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$$\begin{bmatrix} P(1) \\ P(2) \\ P(3) \\ \vdots \\ P(n+2k) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1^2 & \cdot & 1 \\ 1 & 2 & 2^2 & \cdot & 2^{n-1} \\ 1 & 3 & 3^2 & \cdot & 3^{n-1} \\ \vdots & . & \vdots & \vdots & \\ 1 & (n+2k) & (n+2k)^2 \cdot & (n+2k)^{n-1} & \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{n-1} \end{bmatrix} \pmod p$$

Berlekamp-Welsh Algorithm

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Idea:
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Solve. Then output $P(x) = Q(x)/E(x)$.

Finding $Q(x)$ and $E(x)$?

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Total unknown coefficient: $n+2k$.

Solving for $Q(x)$ and $E(x)$...

For all points $1, \ldots, i, n+2k$,

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Solve for coefficients of $Q(x)$ and $E(x)$.

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Solving for $Q(x)$ and $E(x)$...and $P(x)$

For all points $1, \ldots, i, n+2k$,

$$Q(i) = R(i)E(i) \pmod{p}$$

Gives $n+2k$ linear equations.

$$\begin{aligned}
a_{n+k-1} + \ldots a_0 &\equiv R(1)(1 + b_{k-1} \cdots b_0) \pmod{p} \\
a_{n+k-1}(2)^{n+k-1} + \ldots a_0 &\equiv R(2)((2)^k + b_{k-1}(2)^{k-1} \cdots b_0) \pmod{p} \\
&\vdots \\
a_{n+k-1}(m)^{n+k-1} + \ldots a_0 &\equiv R(m)((m)^k + b_{k-1}(m)^{k-1} \cdots b_0) \pmod{p}
\end{aligned}$$

..and $n+2k$ unknown coefficients of $Q(x)$ and $E(x)$!

Solve for coefficients of $Q(x)$ and $E(x)$.

Find $P(x) = Q(x)/E(x)$.

Example.

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$$a_3 + a_2 + a_1 + a_0 \equiv 3(1 - b_0) \pmod 7$$

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\begin{aligned}
a_3 + a_2 + a_1 + a_0 &\equiv 3(1 - b_0) \pmod 7 \\
a_3 + 4a_2 + 2a_1 + a_0 &\equiv 1(2 - b_0) \pmod 7
\end{aligned}
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a_3 + 2a_2 + 4a_1 + a_0 &\equiv 0(4 - b_0) \pmod 7 \\
6a_3 + 4a_2 + 5a_1 + a_0 &\equiv 3(5 - b_0) \pmod 7
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$a_3 = 1$, $a_2 = 6$, $a_1 = 6$, $a_0 = 5$ and $b_0 = 2$.

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$a_3 = 1$, $a_2 = 6$, $a_1 = 6$, $a_0 = 5$ and $b_0 = 2$.

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Example: finishing up.

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```
          ------------------
x - 2 ) x^3  + 6 x^2 + 6 x + 5
```

Example: finishing up.

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$E(x) = x - 2.$

```
                  1 x^2
           -----------------
x - 2 )  x^3   + 6 x^2 + 6 x + 5
         x^3   - 2 x^2
```

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$Q(x) = x^3 + 6x^2 + 6x + 5.$
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```
                  1 x^2
         -----------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
                1 x^2 + 6 x + 5
```

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```
                  1 x^2 + 1 x
         -----------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
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```

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```
                1 x^2 + 1 x
         -----------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
                1 x^2 + 6 x + 5
                1 x^2 - 2 x
                ---------------
                          x + 5
```

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                    1 x^2 + 1 x + 1
              -----------------
    x - 2 ) x^3  + 6 x^2 + 6 x + 5
            x^3  - 2 x^2
            ----------
                   1 x^2 + 6 x + 5
                   1 x^2 - 2 x
                   ---------------
                             x + 5
                             x - 2
```

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
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```
                 1 x^2 + 1 x + 1
        -----------------
x - 2 ) x^3  + 6 x^2 + 6 x + 5
        x^3  - 2 x^2
        ----------
               1 x^2 + 6 x + 5
               1 x^2 - 2 x
               ---------------
                         x + 5
                         x - 2
                         -----
                             0
```

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                1 x^2 + 1 x + 1
         -----------------
x - 2 )  x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
                1 x^2 + 6 x + 5
                1 x^2 - 2 x
                ---------------
                          x + 5
                          x - 2
                          -----
                              0
```

$P(x) = x^2 + x + 1$

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                   1 x^2 + 1 x + 1
          -----------------
x - 2 ) x^3  + 6 x^2 + 6 x + 5
        x^3  - 2 x^2
        ----------
                   1 x^2 + 6 x + 5
                   1 x^2 - 2 x
                   ---------------
                             x + 5
                             x - 2
                             -----
                                 0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6.$

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                 1 x^2 + 1 x + 1
           -----------------
    x - 2 ) x^3  + 6 x^2 + 6 x + 5
            x^3  - 2 x^2
            ----------
                 1 x^2 + 6 x + 5
                 1 x^2 - 2 x
                 ---------------
                           x + 5
                           x - 2
                           -----
                               0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6.$

What is $\frac{x-2}{x-2}$?

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
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```
                1 x^2 + 1 x + 1
         -----------------
 x - 2 ) x^3  + 6 x^2 + 6 x + 5
         x^3  - 2 x^2
         ----------
                1 x^2 + 6 x + 5
                1 x^2 - 2 x
                ---------------
                          x + 5
                          x - 2
                          -----
                              0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6.$

What is $\frac{x-2}{x-2}$? 1 Except at $x = 2$?

Example: finishing up.

$Q(x) = x^3 + 6x^2 + 6x + 5.$
$E(x) = x - 2.$

```
                  1 x^2 + 1 x + 1
         -----------------
x - 2 ) x^3  + 6 x^2 + 6 x + 5
        x^3  - 2 x^2
        ----------
                  1 x^2 + 6 x + 5
                  1 x^2 - 2 x
                  ---------------
                            x + 5
                            x - 2
                            -----
                                0
```

$P(x) = x^2 + x + 1$
Message is $P(1) = 3, P(2) = 0, P(3) = 6.$

What is $\frac{x-2}{x-2}$? 1 Except at $x = 2$? Hole there?

Error Correction: Berlekamp-Welsh

Message: $m_1, \ldots, m_n$.
Sender:

1. Form degree $n-1$ polynomial $P(x)$ where $P(i) = m_i$.
2. Send $P(1), \ldots, P(n+2k)$.

Receiver:

1. Receive $R(1), \ldots, R(n+2k)$.
2. Solve $n+2k$ equations, $Q(i) = E(i)R(i)$ to find $Q(x) = E(x)P(x)$ and $E(x)$.
3. Compute $P(x) = Q(x)/E(x)$.
4. Compute $P(1), \ldots, P(n)$.

Check your undersanding.

You have error locator polynomial!

Check your undersanding.

You have error locator polynomial!

Where oh where can my bad packets be?...

Check your undersanding.

You have error locator polynomial!

Where oh where can my bad packets be?...

Factor?

Check your undersanding.

You have error locator polynomial!

Where oh where can my bad packets be?...

Factor? Sure.

Check your undersanding.

You have error locator polynomial!

Where oh where can my bad packets be?...

Factor? Sure.
Check all values?

Check your undersanding.

You have error locator polynomial!

Where oh where can my bad packets be?...

Factor? Sure.
Check all values? Sure.

Check your undersanding.

You have error locator polynomial!

Where oh where can my bad packets be?...

Factor? Sure.
Check all values? Sure.

Check your undersanding.

You have error locator polynomial!

Where oh where can my bad packets be?...

Factor? Sure.
Check all values? Sure.

Efficiency?

Check your undersanding.

You have error locator polynomial!

Where oh where can my bad packets be?...

Factor? Sure.
Check all values? Sure.

Efficiency? Sure.

Check your undersanding.

You have error locator polynomial!

Where oh where can my bad packets be?...

Factor? Sure.
Check all values? Sure.

Efficiency? Sure. Only $n+k$ values.

Check your undersanding.

You have error locator polynomial!

Where oh where can my bad packets be?...

Factor? Sure.
Check all values? Sure.

Efficiency? Sure. Only $n+k$ values.
See where it is 0.

Hmmm...

Is there one and only one $P(x)$ from Berlekamp-Welsh procedure?

Hmmm...

Is there one and only one $P(x)$ from Berlekamp-Welsh procedure?

Existence: there is a $P(x)$ and $E(x)$ that satisfy equations.

Unique solution for $P(x)$

Uniqueness: any solution $Q'(x)$ and $E'(x)$ have

$$\frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)} = P(x). \tag{1}$$

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Proof:

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Proof:
We claim

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Proof:
We claim

$$Q'(x)E(x) = Q(x)E'(x) \text{ on } n+2k \text{ values of } x. \tag{2}$$

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Equation 2 implies 1:

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Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$
and agree on $n+2k$ points

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Equation 2 implies 1:

$Q'(x)E(x)$ and $Q(x)E'(x)$ are degree $n+2k-1$
and agree on $n+2k$ points
$E(x)$ and $E'(x)$ have at most k zeros each.

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Can cross divide at n points.

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Can cross divide at n points.
$\implies \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)}$ equal on n points.

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Both degree $\leq n$

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Can cross divide at n points.
$\implies \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)}$ equal on n points.
Both degree $\leq n \implies$ Same polynomial!

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$\implies \frac{Q'(x)}{E'(x)} = \frac{Q(x)}{E(x)}$ equal on n points.
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Last bit.

Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n+2k$ values of x.

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Proof: Construction implies that

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Last bit.

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Proof: Construction implies that

$$Q(i) = R(i)E(i)$$

$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \ldots n+2k\}$.

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Proof: Construction implies that

$$Q(i) = R(i)E(i)$$
$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \ldots n+2k\}$.

If $E(i) = 0$, then $Q(i) = 0$.

Last bit.

Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n+2k$ values of x.

Proof: Construction implies that

$$Q(i) = R(i)E(i)$$
$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \ldots n+2k\}$.

If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$.

Last bit.

Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n+2k$ values of x.

Proof: Construction implies that

$$Q(i) = R(i)E(i)$$
$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \ldots n+2k\}$.

If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$.
$\implies$ $Q(i)E'(i) = Q'(i)E(i)$ holds when $E(i)$ or $E'(i)$ are zero.

Last bit.

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$$Q(i) = R(i)E(i)$$
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When $E'(i)$ and $E(i)$ are not zero

Last bit.

Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n+2k$ values of x.

Proof: Construction implies that

$$Q(i) = R(i)E(i)$$

$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \dots n+2k\}$.

If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$.
$\implies Q(i)E'(i) = Q'(i)E(i)$ holds when $E(i)$ or $E'(i)$ are zero.

When $E'(i)$ and $E(i)$ are not zero

$$\frac{Q'(i)}{E'(i)} = \frac{Q(i)}{E(i)} = R(i).$$

Last bit.

Fact: $Q'(x)E(x) = Q(x)E'(x)$ on $n+2k$ values of x.

Proof: Construction implies that

$$Q(i) = R(i)E(i)$$
$$Q'(i) = R(i)E'(i)$$

for $i \in \{1, \ldots n+2k\}$.

If $E(i) = 0$, then $Q(i) = 0$. If $E'(i) = 0$, then $Q'(i) = 0$.
$\implies Q(i)E'(i) = Q'(i)E(i)$ holds when $E(i)$ or $E'(i)$ are zero.

When $E'(i)$ and $E(i)$ are not zero

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Points to polynomials, have to deal with zeros!

Example: dealing with $\frac{x-2}{x-2}$ at $x = 2$.

Berlekamp-Welsh algorithm decodes correctly when k errors!

Summary: polynomials.

Set of $d+1$ points determines degree d polynomial.

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Error Locator Polynomial.

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Linear code.

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Relations:
Linear code.
Almost any coding matrix works.

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Other Algebraic-Geometric codes.

Farewell to modular arithmetic...

Modular arithmetic modulo a prime.

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Add, subtract, commutative, associative,

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Why not modular arithmetic all the time?

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$4 > 3$?

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For modular arithmetic...no Calculus. Sad face!

Next up: how big is infinity.

Next up: how big is infinity.

- Countable
- Countably infinite.
- Enumeration

How big are the reals or the integers?

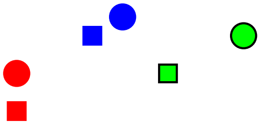
Infinite!

How big are the reals or the integers?

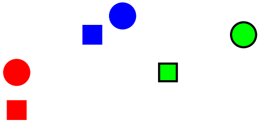
Infinite!

Is one bigger or smaller?

Same size?



Same size?



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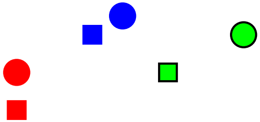
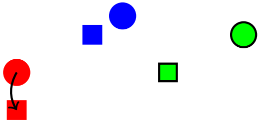


Same number?

Make a function f : Circles $\to$ Squares.

Same size?

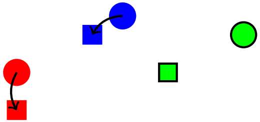


Same number?

Make a function f : Circles $\rightarrow$ Squares.

f(red circle) $=$ red square

Same size?



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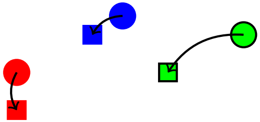
f(red circle) $=$ red square

f(blue circle) $=$ blue square

Same size?



Same number?

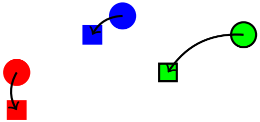
Make a function f : Circles $\to$ Squares.

f(red circle) $=$ red square

f(blue circle) $=$ blue square

f(circle with black border) $=$ square with black border

Same size?



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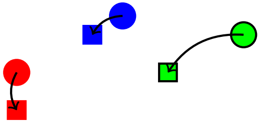
One to one.

Same size?



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Make a function f : Circles $\rightarrow$ Squares.

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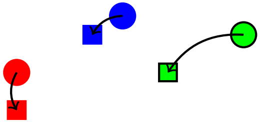
One to one. Each circle mapped to different square.

Same size?



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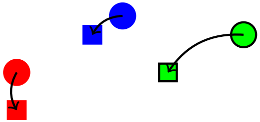

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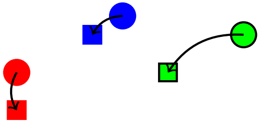

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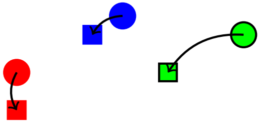

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Onto: For all $s \in R$, $\exists c \in D, s = f(c)$.

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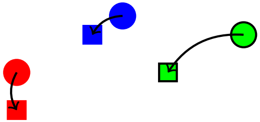

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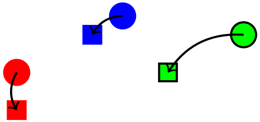

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Isomorphism principle: If there is $f : D \to R$ that is one to one and onto, then, $|D| = |R|$.

Isomorphism principle.

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Countable.

How to count?

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0,

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How to count?

0, 1,

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How to count?

0, 1, 2,

Countable.

How to count?

0, 1, 2, 3,

Countable.

How to count?

$0, 1, 2, 3, \ldots$

Countable.

How to count?

0, 1, 2, 3, ...

The Counting numbers.

Countable.

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0, 1, 2, 3, ...

The Counting numbers.
The natural numbers! N

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Definition: S is **countable** if there is a bijection between S and some subset of N.

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If the subset of N is finite, S has finite **cardinality**.

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If the subset of N is finite, S has finite **cardinality**.

If the subset of N is infinite, S is **countably infinite**.

Where’s 0?

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But.. but Where's zero? "Comes from 1."

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Bijection from A to $B \implies$ a bijection from B to A.

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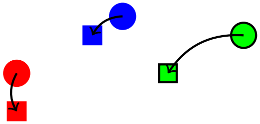

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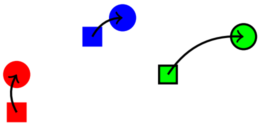

Inverse function!

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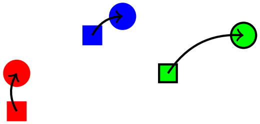

Inverse function!

Can prove equivalence either way.

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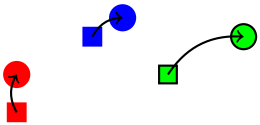

Inverse function!

Can prove equivalence either way.
Bijection to or from natural numbers implies countably infinite.

More large sets.

E - Even natural numbers?

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$f : N \rightarrow E$.

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$f : N \to E$.

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Evens are countably infinite.
Evens are same size as all natural numbers.

All integers?

What about Integers, Z?

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Define $f : N \to Z$.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n+1)/2 & \text{if n odd.} \end{cases}$$

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Define $f : N \rightarrow Z$.

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Integers and naturals have same size!

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61A

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61A —- streams!

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All countably infinite sets have the same cardinality.

Enumeration example.

All binary strings.

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Never get to 1.

More fractions?

Enumerate the rational numbers in order...

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Can't list in "order".

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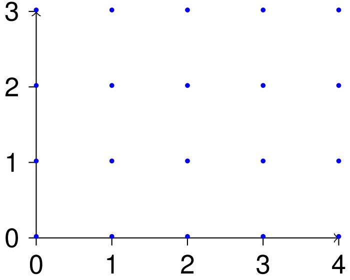

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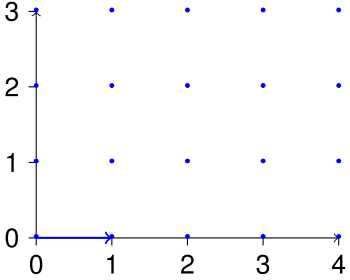

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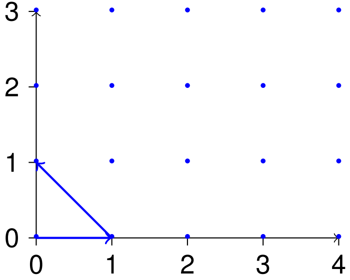

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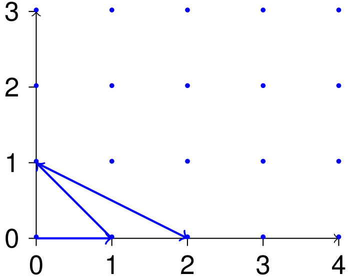

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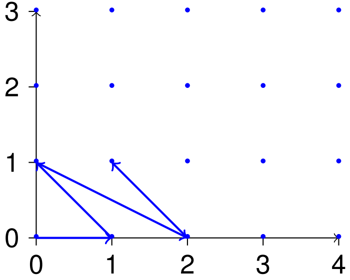
Enumerate in list:
$(0,0),(1,0),(0,1),(2,0),(1,1),(0,2),\ldots\ldots$



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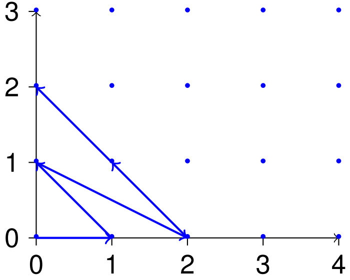

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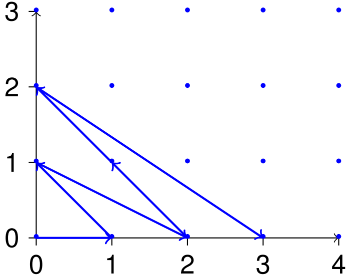

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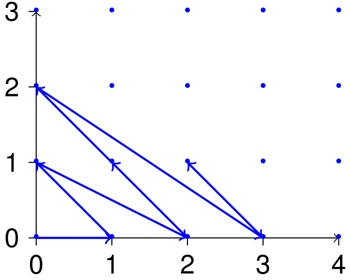

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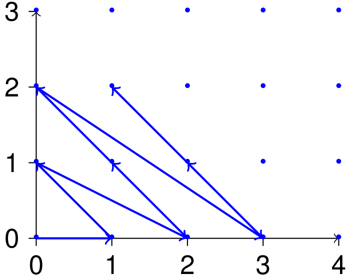

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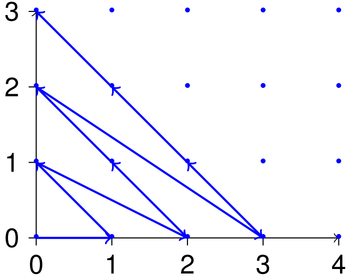

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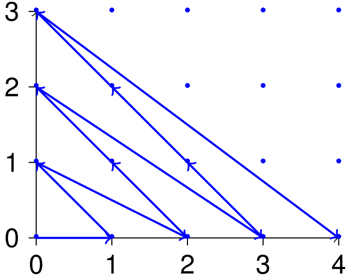

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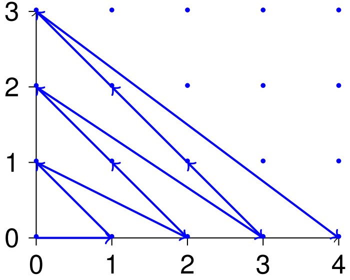

The pair (a,b), is in first $(a+b+1)(a+b)/2$ elements of list!

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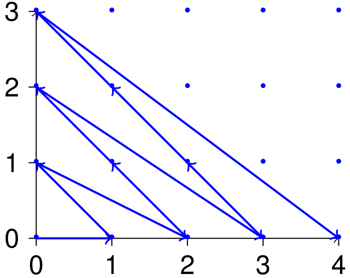

The pair (a,b), is in first $(a+b+1)(a+b)/2$ elements of list!
(i.e., "triangle").

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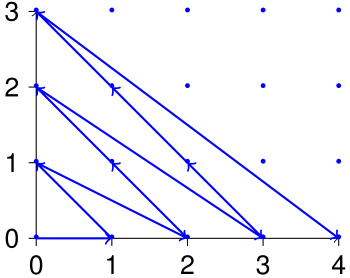

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Countably infinite.

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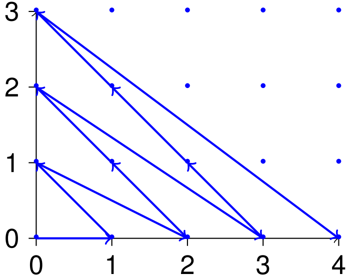

The pair (a,b), is in first $(a+b+1)(a+b)/2$ elements of list!
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Countably infinite.

Same size as the natural numbers!!

Rationals?

Positive rational number.

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The rationals are countably infinite.

Real numbers..

Real numbers are same size as integers?

The reals.

Are the set of reals countable?

The reals.

Are the set of reals countable?

Lets consider the reals $[0,1]$.

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Diagonalization.

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$\vdots$

Construct "diagonal" number: .7

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0: .500000000...
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4: .345212312...
⋮

Construct “diagonal” number: .77

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Subset $[0,1]$ is not countable!!

All reals?

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All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
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Any subset of a countable set is countable.

All reals?

Subset $[0, 1]$ is not countable!!

What about all reals?
No.

Any subset of a countable set is countable.

If reals are countable then so is $[0, 1]$.

Diagonalization.

1. Assume that a set S can be enumerated.

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5. Show that t is in S.
6. Contradiction.

Another diagonalization.

The set of all subsets of N.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$,

Another diagonalization.

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Example subsets of N: $\{0\}$, $\{0, \ldots, 7\}$,

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Assume is countable.

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Assume is countable.

There is a listing, L, that contains all subsets of N.

Another diagonalization.

The set of all subsets of *N*.

Example subsets of *N*: $\{0\}$, $\{0,\ldots,7\}$,
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Define a diagonal set, *D*:

Another diagonalization.

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Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.

Another diagonalization.

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D is different from ith set in L for every i.

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Contradiction.

Another diagonalization.

The set of all subsets of N.

Example subsets of N: $\{0\}$, $\{0,\ldots,7\}$,
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Assume is countable.

There is a listing, L, that contains all subsets of N.

Define a diagonal set, D:
If ith set in L does not contain i, $i \in D$.
otherwise $i \notin D$.

D is different from ith set in L for every i.
$\implies$ D is not in the listing.

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(The set of all subsets of S, is the **powerset** of N.)

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First of Hilbert's problems!

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$[0,1]$ is same cardinality as nonnegative reals!

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There is no infinite set whose cardinality is between the cardinality of an infinite set and its power set.

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The powerset of a set is the set of all subsets.

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Self reference.

More on...

...Tuesday..

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