CS 70	Discrete Mathematics and Probability Theory
Spring 2016	Rao and Walrand — Discussion 9B Sol

1. **A roulette of apples** You bought 20 apples at your local farmer's market. Due to the organic nature of the apples, they are infested with worms. In particular, each apple contains a single worm with probability 0.3 (mutually independent). (**leave answers unevaluated**)

a) You pick up two apples a_1 and a_2.

i) What is the probability that there are exactly 2 worms? What is the probability that there is exactly 1 worm? 0 worms?
$P(\text{2 worms}) = 0.09$, $P(\text{1 worm}) = 0.42$, $P(\text{0 worms}) = 0.49$

ii) What is $P(a_1 \text{ has worm} | a_2 \text{ has worm})$?
0.3 by independence

iii) What is $P(a_1 \text{ has worm} | \text{there is exactly 1 worm among } a_1 \text{ and } a_2)$?
$\frac{P(a_1 \text{ has worm, } a_2 \text{ has no worm})}{P(\text{exactly 1 worm})} = \frac{0.3*0.7}{0.42} = 0.5$

b) You eat all 20 apples.

i) What is the probability that you end up eating no worms?
$0.7^{20} \approx 0.0008$

ii) what is the probability that you end up eating exactly 1 worm?
$\binom{20}{1} * 0.7^{19} * 0.3 \approx 0.007$

iii) what is the probability that you end up eating exactly 2 worms?
$\binom{20}{2} * 0.7^{18} * 0.3^2 \approx 0.028$

iv) How many apples can you eat if you want the probability of eating no worms to be at least 0.2?
Let k be the number of apples you eat. We want $0.7^k > 0.2$ so $k < \frac{\log(0.2)}{\log(0.7)} \approx 4.5$, so you can eat at most 4 apples.

c) You pick a single apple at random and slice it into 3 slices. If the apple has a worm, it will be hidden in one of the slices. You bravely eat the slices one by one. Let s_1, s_2, s_3 denote the three slices.

i) What is $P(s_1 \text{ has no worm} | \text{apple has worm})$
$2/3$

ii) What is $P(s_1 \text{ has no worm})$?

$$\begin{aligned}
& P(s_1 \text{ has no worm}) \\
\textbf{(by total probability)} = \; & P(s_1 \text{ has no worm} \mid \text{apple has no worm}) \times P(\text{apple has no worm}) \\
& + P(s_1 \text{ has no worm} \mid \text{apple has worm}) \times P(\text{apple has worm}) \\
= \; & 1 \times 0.7 + 2/3 \times 0.3 \\
= \; & 0.9
\end{aligned}$$

iii) What is $P(\text{apple has no worm} | s_1 \text{ has no worm})$. Compare your answer to $P(\text{apple has no worm})$.
$\frac{P(\text{apple has no worm, } s_1 \text{ has no worm})}{P(s_1 \text{ has no worm})} = \frac{P(\text{apple has no worm})}{P(s_1 \text{ has no worm})} = \frac{0.7}{0.9} \approx 0.778$

Why is the first equality true? Because if the apple has no worm means that the first slice has no worm. We can see this using a similar argument to ii)

$$\begin{aligned}&P(\text{apple has no worm , } s_1 \text{ has no worm})\\&= P(\text{apple has no worm}) * P(s_1 \text{ has no worm | apple has no worm})\\&= P(\text{apple has no worm}) * 1\\&= P(\text{apple has no worm})\end{aligned}$$

Note how it is higher than $P(\text{apple has no worm}) = 0.7$

iv) What is $P(s_2 \text{ has no worm}|s_1 \text{ has no worm})$ Compare your answer to $P(s_2 \text{ has no worm})$.
$P(s_2 \text{ has no worm, } s_1 \text{ has no worm}) = 0.7 + 0.3 * 1/3 = 0.8$
note that $0.3 * 1/3$ came from $P(\text{apple has worm}) * P(s_1 \text{ has no worm, } s_2 \text{ has no worm | apple has worm})$
Using bayes rule, we get:
$P(s_2 \text{ has no worm, } s_1 \text{ has no worm}|s_1 \text{ has no worm}) = \frac{P(s_2 \text{ has no worm, } s_1 \text{ has no worm})}{P(s_1 \text{ has no worm})} = \frac{0.8}{0.9} \approx 0.8888$
Interestingly, the probability got lower!

v) Using your previous answer, what is the safest way to eat 2 slices of apples?
Take 2 apples, slice each into 3 slices, eat one slice from each apple.

2. Independence in balls and bins

You have k balls and n bins labelled $1, 2, \ldots, n$, where $n \geq 2$. You drop each ball uniformly at random into the bins.

a. What is the probability that bin n is empty?
$(\frac{n-1}{n})^k$

b. What is the probability that bin 1 is non-empty?
$1 - (\frac{n-1}{n})^k$

c. What is the probability that both bin 1 and bin n are empty?
$(\frac{n-2}{n})^k$

d. What is the probability that bin 1 is non-empty and bin n is empty?

$$\Pr[\text{bin 1 nonempty} \cap \text{bin } n \text{ empty}] = \Pr[\text{bin } n \text{ empty}] - \Pr[\text{bin 1 empty} \cap \text{bin } n \text{ empty}] = \left(\frac{n-1}{n}\right)^k - \left(\frac{n-2}{n}\right)^k$$

e. What is the probability that bin 1 is non-empty given that bin n is empty?

$$\Pr[\text{bin 1 nonempty} \mid \text{bin } n \text{ empty}] = \frac{\Pr[\text{bin 1 nonempty} \cap \text{bin } n \text{ empty}]}{\Pr[\text{bin } n \text{ empty}]} = \frac{\left(\frac{n-1}{n}\right)^k - \left(\frac{n-2}{n}\right)^k}{\left(\frac{n-1}{n}\right)^k} = 1 - \left(\frac{n-2}{n-1}\right)^k$$

f. What does this tell us about the independence of the two events, A: bin 1 is non-empty and B: bin n is non-empty?
If two events A and B are independent, we have $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$, which is equivalent to $\Pr[A] = \frac{\Pr[A \cap B]}{\Pr[B]} = \Pr[A \mid B]$. Since $\Pr[A \mid B] \neq \Pr[A]$ in this case, it shows us that these two events are not independent.