(E[X]-3)$$

We know that $E[X] = 3\frac{r+g+b}{r}$.

$$E[Y] = \frac{b}{b+g}(3\frac{r+g+b}{r}-3)$$