1. Stable Marriage

Consider the following list of preferences:

Men	Preferences
A	$4>2>1>3$
B	$2>4>3>1$
C	$4>3>1>2$
D	$3>1>4>2$

Women	Preferences
1	$A>D>B>C$
2	$D>C>A>B$
3	$C>D>B>A$
4	$B>C>A>D$

1. Is $\{(A,4),(B,2),(C,1),(D,3)\}$ a stable pairing?
   No. Rogue pair: $(C,3)$.
2. Find a stable matching by running the Traditional Propose & Reject algorithm.
   A men-optimal pairing: $\{(A,2),(B,4),(C,3),(D,1)\}$.
3. Show that there exist a stable matching where women 1 is matched to men A.
   A pairing can be: $\{(A,1),(B,4),(C,3),(D,2)\}$. This is stable because each woman gets their first preference.

2. True or False

For each of the following statements about the traditional stable marriage algorithm with men proposing, indicate whether the statement is True or False and justify your answer with a short 2-3 line explanation:

1. (True/False) In a stable marriage algorithm execution which takes n days, there is a woman who did not receive a proposal on the $(n-1)$th day.
   True: The algorithm will terminate once there is exactly one man proposing to each woman. This means that there will be at least one woman who is not proposed to every day before the algorithm terminates. This includes day $(n-1)$.
2. (True/False) In a stable marriage algorithm execution, if a woman receives a proposal on day k, she receives a proposal on every subsequent day until termination.
   True: This is true by the improvement lemma, since once a woman has a suitor on a string, she always has some suitor on a string and therefore must receive a proposal each day.
3. (True/False) There is a set of preferences for n men and n women, such that in a stable marriage algorithm execution every man ends up with his least preferred woman.
   False: If this were to occur it would mean that at the end of the algorithm, every man would have proposed to every woman on his list and has been rejected $n-1$ times. This would also require every woman to reject $n-1$ suitors. We know this is impossible though if we consider what we learned in parts (a) and (b). There must be at least one woman who is not proposed to until the very last day.
4. (True/False) There is a set of preferences for n men and n women, such that in a stable marriage algorithm execution every woman ends up with her least preferred man.
   True: One example occurs when every woman has a different least favorite man, who happens to prefer her over all other women. Then the stable marriage algorithm will end in one day.

5. (True/False) In a stable marriage algorithm execution, if woman W receives no proposal on day i, then she receives no proposal on any previous day j which is less than i.

   True: This is the contrapositive of part (b).

3. Large Number of Stable Pairings

How many different stable pairings can there be for an instance with n men and n women? In this question we will see how to construct instances with a very large number of stable pairings.

The overall plan is as follows: imagine we have already constructed an instance of stable marriage with m men and m women which admits X different stable pairings. We will show how to use this to construct a new instance with $2m$ men and $2m$ women which has at least X^2 stable pairings.

1. Construct an instance with $n = 2$ and 2 different stable pairings. Now assuming construction in the overall plan above, show that there is an instance with $n = 4$ and 4 different stable pairings. What does that tell you about $n = 8$? In general if we continue this for $n = 2^k$, how many different stable pairings do we get? Express that as a function of n.

   The $n = 2$ case can be the following.

Men	Preferences
A	$1 > 2$
B	$2 > 1$

Women	Preferences
1	$B > A$
2	$A > B$

   In this instance both $\{(A,1),(B,2)\}$ and $\{(A,2),(B,1)\}$ are stable.

   Now using the previous part for $n = 4$ we get an instance with $2^2 = 4$ stable pairings. For $n = 8$ we get an instance with $4^2 = 16$ stable pairings. In general it seems like for $n = 2^k$ we get $2^{2^{k-1}} = 2^{n/2}$ different pairings. So let us prove that using induction on k.

   The base case of $k = 1$ is correct because for $n = 2$ we have $2^{2^{1-1}} = 2$ different stable pairings. Now if the statement is true for k, we know that for $k+1$ we get the square of what we had for k. So for $k+1$ we have $(2^{2^{k-1}})^2 = 2^{2\times 2^{k-1}} = 2^{2^k}$ different stable pairings.

2. Implement the overall plan: square the number of stable pairings at the cost of doubling the size of the instance. Start with an instance of stable marriage with m men and m women which admits X different stable pairings, and create a new instance with $2m$ men and $2m$ women. To do this, create two copies of each person in the instance you start with. Call one of them the original, and the other the alternate. Let original people prefer original people above alternate people, and alternate people prefer alternate people above originals, but in every other way let preferences remain as they were in the instance you started with. Thus, for example, the preference list of an alternate man would look like the preference list repeated twice of the man he was cloned from in the original instance, with the first half consisting of alternate women and the second half consisting of original women. Prove that the number of stable pairings in this new instance is at least X^2. To do so, it suffices to exhibit that for any pair of stable pairings of the instance you started with there is a unique stable pairing in the new instance.

   Given two stable pairings, we use one of them to pair the original people, and one of them to pair the alternate people. Note that we end up with no original-alternate couple. Now it is easy to see that there are no rogue couples; if (M,W) is a rogue couple, then they can't both be original or both alternate (because the two pairings we used were stable). And they can't be original-alternate, because then they would prefer their current partners over each other.

   Note that for any pair of starting stable pairings, we end up with a unique pairing (because we can get back the original pairings in the smaller instance, by just narrowing our view to original/alternate people). So by doing this we produce X^2 different pairings.

For home Good, Better, Best

In a particular instance of the stable marriage problem with n men and n women, it turns out that there are exactly three distinct stable matchings, M_1, M_2, and M_3. Also, each man m has a different partner in the three matchings. Therefore each man has a clear preference ordering of the three matchings (according to the ranking of his partners in his preference list). Now, suppose for man m_1, this order is $M_1 > M_2 > M_3$.

Prove that every man has the same preference ordering $M_1 > M_2 > M_3$.

In class, you were given the traditional propose-and-reject algorithm, which was guaranteed to produce a male-optimal matching. By switching men's and women's roles, you would be guaranteed to produce a female-optimal matching, which, by a lemma from class, would also be male-pessimal. By the very fact that these algorithms exist and have been proven to work in this way, you're guaranteed that a male-optimal and a male-pessimal matching always exist.

Since there are only three matchings in this particular stable matching instance, we thus know that one of them must be male-optimal and one must be male-pessimal. Since m_1 prefers M_1 above the other stable matchings, only that one can be male-optimal by definition of male-optimality. Similarly, since m_1 prefers M_3 the least, it must be the male-pessimal. Therefore, again from definitions of optimality/pessimality, since each men have different matches in the three stable matchings, they *must* strictly prefer M_1 to both of the others, and they *must* like M_3 strictly less than both of the others. Thus, each man's preference order of stable matchings must be M_1, M_2, M_3.