Today.

Polynomials.

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Secret Sharing.

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Polynomials

A **polynomial**

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0.$$

is specified by **coefficients** $a_d, \ldots a_0$.

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Polynomials $P(x)$ **with arithmetic modulo** p: [1] $a_i \in \{0, \ldots, p-1\}$ and

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0 \pmod{p},$$

for $x \in \{0, \ldots, p-1\}$.

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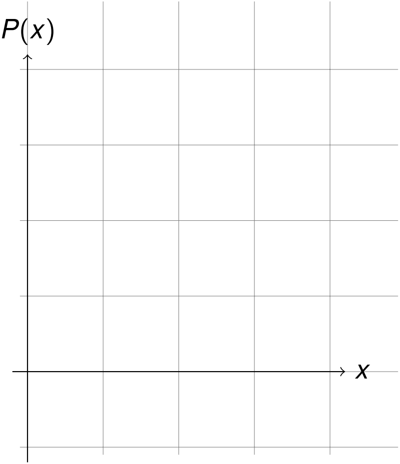

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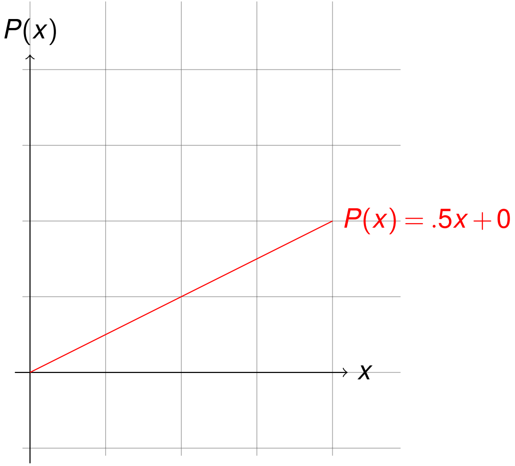

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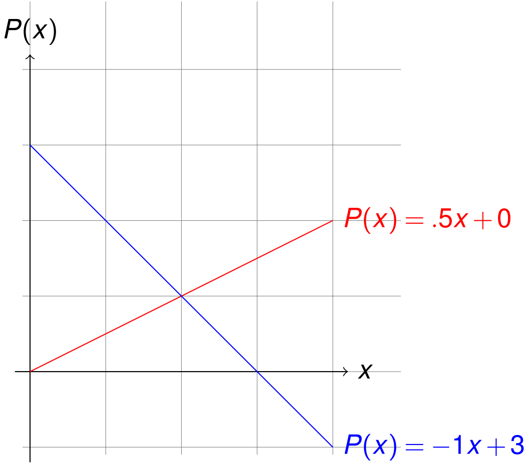

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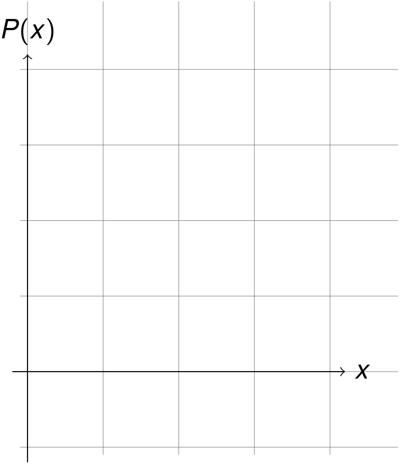

Parabola: $P(x) = a_2 x^2 + a_1 x + a_0$

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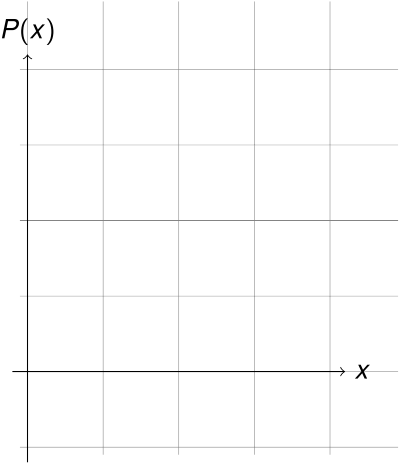

Parabola: $P(x) = a_2 x^2 + a_1 x + a_0 = ax^2 + bx + c$

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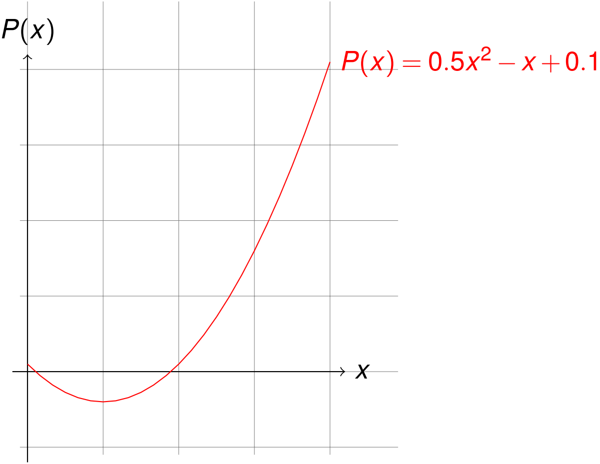

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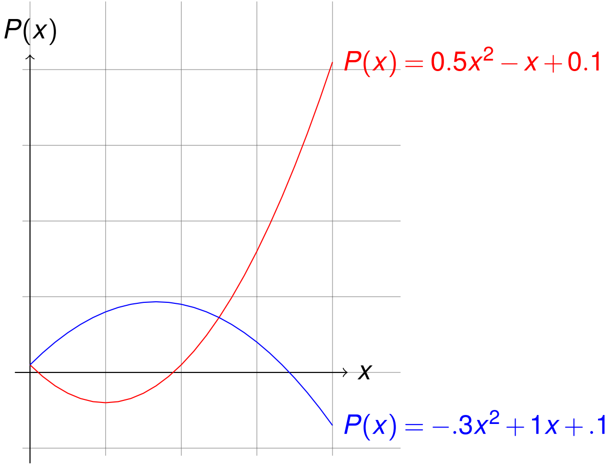

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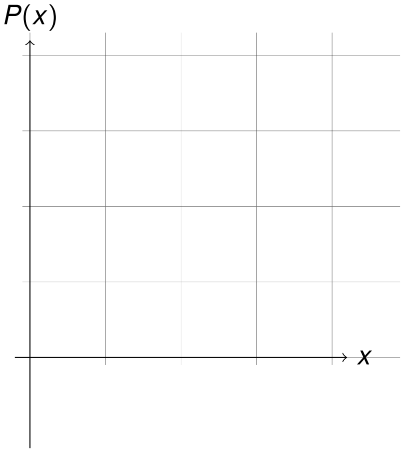

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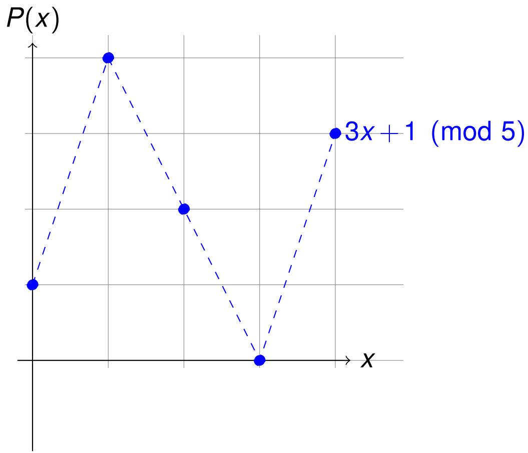

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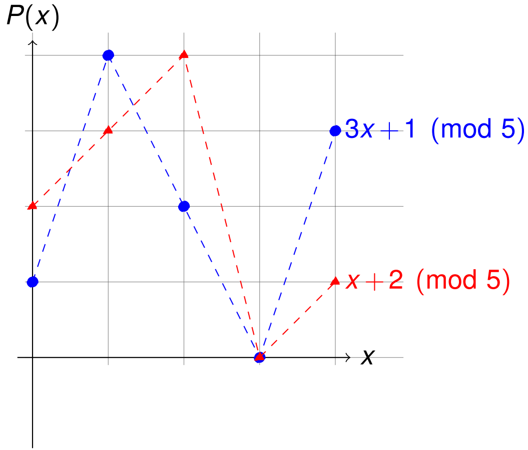

Finding an intersection.
$x + 2 \equiv 3x + 1 \pmod 5$
$\implies 2x \equiv 1 \pmod 5$

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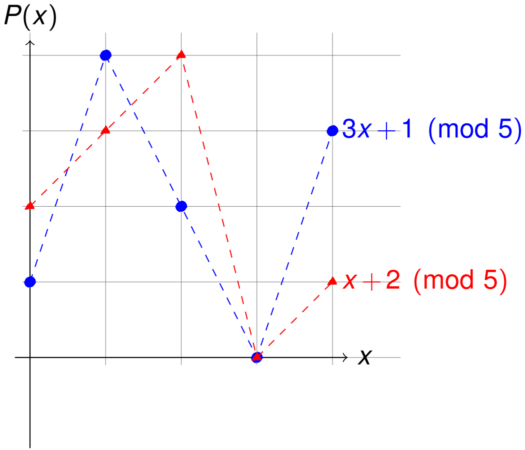

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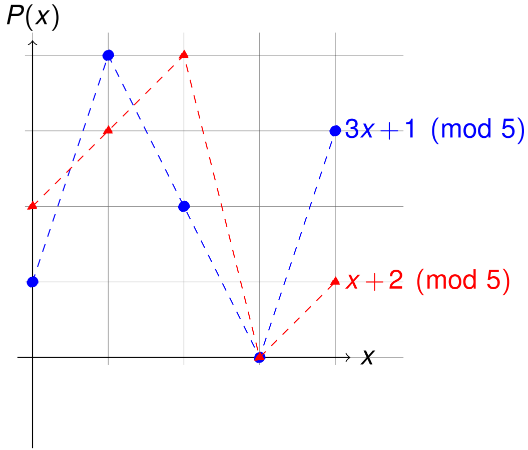

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Good when modulus is prime!!

Two points make a line.

Fact: Exactly 1 degree $\leq d$ polynomial contains $d+1$ points. [2]

[2] Points with different x values.

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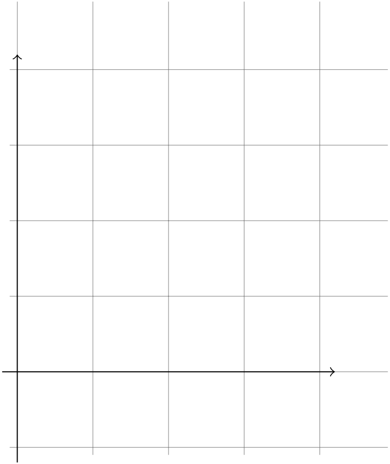

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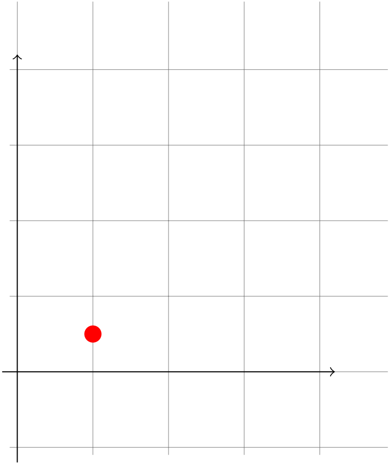

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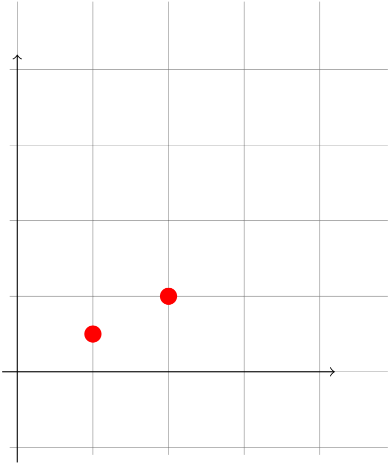

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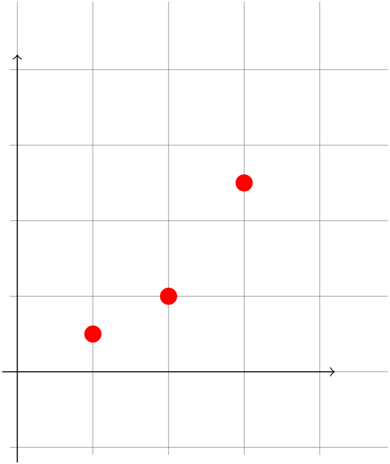

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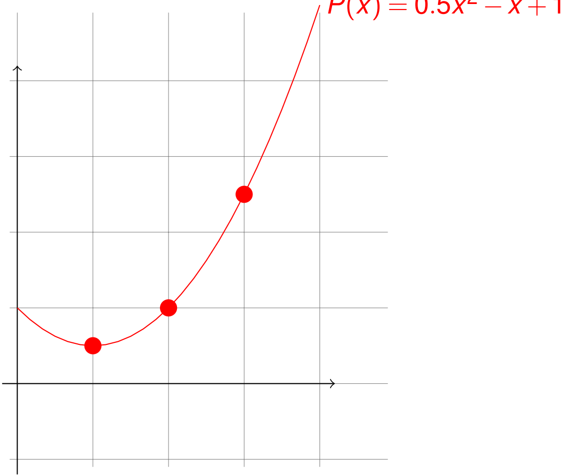

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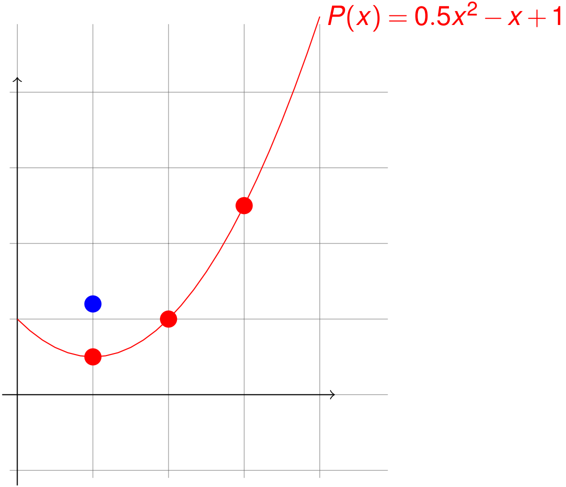

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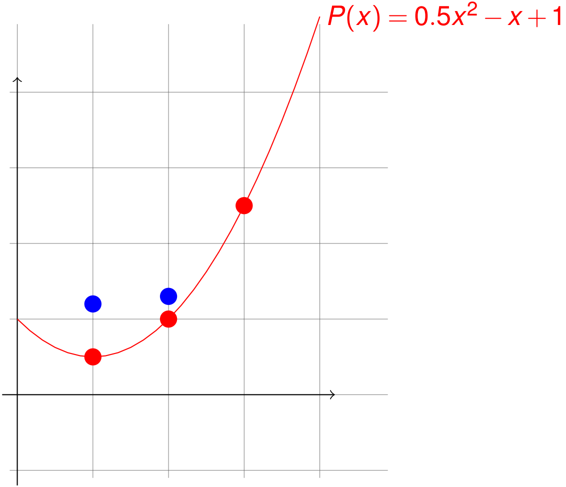

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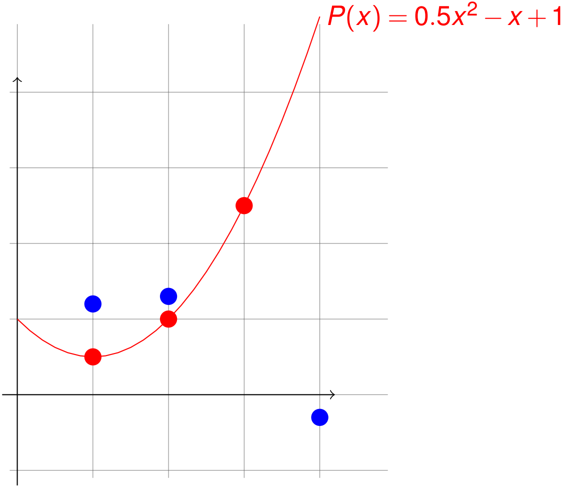

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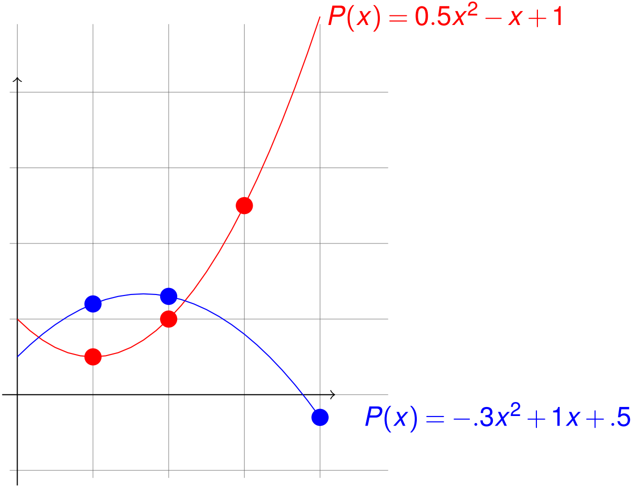

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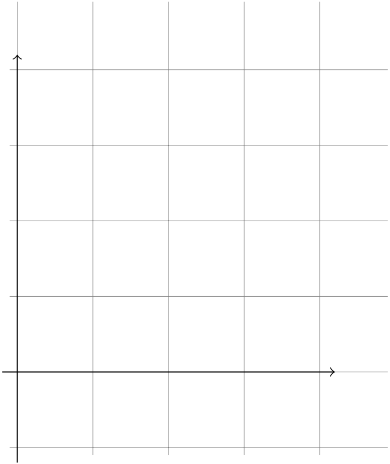

There is $P(x)$ contains blue points and *any* $(0, y)$!

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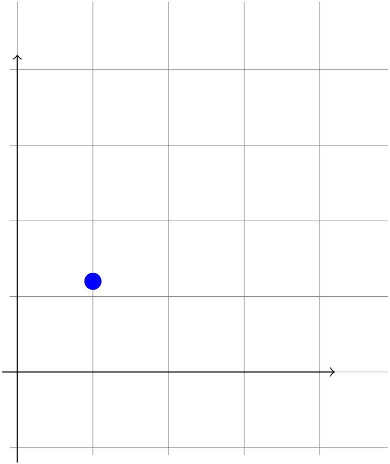

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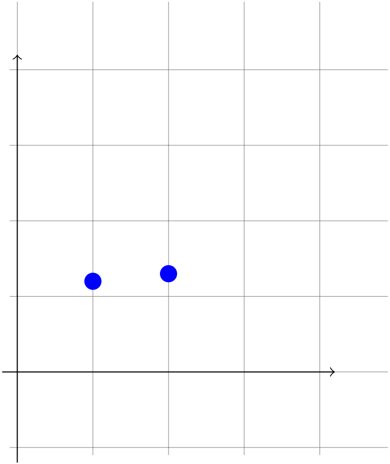

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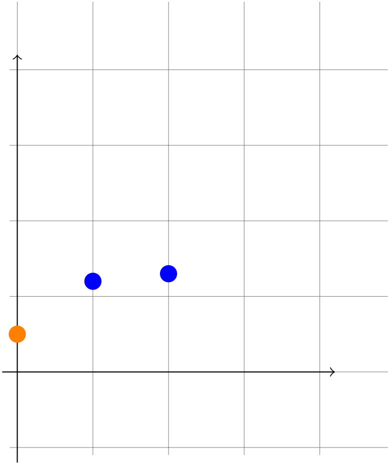

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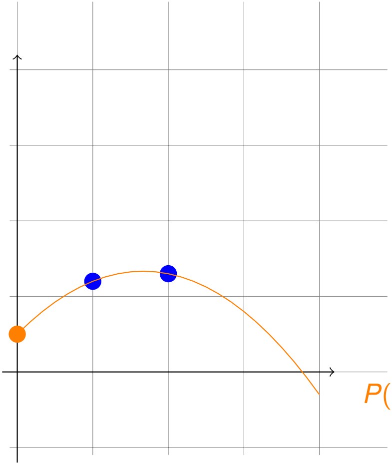

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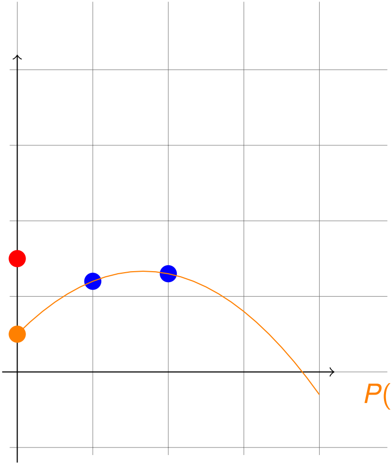

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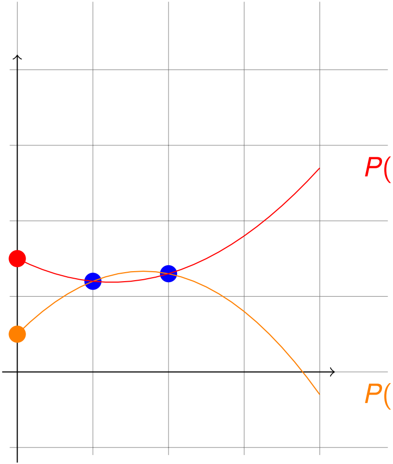

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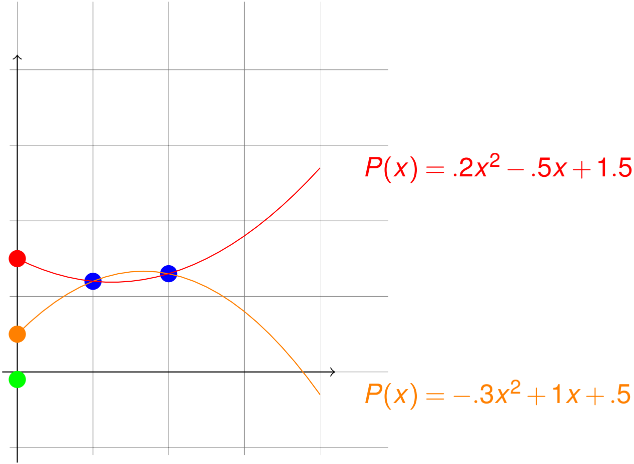

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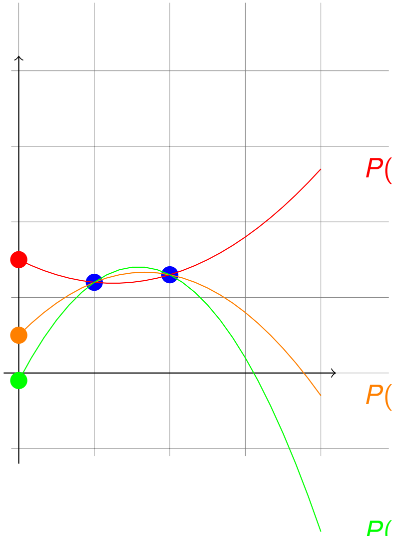

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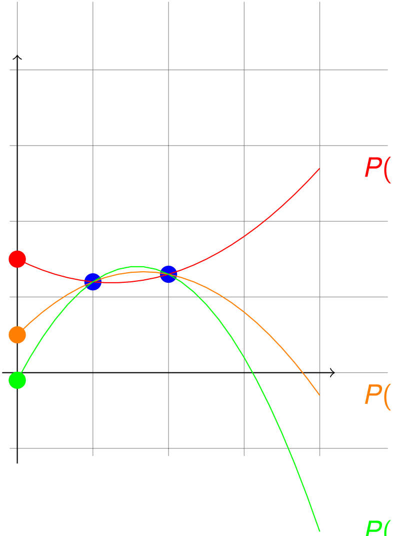

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Knowing k pts $\implies$ only one $P(x)$ $\implies$ evaluate $P(0)$.
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2. Let $P(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots a_0$ with $a_0 = s$.
3. Share i is point $(i, P(i) \mod p)$.

Roubustness: Any k shares gives secret.
Knowing k pts $\implies$ only one $P(x)$ $\implies$ evaluate $P(0)$.
Secrecy: Any $k-1$ shares give nothing.
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Modular Arithmetic Fact and Secrets

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Shamir's k out of n Scheme:
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We will work with polynomials with arithmetic modulo p.

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Construction proves the existence of a polynomial!

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Put the delta functions together.

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So polynomial is $2x^2 + 1x + 4 \pmod 5$

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Construction proves the existence of the polynomial!

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Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

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Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

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(Almost) the same as what is missing: one $P(i)$.

Runtime.

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Runtime: polynomial in k, n, and $\log p$.

1. Evaluate degree $k-1$ polynomial n times using $\log p$-bit numbers.
2. Reconstruct secret by solving system of k equations using $\log p$-bit arithmetic.

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Erasure Codes.

Satellite

GPS device

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Satellite

3 packet message.

GPS device

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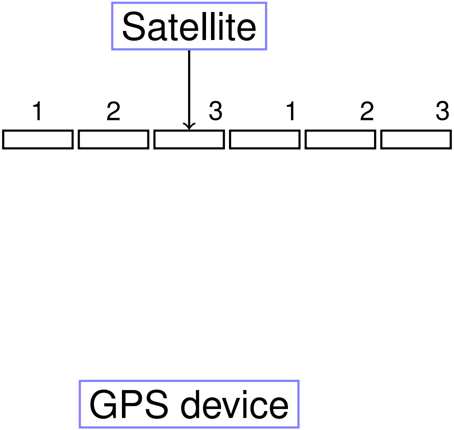
Satellite

3 packet message.

Lose 3 out 6 packets.

GPS device

Erasure Codes.



3 packet message. So send 6!

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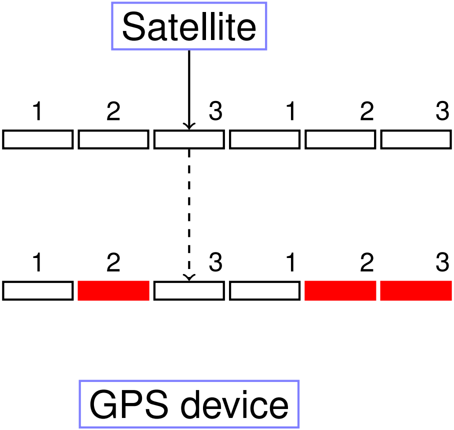

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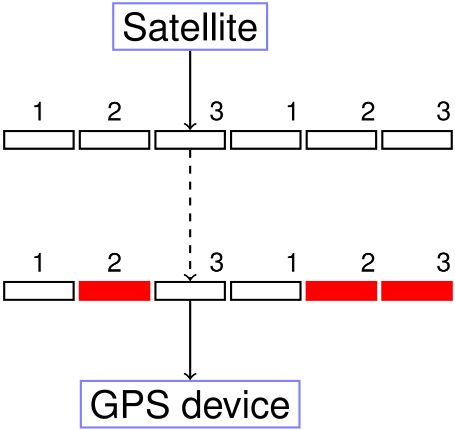

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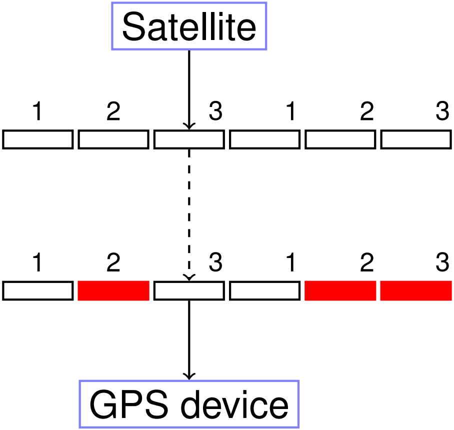

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Gets packets 1,1,and 3. :(

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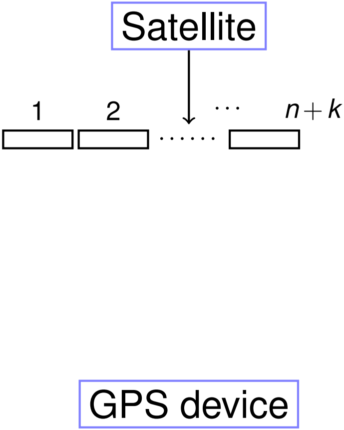

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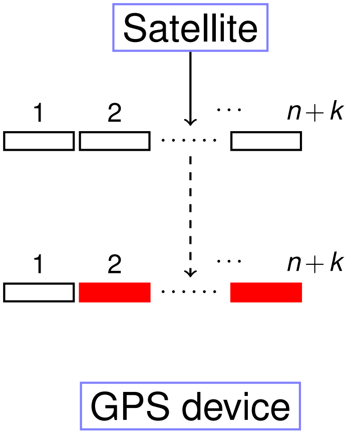

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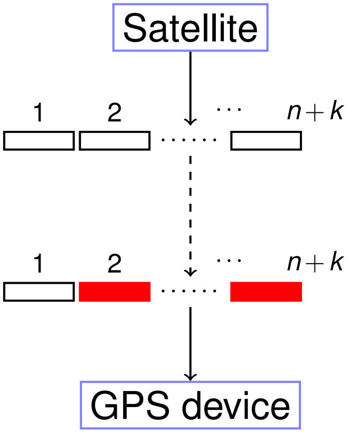

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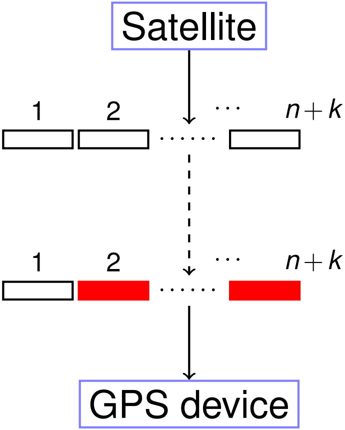

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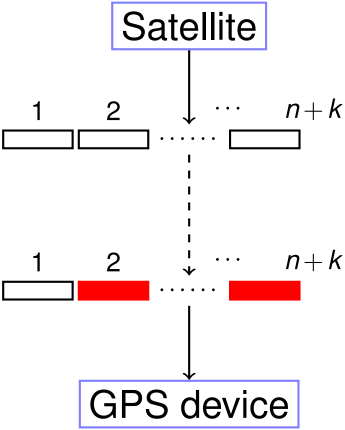

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Polynomials.

- ..give Secret Sharing.
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Next Time: Error Correction.

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Additional Challenge: Finding which packets are corrupt.

Erasure Codes.

Satellite

GPS device

Erasure Codes.

Satellite

3 packet message.

GPS device

Erasure Codes.

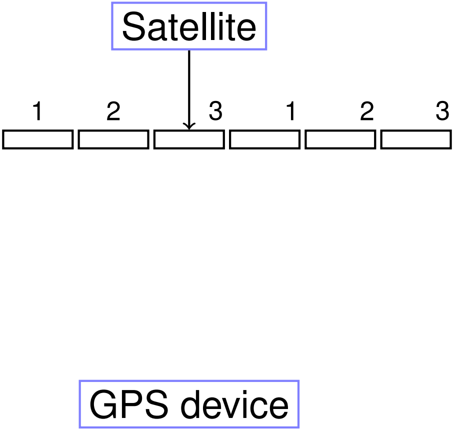
Satellite

3 packet message.

Lose 3 out 6 packets.

GPS device

Erasure Codes.



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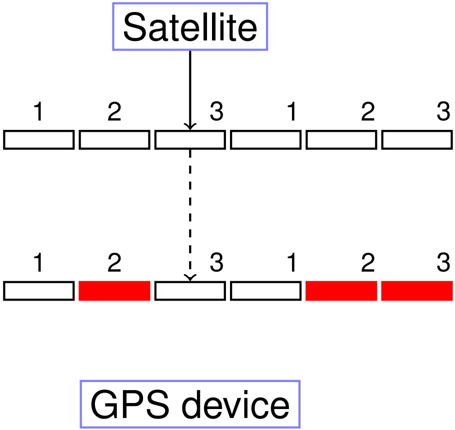

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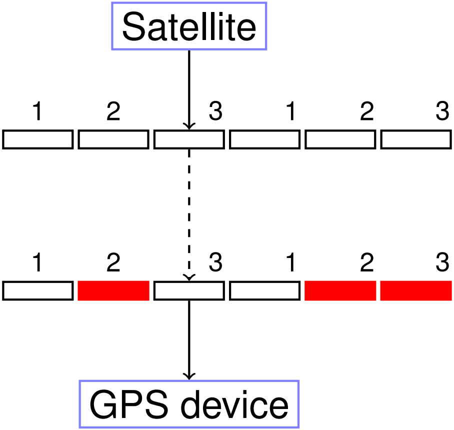

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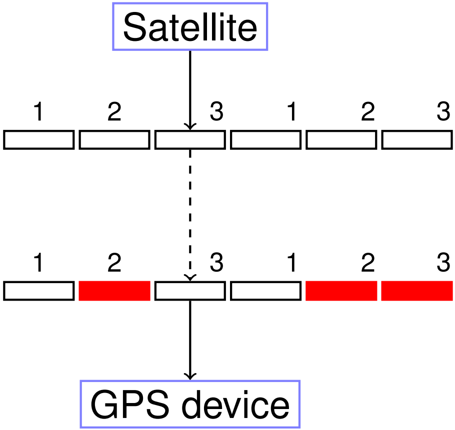

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n packet message, channel that loses k packets.

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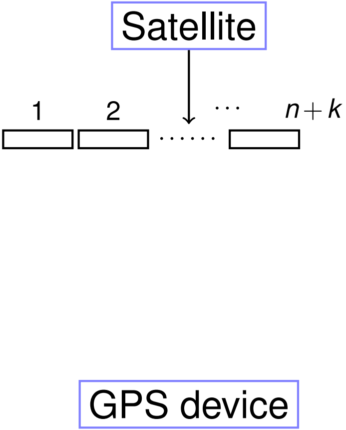

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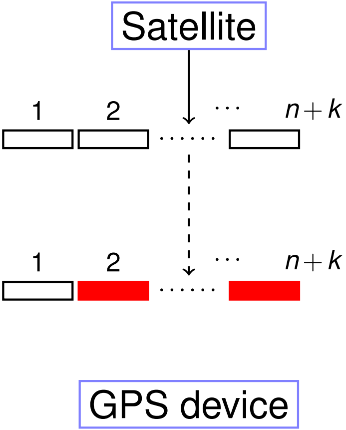

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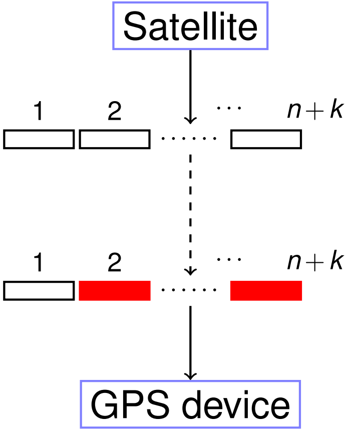

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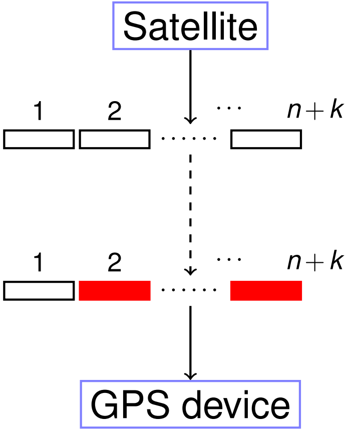

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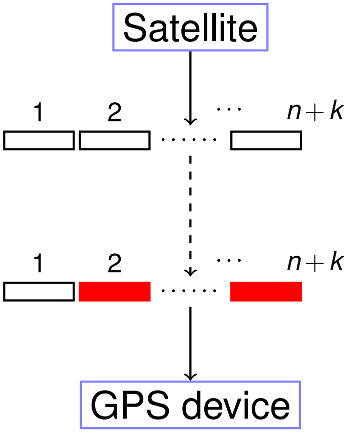

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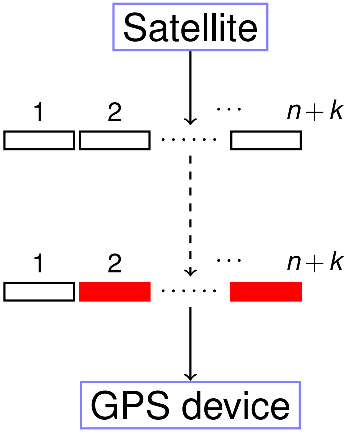

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Example

Make polynomial with $P(1) = 1$, $P(2) = 4$, $P(3) = 4$.

Modulo 7 to accommodate at least 6 packets.

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Packets: $(1,1),(2,4),(3,4),(4,7),(5,2),(6,0)$

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Notice that packets contain “x-values”.

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Channeling Sahai

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Message? $P(1) = 1, P(2) = 4, P(3) = 4.$

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You want to encode a secret consisting of 1,4,4.

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Send n packets b-bit packets, with k errors.

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Send n packets b-bit packets, with k errors.
Modulus should be larger than $n+k$ and also larger than 2^b.

Polynomials.

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Additional Challenge: Finding which packets are corrupt.

Error Correction

Satellite

GPS device

Error Correction

Satellite

3 packet message.

GPS device

Error Correction

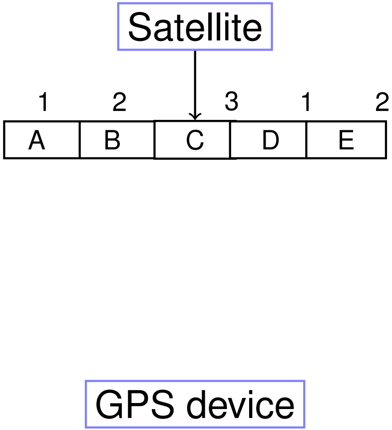
Satellite

3 packet message.

Corrupts 1 packets.

GPS device

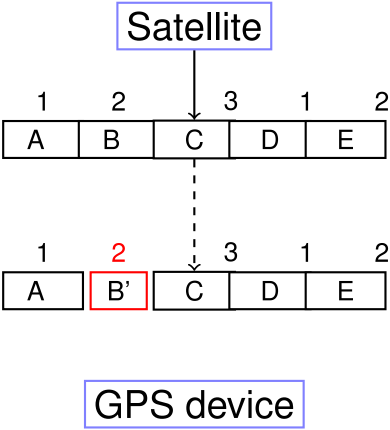
Error Correction



3 packet message. Send 5.

Corrupts 1 packets.

Error Correction



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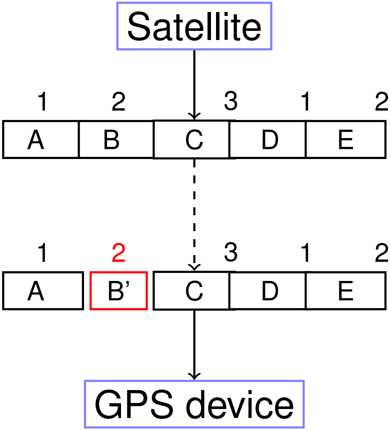

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The Scheme.

Problem: Communicate n packets $m_1, \ldots, m_n$ on noisy channel that corrupts $\leq k$ packets.

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Properties: proof.

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(2) Degree $n-1$ polynomial $Q(x)$ consistent with $n+k$ points.

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(2) Degree $n-1$ polynomial $Q(x)$ consistent with $n+k$ points.
$Q(x)$ agrees with $R(i)$, $n+k$ times.

Properties: proof.

$P(x)$: degree $n-1$ polynomial.
Send $P(1),\ldots,P(n+2k)$
Receive $R(1),\ldots,R(n+2k)$
At most k i's where $P(i) \neq R(i)$.

Properties:

(1) $P(i) = R(i)$ for at least $n+k$ points i,
(2) $P(x)$ is unique degree $n-1$ polynomial
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Total points contained by both: $2n+2k$.

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Receive $R(1) = 3, R(2) = 1, R(3) = 6, R(4) = 0, R(5) = 3$.

$P(i) = R(i)$ for $n + k = 3 + 1 = 4$ points.

Slow solution.

Brute Force:
For each subset of $n+k$ points

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Reconstructs $P(x)$ and only $P(x)$!!

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Find $P(x) = p_2x^2 + p_1x + p_0$ that contains $n + k = 3 + 1$ points.

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Find $P(x) = p_2 x^2 + p_1 x + p_0$ that contains $n + k = 3 + 1$ points.

All equations..

$$\begin{aligned}
p_2 + p_1 + p_0 &\equiv 3 \pmod 7 \\
4p_2 + 2p_1 + p_0 &\equiv 1 \pmod 7 \\
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Assume point 1 is wrong and solve..

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Assume point 2 is wrong

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Something like $(n/k)^k$ ...Exponential in k!.

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How do we find where the bad packets are efficiently?!?!?!

Ditty...

Ditty...

Where oh where

Ditty...

Where oh where can my bad packets be ...

Ditty...

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On Monday!!!!