CS 70 Discrete Mathematics and Probability Theory
Spring 2016 Rao and Walrand Discussion 7B Solutions

1. **Clothes and stuff**

   (a) Say we've decided to do the whole capsule wardrobe thing and we now have only 5 different items of clothing that we wear (jeans, tees, shoes, jackets, and floppy hats, obvi). We have 3 variations on each of the items, and we wear one of each item every day. How many different outfits can we make?

   Answer: 3^5

   (b) It turns out 3 floppy hats really isn't enough of a selection, so we've bought 11 more, and we now have 14 floppy hats. Now how many outfits can we make?

   Answer: $14 \cdot 3^4$

   (c) If we own k different items of clothing, with n_1 varitions of the first item, n_2 variations of the second, n_3 of the third, and so on, how many outfits can we make?

   Answer: $n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_k$

   (d) We love our floppy hats so much that we've decided to also use them as wall art, so we're picking 4 of our 14 hats to hang in a row on the wall. How many such arrangements could we make? (Order matters, because no one really wants to see that burgundy one next to our favorite forest green fedora.)

   Answer: $14!/10!$

   (e) Ok, now we're packing for vacation to Iceland, and we only have space for 4 of our 14 floppy hats. How many sets of 4 could we bring? (Yeah, yeah, we knew you were going to use that notation. Now tell us the number as a function of a, your answer from the previous part.)

   Answer: $\binom{14}{4}$ or written as a function of the previous part, $a/4!$

   (f) Ok, turns out the check-in person for our flight to Iceland is being *very* unreasonable about the luggage weight restrictions, and we're going to have to leave some hats behind. Despite our best intentions, and having packed only 4 hats, we actually bought 18 additional floppy hats at the airport (6 in burgundy, 6 in forest green, and 6 in classic black). We'll keep our 4 hats that we brought from home, but we'll have to return all but 6 of the airport hats. How many color configurations can there be for the 6 airport hats that we keep?

   Answer: $\binom{8}{6}$. But let's be serious, you should just keep the black ones – so much more versatile.

2. **Counting practice**

   Leave your answers as (tidy) expressions involving factorials, binomial coefficients, etc., rather than evaluating them as decimal numbers (though you are welcome to perform this last step as well for your own interest if you like, provided it is clearly separated from your main answer). Explain clearly how you arrived at your answers.

   (a) How many 75-bit strings are there that contain more ones than zeros?

(b) How many bridge hands contain exactly 4 hearts?

(c) How many anagrams of OKLAHOMA are there? (An anagram of OKLAHOMA is any re-ordering of the letters of OKLAHOMA; i.e., any string made up of the eight letters O, K, L, A, H, O, M and A, in any order. The anagram does not have to be an English word.)

(d) How many different anagrams of MISSISSIPPI are there?

(e) How many rolls of 6 dice with at most 5 distinct values?

(f) How many increasing sequences of k numbers from $\{1,\dots,n\}$? ($2,3,5,5,7$ is not an increasing sequence, but $2,3,5,7$ is.)

(g) How many distinct degree $\leq d$ polynomials modulo p, where $d \leq p-1$?

(h) How many distinct degree $\leq d$ polynomials with exactly d roots modulo p where $d \leq p-1$? (Brief explanation ok.)

(i) How many values of $x \in \{0,\dots,11\}$ satisfy $4x = 6 \pmod{12}$?

(j) How many values of $x \in \{0,\dots,35\}$ satisfy $8x = 4 \pmod{36}$?

Answer:

(a) $\boxed{2^{74}}$. Since 75 is odd, there are either more zeros than ones or more ones than zeros. By symmetry, the number of strings in each case is equal. Thus there are $2^{75}/2$ strings with more ones than zeros.

(b) $\boxed{\binom{13}{4}\binom{39}{9}}$ We first choose the 4 hearts, then the rest of the hand, which should not contain any more hearts.

(c) $\boxed{\dfrac{8!}{2!2!}}$ O and A are both repeated twice.

(d) $\boxed{\dfrac{11!}{4!4!2!}}$ Same reasoning as above.

(e) $\boxed{6^6 - 6!}$ This is the same as all possible rolls of 6 dice minus the rolls with 6 distinct values.

(f) $\boxed{\binom{n}{k}}$ We choose any k distinct numbers and arrange them in increasing order.

(g) $\boxed{p^{d+1}}$ A polynomial of degree $\leq d$ is uniquely described by $d+1$ coefficients (for $x^0, x^1, \dots, x^d$). For each coefficient there are p choices. Thus the total number is p^{d+1}.

(h) $\boxed{(p-1)\binom{p}{d}}$ We can pick the roots in $\binom{p}{d}$ ways. For any choice of d roots, there are exactly $p-1$ polynomials of degree $\leq d$ with those roots, because given the value of the polynomial on any additional point, the polynomial is determined uniquely. Note that the value of the polynomial on the additional point cannot be zero (since that would result in the polynomial being identical to zero, which has more than d roots). Therefore the value of the polynomial on the additional point can be any of $1,2,\dots,p-1$.

(i) $\boxed{0}$ If there was such an x, then 12 would divide $4x-6 = 2(2x-3)$. But note that $2x-3$ is an odd number, therefore $2(2x-3)$ is not divisible by 4 and therefore it's not divisible by 12 either.

(j) $\boxed{4}$ We can use the Chinese remainder theorem. Note that $36 = 4 \times 9$. So we just need to find out the number of values mod 4 that satisfy the equation and the number of values mod 9 that satisfy the equation and multiply the two together (because values mod 36 correspond to pairs of values mod 4 and mod 9). When looking at the equation mod 4, it becomes $0 = 0 \pmod 4$ which always holds. So all values mod 4 are answers (there are 4 of them). When looking at the equation mod 9, note that 8 has a multiplicative inverse (itself). So there the equation becomes $8x = 4 \pmod 9$ which is equivalent to $x = 5 \pmod 9$. So there is one answer mod 9. Therefore the total number of answers mod 36 is $4 \times 1 = 4$.