1. **Monty Hall Again**

In the three-door Monty Hall problem, there are two stages to the decision, the initial pick followed by the decision to stick with it or switch to the only other remaining alternative after the host has shown an incorrect door. An extension of the basic problem to multiple stages goes as follow.
Suppose there are four doors, one of which is a winner. The host says: "You point to one of the doors, and then I will open one of the other non-winners. Then you decide whether to stick with your original pick or switch to one of the remaining doors. Then I will open another (other than the current pick) non-winner. You will then make your nal decision by sticking with the door picked on the previous decision or by switching to the only other remaining door.

(a) How many possible strategies are there?

Answer: In the original Monty Hall problem, there was only two strategies - the sticking strategy where you would stick with your original pick after Monty opens the first door, or the switching strategy, where you would change your original pick after Monty opens the first door. Now, there are two stages where a decision needs to be made, and at each stage there are two choices. Either you can stick with your current door, or you can switch to another door (at stage one where there are two possible doors to choose from, choose one of them randomly). So, in all, there are $2 \cdot 2 = 4$ different strategies.

(b) Find the best strategy and compute its probability of winning. You can do this using any method you want. Enumeration is a valid approach, but a less tedious method is to consider a simpler problem: what if you have a 3-door monty hall problem, except the probability of picking the right door at first is not $\frac{1}{3}$ but some general value p? What is the best strategy if $p = 0$? What is the best strategy if $p = 1$? Having done that, how would you reduce the 4-door problem to the above 3-door problem?

Answer:
Solution 1
We calculate probability of winning given that we play with a specic strategy. We use RRR to denote picking the right door all 3 times. WRW then means picking the wrong door for the first time, right for the second time, and wrong for the third time. Thus, we win if the third letter is R.

Note the notation of P(WWW ; S1), which means "probability of WWW under strategy S1", instead of P(WWW | S1), i.e. "probability of WWW conditioned on S1", because S1 is not random.

S1: Stick and stick strategy

Case 1 (RRR): Pick the right door at the beginning and stick with it, so we pick the right door all three times. **P(RRR ; S1) = P(pick the right door at the beginning) = 1/4**

Case 2 (WWW): Pick one of the wrong doors at the beginning. **P(WWW ; S1) = 3/4**

Notice that the sum of the two cases is 1, so we do not miss any case.

The reason that we only need to consider 2 cases is because we stick to the same door throughout, so we either always picked the right door, or always picked the wrong door.

P(win ; S1) = P(RRR;S1) = 1/4

S2: Stick and switch strategy

Case 1 (RRW): pick the right one at the beginning. **P(RRW ; S2) = 1/4**

Case 2 (WWR): Pick one of the wrong doors at the beginning. Then, the host will open another wrong one. We first stick with our door, so the host will open yet another wrong door. Thus, we are left with the right door to switch to. In short, if we pick a wrong one at the beginning, it is guaranteed that we will pick the right one at the end with this strategy. **P(WWR; S2) = P(pick a wrong door at the beginning) = 3/4**.

P(win ; S2) = P(WWR;S2) = 3/4

As an example, we will formally show the derivation of P(RRW; S2):

- P(RR ; S2) = P(R ; S2) = P(pick the right door at the beginning) = 1/4 since we are sticking the first round (so picking the right door at first = picking the right door the first 2 times), and the probability is just 1/4.

- P(RRW | RR; S2) = 1, because conditioned on us picking the right door, the only other door that remains closed must be wrong, so we are guaranteed to pick the wrong door if we switch.

- P(RRW; S2) = P(RR; S2)*P(RRW|RR; S2) = 1/4 * 1 = 1/4 by bayes rule.

For P(WWR; S2): - P(WW; S2) = P(W; S2) = 3/4, once again because we stick the first - P(RRW | RR; S2) = 1, because conditioned on us picking the right door, the only other door that remains closed must be right, so we are guaranteed to pick the right door if we switch.

- P(WWR; S2) = P(WWR|WW; S2) * P(WW; S2) = 1 * 3/4 = 3/4 by bayes rule.

Similar reasoning applies to the other cases.

S3: Switch and stick strategy

Case 1: **P(RWW ; S3) = 1/4**

Case 2: **P(WWW ; S3) = 3/4 * 1/2**. This comes from 3/4 probability of picking one of the wrong doors at first. Host opens another wrong door. Now there are 2 doors left, so the probability of picking the wrong door the second time is 1/2, then we stick with the wrong door.

Case 3: **P(WRR ; S3) = 3/4 * 1/2**. Follow the similar logic as case 2.

P(win ; S3) = P(WRR ; S3) = 3/8

S4: Switch and Switch strategy

Case 1: **P(RWR ; S4) = 1/4**, which is the probability of picking the right door at the first time. If we pick the right one the first time, host opens one of the wrong doors. We then switch to another wrong door. Host opens the last remaining wrong door. Finally, we can only switch to the original door, which is the right door.

Case 2: **P(WWR ; S4) = 3/4 * 1/2**. This come from 3/4 probability of picking one of the wrong doors at first. Host opens another wrong door. There are 2 doors left, so the probability of picking the wrong door the second time is 1/2. Since our second pick is wrong, the host must open the door we pick the first time, so we will switch to the only one door left, which is the right door.

Case 3: **P(WRW ; S4) = 3/4 * 1/2**. Follow the similar logic as case 2.

P(win ; S4) = P(RWR ; S4) + P(WWR ; S4) = 5/8

Thus, stick-and-switch strategy is the best.

Solution 2

Let Alice be a player and Bob be a host. There are four doors: 1, 2, 3, and 4. We let the first door Alice picks be door 1 without loss of generality.

Assume Alice always sticks to door 1. She has a probability $1/4$ of winning.

Assume Alice sticks to door 1 and then switches the second time. After Alice has picked door 1 twice, Bob will open two other doors that do not have a prize. Since the probability of the prize not behind door 1 is $3/4$, and we know that the other two doors do not have a prize for sure, the other door left thus has $3/4$ probability of having the prize behind it. Therefore, Alice has $3/4$ probability of winning.

Assume Alice switches to one of the two other doors that Bob did not open. The probability that she picks the door that has the prize is $(3/4) \times (1/2)$. Indeed, this happens if the prize is not behind door 1 (probability $3/4$) and if she also picks the one door out of two that has the prize (probability $1/2$).

If Alice sticks to her second choice, she then has a probability $(3/4) \times (1/2) = 3/8$ of winning.

Assume she switches again after Bob shows her a second door that has no prize. The claim is that she wins with probability $5/8$. To see this, note that she always wins if her first choice was the door with the prize. Indeed, say the prize was behind door 1 that Alice picks. Bob opens door 4, Alice switches to door 2, Bob opens door 3, Alice switches back to door 1. Also, we claim that if Alice had not picked the correct door, then she wins with probability $1/2$. Indeed, say the prize is not behind door 1 that Alice first picks. Bob opens one of the doors (2, 3, 4), say door 4. Alice picks door 2. With probability $1/2$, the prize is behind door 2 and Bob opens door 3, Alice switches to door 1 and loses. With probability $1/2$, the prize is behind door 3, Bob opens door 1, Alice switches to door 3 and wins. Thus, if she switches twice, Alice wins with probability $1/4 + (3/4)(1/2) = 5/8$.

Among these four strategies, stick and switch strategy has the highest probability of winning.

2. **Drunk man**

Imagine that you have a drunk man moving along the horizontal axis (that stretches from $x = -\infty$ to $x = +\infty$). At time $t = 0$, his position on this axis is $x = 0$. At each time point $t = 1$, $t = 2$, etc., the man moves forward (that is, $x(t+1) = x(t) + 1$) with probability 0.5, backward (that is, $x(t+1) = x(t) - 1$) with probability 0.3, and stays exactly where he is (that is, $x(t+1) = x(t)$) with probability 0.2.

(a) What are all his possible positions at time t, $t \geq 0$?

Answer: Clearly, by time t, the man could have moved at most t positions to the right, and at most t positions to the left. Furthermore, within this range $[-t, t]$, the man could be occupying any integer position. Therefore, the possible values for the position $x(t)$ of the man at time t are exactly the integers in the closed range $[-t, t]$.

(b) Calculate the probability of each possible position at $t = 1$.

Answer: Clearly, at time $t = 1$, the man could be either in position -1, or in position 0, or in position 1. We know the man starts at position 0 at $t = 0$, and at time $t = 1$, he has taken at most 1 step; if this step were taken backward (w.p. 0.3), he would be in position -1, and if this step were forward (w.p. 0.5), he would be in position $+1$. And if he had chosen to remain wherever he was (w.p. 0.2), he would be in position 0. There is no other way he could have been in any of these positions. So, his possible positions are $[-1,\ 0,\ 1]$, with probabilities $[0.3,\ 0.2,\ 0.5]$ respectively.

(c) Calculate the probability of each possible position at $t = 2$.

Answer: From the discussion above, at time $t = 2$, the man can be in any one of the 5 positions $[-2,\ -1,\ 0,\ 1,\ 2]$. The probability associated with each of these positions can be calculated from the probabilities that we just computed above (for the man's position at time $t = 1$).

For example, what is the probability that the man is in position -2 at time 2? Clearly, this can happen under only one circumstance: the man should have been in position -1 at time 1, and moved backwards at time 2. Thus we have:

$$\begin{aligned}
P(x(2) = -2) &= P\left(x(1) = -1 \cap \text{man moves backward at } t = 2\right) \\
&= P\left(\text{man moves backward at } t = 2 \mid x(1) = -1\right) \times P(x(1) = -1) \\
&= 0.3 \times 0.3 \\
&= 0.09.
\end{aligned}$$

In general, using the law of total probability, we can write the probability that the man is in position i at time $t+1$ as

$$\begin{aligned}
&P(x(t+1) = i) \\
=&P(x(t+1) = i|x(t) = i-1) \times P(x(t) = i-1) + P(x(t+1) = i|x(t) = i+1) \times P(x(t) = i+1) \\
&+ P(x(t+1) = i|x(t) = i) \times P(x(t) = i) \\
=&0.5 \times P(x(t) = i-1) + 0.3 \times P(x(t) = i+1) + 0.2 \times P(x(t) = i).
\end{aligned} \tag{1}$$

Now, using (1),

$$\begin{aligned}
&P(x(2) = -1) \\
=&P(x(2) = -1|x(1) = -2) \times P(x(1) = -2) + P(x(2) = -1|x(1) = 0) \times P(x(1) = 0) \\
&+ P(x(2) = -1|x(1) = -1) \times P(x(1) = -1) \\
=&0 + 0.3 \times 0.2 + 0.2 \times 0.3 \\
=&0.12.
\end{aligned}$$

Doing this for the rest of the values, we get

$$\begin{aligned}
&P(x(2)=0)\\
=&P(x(2)=0|x(1)=-1)\times P(x(1)=-1)+P(x(2)=0|x(1)=1)\times P(x(1)=1)\\
&+P(x(2)=0|x(1)=0)\times P(x(1)=0)\\
=&0.5\times 0.3+0.3\times 0.5+0.2\times 0.2\\
=&0.34.
\end{aligned}$$

$$\begin{aligned}
&P(x(2)=1)\\
=&P(x(2)=1|x(1)=0)\times P(x(1)=0)+P(x(2)=1|x(1)=2)\times P(x(1)=2)\\
&+P(x(2)=1|x(1)=1)\times P(x(1)=1)\\
=&0.5\times 0.2+0+0.2\times 0.5\\
=&0.2.
\end{aligned}$$

$$\begin{aligned}
&P(x(2)=2)\\
=&P(x(2)=2|x(1)=1)\times P(x(1)=1)+P(x(2)=2|x(1)=2)\times P(x(1)=2)\\
=&0.5\times 0.5+0\\
=&0.25.
\end{aligned}$$

Notice that the 5 probabilities above add up to 1, as we would expect.

(d) Calculate the probability of each possible position at $t=3$.

Answer: From the discussion above, at time $t=3$, the man can be in any one of the 7 positions -3, -2, -1, 0, 1, 2, or 3. The probability associated with each of these positions can be calculated from the probabilities that we just computed above (for the man's position at time $t=2$).

The calculations are carried out in exactly the same way as in the previous part, by considering all possible ways in which the man can occupy position x at time 3, for each x satisfying $-3\le x\le 3$.

$$\begin{aligned}
&P(x(3)=-3)\\
=&P(x(3)=-3|x(2)=-2)\times P(x(2)=-2)+P(x(3)=-3|x(2)=-3)\times P(x(2)=-3)\\
=&0.3\times 0.09+0\\
=&0.027.
\end{aligned}$$

$$\begin{aligned}
&P(x(3)=-2)\\
=&P(x(3)=-2|x(2)=-3)\times P(x(2)=-3)+P(x(3)=-2|x(2)=-1)\times P(x(2)=-1)\\
&+P(x(3)=-2|x(2)=-2)\times P(x(2)=-2)\\
=&0.3\times 0.12+0.2\times 0.09\\
=&0.054.
\end{aligned}$$

Proceeding in a similar fashion, the probabilities for the man to be in positions -3, -2, -1, 0, 1, 2, and 3 are 0.027, 0.054, 0.171, 0.188, 0.285, 0.15, and 0.125 respectively for $t=3$. Again, as expected, these probabilities add up to 1.

(e) If you know the probability of each position at time t, how will you nd the probabilities at time $t+1$?

Answer: The answerution to the previous part of the problem suggests a nice algorithm for computing the probability of each position the man can take at time $t+1$, provided these probabilities are known for time t.

Let X_t be the list of all possible positions that the man can be in at time t. From the arguments above, we know that:

$$X_t = [-t,\ -(t-1),\ \ldots,\ -1,\ 0,\ 1,\ \ldots,\ (t-1),\ t]\ .$$

Let Pt denote a list of probabilities corresponding to the positions X_t. Our goal is to nd a way to calculate P_{t+1} (the *next probabilities*) given P_t (the *current probabilities*).

```
#! /usr/bin/env python2

import sys

def next_pvec (pvec, pf, pb, pc):

    qvec = []
    for idx in range(len(pvec)+2):
        q = pvec[idx]*pb if 0 <= idx < len(pvec) else 0
        q += pvec[idx-1]*pc if 0 <= idx-1 < len(pvec) else 0
        q += pvec[idx-2]*pf if 0 <= idx-2 < len(pvec) else 0
        qvec.append(q)

    return qvec


if __name__ == '__main__':

    (pf, pc, pb) = (0.5, 0.2, 0.3)
    tf = int(sys.argv[1])

    (t, pvec) = (0, [1])
    while t < tf:
        pvec = next_pvec (pvec, pf, pb, pc)
        t += 1

    print(pvec)
```

The figure above shows Python code for calculating the above next probabilities. The function `next_pvec` takes as input the current list of probabilities `pvec` (at time t), and values for `pf`, `pb`, and `pc` (the forward, backward, and "stay put" probabilities), and it produces as output a list of the next probabilities (at time $t+1$).

First of all, observe that the length of the list X_{t+1}, and hence P_{t+1} is two more than the length of X_t (hence P_t). This is because, at time $t+1$ the man can be in two additional possible positions that he could not have been in at time t.

Also, for each possible position at time $t+1$, there are at most 3 possible positions the man could have been in at time t. Therefore, the rules described above for multiplying the relevant probabilities and adding up these products generalize quite readily.

Thus, given the positional probabilities at time t, the man's positional probabilities at time $t+1$ can be readily calculated. And the man's initial position is known to be $x(0) = 0$. Therefore, starting from this initial condition, the positional probabilities can be calculated at any desired future time. Indeed, the `main` part of the above program does exactly this; it accepts a future time `tf` from the user and prints out a list of probabilities corresponding to every possible position the man can be in at time `tf`.

Note: Those of you who are familiar with Linear Algebra will readily recognize that the "next probabilities" list is simply a linear combination of the "current probabilities" list, which corresponds to pre-multiplying the current probabilities list by a (tall and thin) rectangular matrix. Indeed, this idea can be used to considerably speed-up the probability calculations above.

The Drunk Man has regained some control over his movement, and no longer stays in the same spot; he only moves forwards or backwards. More formally, let the Drunk Man's initial position be $x(0) = 0$. Every second, he either moves forward one pace or backwards one pace, *i.e.*, his position at time $t+1$ will be one of $x(t+1) = x(t)+1$ or $x(t+1) = x(t)-1$.

We want to compute the number of paths in which the Drunk Man returns to 0 at time t and it is his first return, *i.e.*, $x(t) = 0$ and $x(s) \neq 0$ for all s where $0 < s < t$. Note, we **no longer** care about probabilities. We are just counting paths here.

(a) How many paths can the Drunk Man take if he returns to 0 at $t = 6$ and it is his first return?

Answer: We use an "F" to represent that the Drunk Man moves forward one pace and a "B" to represent that the Drunk Man moves backward one pace.

$\boxed{4}$ possible paths: FFFBBB, FFBFBB, BBBFFF, and BBFBFF. The last two paths can also be obtained by exchanging F's and B's in the first two paths.

(b) How many paths can the Drunk Man take if he returns to 0 at $t = 7$ and it is his first return?

Answer: $\boxed{0}$ possible path because it needs the same number of forward paces and backward paces.

(c) How many paths can the Drunk Man take if he returns to 0 at $t = 8$ and it is his first return?

Answer: $\boxed{10}$ possible paths: FFFFBBBB, FFFBFBBB, FFFBBFBB, FFBFFBBB, FFBFBFBB, and the other ve by exchanging F's and B's.

(d) How many paths can the Drunk Man take if he returns to 0 at $t = 2n+1$ for $n \in \mathbb{N}$ and it is his first return?

Answer: $\boxed{0}$ possible path because it needs the same number of forward paces and backward paces.

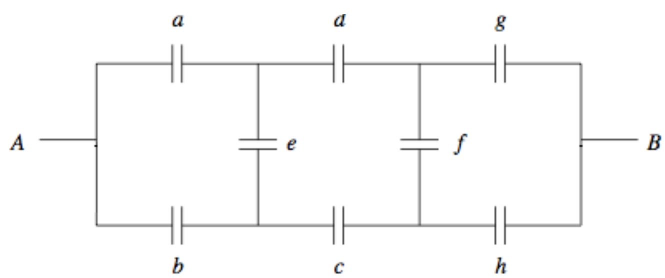
(e) How many paths can the Drunk Man take if he returns to 0 at $t = 2n+2$ for $n \in \mathbb{N}$ and it is his first return?

Answer: From Wikipedia, "a Catalan number C_n is the number of monotonic paths along the edges of a grid with $n \times n$ square cells, which **do not pass above** the diagonal. A monotonic path is one which starts in the lower left corner, nishes in the upper right corner, and consists entirely of edges pointing rightwards or upwards." We can regard an F as an edge pointing rightwards, a B as an edge pointing upwards, and a possible path as a monotonic path of a grid with $(n+1) \times (n+1)$ square cells, which **does not touch** the diagonal. If the first pace is an

F, then the $(2n+2)$-th path (the last pace) must be B; otherwise, the Drunk Man must have returned 0 before $t = 2n+2$. Therefore, in this case, we can focus on the second pace to the $(2n+1)$-th pace, and the number of possible paths is C_n because it is the number of monotonic paths, which do not pass above the diagonal of $n \times n$ square cells, *i.e.*, do not touch the diagonal of $(n+1) \times (n+1)$ square cells if the first pace is an F and the last pace is a B. Considering the other case that the first pace is a B and the last pace is an F, the total number of possible paths is $2C_n = \boxed{\frac{2}{n+1}\binom{2n}{n}}$.

3. **Communication network**

   In the communication network shown below, link failures are independent, and each link has a probability of failure of p. Consider the physical situation before you write anything. A can communicate with B as long as they are connected by at least one path which contains only in-service links.



   (a) Given that exactly 5 links have failed, determine the probability that A can still communicate with B.

   (b) Given that exactly 5 links have failed, determine the probability that either g or h (*but not both*) is still operating properly.

   (c) Given that a, d and h have failed (but no information about the information of the other links), determine the probability that A can communicate with B.

   Answer:

   (a) There are only two paths of 3 links from A to B. And there are $\binom{8}{5}$ ways of the links messing up. So the probability is $\frac{2}{56} = \frac{1}{28}$.
   This is because every single case of exactly 5 links being down have the same probability. So it's a uniform distribution over all possibilities.

   (b) Fix g as down and h as working. There are $\binom{6}{4}$ ways to have 4 out of the remaining go down. Symmetric argument for h down and g up.
   So probability is $\frac{30}{56} = \frac{15}{28}$.

   (c) We would just want the 4 on the only remaining path from A to B not to be down.
   The probability of this happening is $(1-p)^4$.