1. Let $P(x)$ and $Q(x)$ denote some propositions involving x. For each statement below, either prove that the statement is correct or provide a counterexample if it is false.

(a) $\forall x\,(P(x) \vee Q(x))$ is equivalent to $(\forall x, P(x)) \vee (\forall x, Q(x))$.

(b) $\forall x\,(P(x) \wedge Q(x))$ is equivalent to $(\forall x, P(x)) \wedge (\forall x, Q(x))$.

(c) $\exists x\,(P(x) \vee Q(x))$ is equivalent to $(\exists x, P(x)) \vee (\exists x, Q(x))$.

(d) $\exists x\,(P(x) \wedge Q(x))$ is equivalent to $(\exists x, P(x)) \wedge (\exists x, Q(x))$.

Answer:

(a) False. One direction of implication is true, namely (prove this as an exercise)

$$(\forall x, P(x)) \vee (\forall x, Q(x)) \quad \text{implies} \quad \forall x\,(P(x) \vee Q(x)).$$

However, the converse is false, so the two statements are not equivalent. As a counterexample, take the universe to be $\mathbb{R}$, take $P(x)$ to be the statement "$x \geq 0$," and $Q(x)$ to be the statement "$x < 0$." Then $\forall x(P(x) \vee Q(x))$ is true, but $(\forall x, P(x)) \vee (\forall x, Q(x))$ is false.

(b) True. Recall that to prove $A \Leftrightarrow B$, we have to prove both $A \Rightarrow B$ and $B \Rightarrow A$.
Suppose the first statement $\forall x\,(P(x) \wedge Q(x))$ is true. This means for all x, $P(x) \wedge Q(x)$ is true, so $P(x)$ and $Q(x)$ are both true. Thus, the statement $(\forall x,\, P(x))$ is true, and similarly the statement $(\forall x,\, Q(x))$ is true, so the conjunction $(\forall x, P(x)) \wedge (\forall x, Q(x))$ is true.
Conversely, suppose the second statement $(\forall x, P(x)) \wedge (\forall x, Q(x))$ is true. This means for all x, both $P(x)$ is true and $Q(x)$ is true, which implies $P(x) \wedge Q(x)$ is true. Thus, the first statement $\forall x\,(P(x) \wedge Q(x))$ is true. This completes the proof of the equivalence.

(c) True. We can prove the equivalence directly as in part (b). Alternatively, we can use our result in part (b) as follows. We know from part (b) that for any propositions $\tilde{P}(x)$ and $\tilde{Q}(x)$, the following is true:

$$\forall x\,(\tilde{P}(x) \wedge \tilde{Q}(x)) \;\text{ is equivalent to }\; (\forall x, \tilde{P}(x)) \wedge (\forall x, \tilde{Q}(x)).$$

Recall that $A \Leftrightarrow B$ is equivalent to $\neg A \Leftrightarrow \neg B$, so we can write the equivalence above as

$$\neg\big(\forall x\,(\tilde{P}(x) \wedge \tilde{Q}(x))\big) \;\text{ is equivalent to }\; \neg\big((\forall x, \tilde{P}(x)) \wedge (\forall x, \tilde{Q}(x))\big).$$

After simplifying the negations, we arrive at

$$\exists x\,(\neg\tilde{P}(x) \vee \neg\tilde{Q}(x)) \;\text{ is equivalent to }\; (\exists x, \neg\tilde{P}(x)) \vee (\exists x, \neg\tilde{Q}(x)).$$

Finally, let us choose $\tilde{P}(x)$ to be $\neg P(x)$ and $\tilde{Q}(x)$ to be $\neg Q(x)$, so we obtain

$$\exists x\,(P(x) \vee Q(x)) \;\text{ is equivalent to }\; (\exists x, P(x)) \vee (\exists x, Q(x)),$$

which is what we want to prove.

(d) False. One direction of implication is true, namely (prove this as an exercise)

$$\exists x\,(P(x) \land Q(x)) \quad \text{implies} \quad (\exists x, P(x)) \land (\exists x, Q(x)).$$

However, the converse is false. Take the same counterexample as in part (a). Then $(\exists x, P(x)) \land (\exists x, Q(x))$ is true but $\exists x\,(P(x) \land Q(x))$ is false.

2. (Proof) A *perfect square* is an integer n of the form $n = m^2$ for some integer m. Prove that every odd perfect square is of the form $8k+1$ for some integer k.

Let $n = m^2$ for some integer m. Since n is odd, m is also odd, i.e., of the form $m = 2l+1$ for some integer l. Then, $m^2 = 4l^2 + 4l + 1 = 4l(l+1) + 1$. Since one of l and $l+1$ must be even, $l(l+1)$ is of the form $2k$ and $n = m^2 = 8k+1$.

3. (Contradiction) Prove that $2^{1/n}$ is not rational for any integer $n > 3$. [Hint : Fermat's Last Theorem and the method of contradiction]

If not, then there exists an integer $n > 3$ such that $2^{1/n} = \frac{p}{q}$ where p, q are positive integers. Thus, $2q^n = p^n$, and this implies,

$$q^n + q^n = p^n$$

, which is a contradiction to the Fermat's Last Theorem.

4. (Problem solving) Prove that if you put $n+1$ apples into n boxes, any way you like, then at least one box must contain at least 2 apples. This is known as the *pigeonhole principle*.

Suppose this is not the case. Then all the boxes would contain at most 1 apple. Then the maximum number of apples we could have would be n, but this is a contradiction since we have $n+1$ apples.

5. Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party.

Answer: Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0, 1, \ldots, n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), and there is one person who has 0 friends (i.e., friends with no one), which is a contradiction.