

## Today.

Finish Counting.  
...and then Professor Walrand.

## Stars and Bars.

How many different 5 star and 2 bar diagrams?

| \* | \* \* \* \*

7 positions in which to place the 2 bars.

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Alice: 0; Bob 1; Eve: 4

| \* | \* \* \* \*

Bars in first and third position.

Alice: 1; Bob 4; Eve: 0

\* | \* \* \* \* |.

Bars in second and seventh position.

$\binom{7}{2}$  ways to do so and

$\binom{7}{2}$  ways to split 5 dollars among 3 people.

## But first..

This statement is a lie. **Neither true nor false!**

Every person who doesn't shave themselves is shaved by the barber.

**Who shaves the barber?**

```
def Turing(P):
  if Halts(P,P):
    while(true):
      pass
  else:
    return
```

...Text of Halt...

Halt Program  $\implies$  Turing Program. ( $P \implies Q$ )

Turing("Turing")? Neither halts nor loops!  $\implies$  No Turing program.

No Turing Program  $\implies$  No halt program. ( $\neg Q \implies \neg P$ )

Program is text, so we can pass it to itself,  
or refer to self.

## 6 or 7???

An alternative counting.

\* \* \* \* \*

Alternative: 6 places "in between" stars.

Each selection of two places "in between" stars maps to an allocation of dollars to Alice, Bob, and Eve.

Ways to choose two different places  $\binom{6}{2}$  plus  
ways to choose same place twice  $\binom{6}{1} = 6$

$$\binom{6}{2} + 6 = 21.$$

$$\binom{7}{2} = 21.$$

For splitting among 4 people, this way becomes a mess.

$$\binom{6}{3} + 2 * \binom{6}{2} + \binom{6}{1}. \quad 20 + 30 + 6 = 56$$

$$\binom{8}{3}. \quad (8 * 7 * 6) / 6 = 56.$$

## Splitting 5 dollars..

How many ways can Alice, Bob, and Eve split 5 dollars.

Alice gets 3, Bob gets 1, Eve gets 1:  $(A, A, A, B, E)$ .

Separate Alice's dollars from Bob's and then Bob's from Eve's.

Five dollars are five stars: \* \* \* \* \*.

Alice: 2, Bob: 1, Eve: 2.

Stars and Bars: \*\*|\*|\*\*.

Alice: 0, Bob: 1, Eve: 4.

Stars and Bars: |\*|\*\*\*\*.

Each split "is" a sequence of stars and bars.

Each sequence of stars and bars "is" a split.

**Counting Rule: if there is a one-to-one mapping between two sets they have the same size!**

## Stars and Bars.

Ways to add up  $n$  numbers to sum to  $k$ ? or

" $k$  from  $n$  with replacement where order doesn't matter."

In general,  $k$  stars  $n-1$  bars.

\*\*|\*|...|\*\*.

$n+k-1$  positions from which to choose  $n-1$  bar positions.

$$\binom{n+k-1}{n-1}$$

Or:  $k$  unordered choices from set of  $n$  possibilities with replacement.  
**Sample with replacement where order doesn't matter.**

## Quick review of the basics.

**First rule:**  $n_1 \times n_2 \cdots \times n_3$ .

$k$  Samples with replacement from  $n$  items:  $n^k$ .

Sample without replacement:  $\frac{n!}{(n-k)!}$

**Second rule: when order doesn't matter divide..when possible.**

Sample without replacement and order doesn't matter:  $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ .  
 "n choose k"

**One-to-one rule: equal in number if one-to-one correspondence.**

Sample with replacement and order doesn't matter:  $\binom{k+n-1}{n-1}$ .

## Balls in bins.



"k Balls in n bins"  $\equiv$  "k samples from n possibilities."

"indistinguishable balls"  $\equiv$  "order doesn't matter"

"only one ball in each bin"  $\equiv$  "without replacement"

5 balls into 10 bins

5 samples from 10 possibilities with replacement

Example: 5 digit numbers.

5 indistinguishable balls into 52 bins only one ball in each bin

5 samples from 52 possibilities without replacement

Example: Poker hands.

5 indistinguishable balls into 3 bins

5 samples from 3 possibilities with replacement and no order

Dividing 5 dollars among Alice, Bob and Eve.

## Sum Rule

Two indistinguishable jokers in 54 card deck.

How many 5 card poker hands?

**Sum rule: Can sum over disjoint sets.**

No jokers "exclusive" or One Joker "exclusive" or Two Jokers

$$\binom{52}{5} + \binom{52}{4} + \binom{52}{3}.$$

Two distinguishable jokers in 54 card deck.

How many 5 card poker hands? Choose 4 cards plus one of 2 jokers!

$$\binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$$

Wait a minute! Same as choosing 5 cards from 54 or

$$\binom{54}{5}$$

**Theorem:**  $\binom{54}{5} = \binom{52}{5} + 2 * \binom{52}{4} + \binom{52}{3}$ .

## Combinatorial Proofs.

**Theorem:**  $\binom{n}{k} = \binom{n}{n-k}$

**Proof:** How many subsets of size  $k$ ?  $\binom{n}{k}$

How many subsets of size  $k$ ?

Choose a subset of size  $n-k$

and what's left out is a subset of size  $k$ .

Choosing a subset of size  $k$  is same

as choosing  $n-k$  elements to not take.

$\Rightarrow \binom{n}{n-k}$  subsets of size  $k$ .

□

## Pascal's Triangle

$$\begin{array}{ccccccc} & & & & 1 & & & & \\ & & & & 1 & & 1 & & \\ & & & 1 & 2 & 1 & & & \\ & & 1 & 3 & 3 & 1 & & & \\ & 1 & 4 & 6 & 4 & 1 & & & \end{array}$$

Row  $n$ : coefficients of  $(1+x)^n = (1+x)(1+x) \cdots (1+x)$ .

Foil (4 terms) on steroids:

$2^n$  terms: choose 1 or  $x$  from each factor of  $(1+x)$ .

Simplify: collect all terms corresponding to  $x^k$ .

Coefficient of  $x^k$  is  $\binom{n}{k}$ : choose  $k$  factors where  $x$  is in product.

$$\begin{array}{cccc} \binom{0}{0} & & & \\ \binom{1}{0} & \binom{1}{1} & & \\ \binom{2}{0} & \binom{2}{1} & \binom{2}{2} & \\ \binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \end{array}$$

Pascal's rule  $\Rightarrow \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ .

## Combinatorial Proofs.

**Theorem:**  $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ .

**Proof:** How many size  $k$  subsets of  $n+1$ ?  $\binom{n+1}{k}$ .

How many size  $k$  subsets of  $n+1$ ?

How many contain the first element?

Chose first element, need to choose  $k-1$  more from remaining  $n$  elements.

$$\Rightarrow \binom{n}{k-1}$$

How many don't contain the first element?

Need to choose  $k$  elements from remaining  $n$  elts.

$$\Rightarrow \binom{n}{k}$$

So,  $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ .

□

## Combinatorial Proof.

**Theorem:**  $\binom{n}{k} = \binom{n-1}{k-1} + \dots + \binom{n-1}{k}$ .

**Proof:** Consider size  $k$  subset where  $i$  is the first element chosen.

$$\{1, \dots, \underline{i}, \dots, n\}$$

Must choose  $k-1$  elements from  $n-i$  remaining elements.

$\implies \binom{n-i}{k-1}$  such subsets.

Add them up to get the total number of subsets of size  $k$  which is also  $\binom{n+1}{k}$ . □

## Binomial Theorem: $x = 1$

**Theorem:**  $2^n = \binom{n}{n} + \binom{n}{n-1} + \dots + \binom{n}{0}$

**Proof:** How many subsets of  $\{1, \dots, n\}$ ?

Construct a subset with sequence of  $n$  choices:  
element  $i$  is in or is not in the subset: 2 poss.

First rule of counting:  $2 \times 2 \times \dots \times 2 = 2^n$  subsets.

How many subsets of  $\{1, \dots, n\}$ ?

$\binom{n}{i}$  ways to choose  $i$  elts of  $\{1, \dots, n\}$ .

Sum over  $i$  to get total number of subsets..which is also  $2^n$ . □

## Simple Inclusion/Exclusion

**Sum Rule: For disjoint sets  $S$  and  $T$ ,**  $|S \cup T| = |S| + |T|$

Used to reason about all subsets

by adding number of subsets of size 1, 2, 3, ...

Also reasoned about subsets that contained or didn't contain an element. (E.g., first element, first  $i$  elements.)

**Inclusion/Exclusion Rule: For any  $S$  and  $T$ ,**

$|S \cup T| = |S| + |T| - |S \cap T|$ .

**Example:** How many 10-digit phone numbers have 7 as their first or second digit?

$S$  = phone numbers with 7 as first digit.  $|S| = 10^9$

$T$  = phone numbers with 7 as second digit.  $|T| = 10^9$ .

$S \cap T$  = phone numbers with 7 as first and second digit.  $|S \cap T| = 10^8$ .

Answer:  $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$ .

## Summary.

First Rule of counting: Objects from a sequence of choices:

$n_i$  possibilities for  $i$ th choice.

$n_1 \times n_2 \times \dots \times n_k$  objects.

Second Rule of counting: If order does not matter.

Count with order. Divide by number of orderings/sorted object.

Typically:  $\binom{n}{k}$ .

Stars and Bars: Sample  $k$  objects with replacement from  $n$ .

Order doesn't matter.

Typically:  $\binom{n+k-1}{k-1}$ .

Inclusion/Exclusion: two sets of objects.

Add number of each subtract intersection of sets.

Sum Rule: If disjoint just add.

Combinatorial Proofs: Identity from counting same in two ways.

Pascal's Triangle Example:  $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$ .

RHS: Number of subsets of  $n+1$  items size  $k$ .

LHS:  $\binom{n}{k-1}$  counts subsets of  $n+1$  items with first item.

$\binom{n}{k}$  counts subsets of  $n+1$  items without first item.

Disjoint – so add!

## Some final advice...

Confused is ok. Apologies for my part in it.

But it is part of learning.

Don't let frustration get in the way.

Terry Tao: Arguably smartest man on the planet.

"I remember crying," Tao said, "and the proctor had to escort me out." He failed.

Also, struggled for a time in grad school.

"I went in and very quickly got out of my depth," he said. "They were asking questions which I had no ability to answer." Immediately after, Tao sat with his adviser, Elias Stein, and felt that he had let him down.

Struggle is fine.

Read his blog. Intimidating often. But nice.