CS70: Jean Walrand: Lecture 15b.

Modeling Uncertainty: Probability Space

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Modeling Uncertainty: Probability Space

1. Key Points
2. Random Experiments
3. Probability Space

Key Points

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- Probability
  - Models knowledge about uncertainty
  - Discovers best way to use that knowledge in making decisions

The Magic of Probability

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Uncertainty:

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Uncertainty: vague,

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Uncertainty: vague, fuzzy,

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Your cost: focused attention and practice on examples and problems.

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Random Experiment: Flip one Fair Coin

Flip a fair coin: (*One flips or tosses a coin*)

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- Likelihoods: $H : 50\%$ and $T : 50\%$

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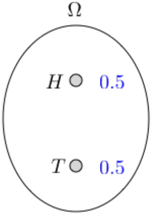
Physical Experiment



Probability Model

Random Experiment: Flip one Fair Coin

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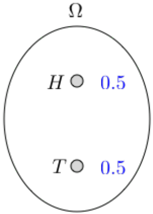

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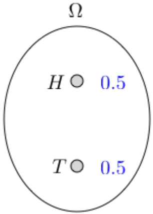

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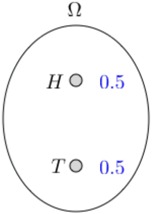

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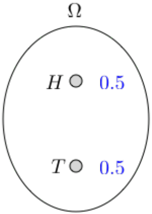

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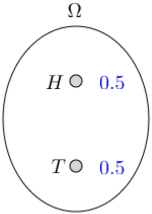

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- The Probability model is simple:
  - A set Ω of outcomes: $\Omega = \{H, T\}$.
  - A probability assigned to each outcome: $Pr[H] = 0.5, Pr[T] = 0.5$.

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- Tautolgy? No: Statistical regularity!

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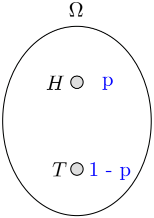

Probability Model

Flip Two Fair Coins

Flip Two Fair Coins

- Possible outcomes:

Flip Two Fair Coins

- Possible outcomes: $\{HH, HT, TH, TT\}$

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- Possible outcomes: $\{HH, HT, TH, TT\} \equiv \{H, T\}^2$.

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Flip Glued Coins

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Flips two coins glued together side by side:

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- Possible outcomes: $\{HH, TT\}$.
- Likelihoods: $HH : 0.5, TT : 0.5$.

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Flips two coins glued together side by side:

Glued coins

50%

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- Possible outcomes: $\{HH, TT\}$.
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- Note: Coins are glued so that they show the same face.

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- Likelihoods: $HT : 0.5, TH : 0.5$.

Flip Glued Coins

Flips two coins glued together side by side:

- Possible outcomes: $\{HT, TH\}$.
- Likelihoods: $HT : 0.5, TH : 0.5$.
- Note: Coins are glued so that they show different faces.

Flip two Attached Coins

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Flips two coins attached by a spring:

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- Likelihoods:

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Flips two coins attached by a spring:

- Possible outcomes: $\{HH, HT, TH, TT\}$.
- Likelihoods: $HH : 0.4, HT : 0.1, TH : 0.1, TT : 0.4$.

Flip two Attached Coins

Flips two coins attached by a spring:

Attached coins

- Possible outcomes: $\{HH, HT, TH, TT\}$.
- Likelihoods: $HH : 0.4, HT : 0.1, TH : 0.1, TT : 0.4$.
- Note: Coins are attached so that they tend to show the same face, unless the spring twists enough.

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Here is a way to summarize the four random experiments:

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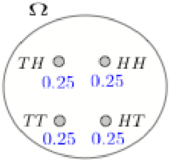
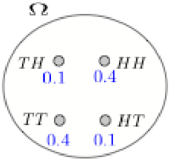


[1]



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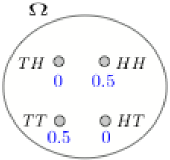

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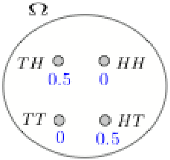

[4]

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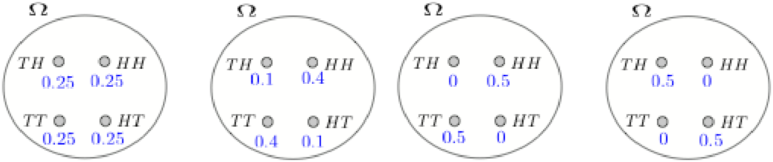

[1] [2] [3] [4]

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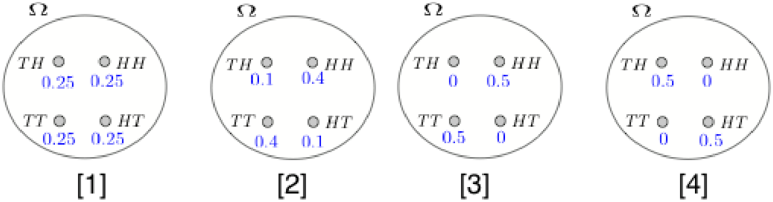

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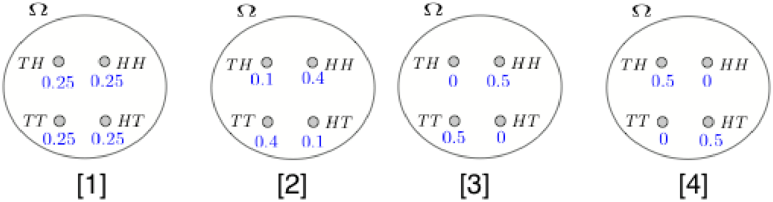

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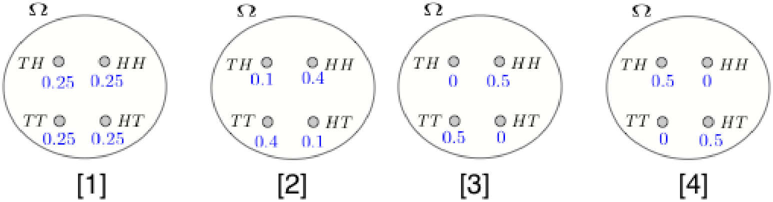

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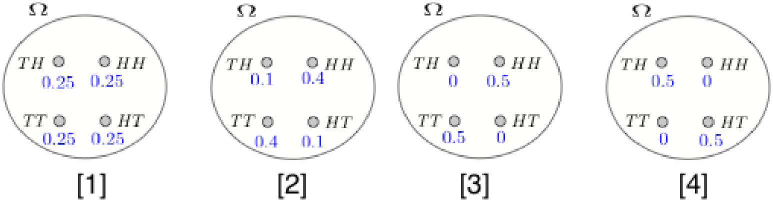

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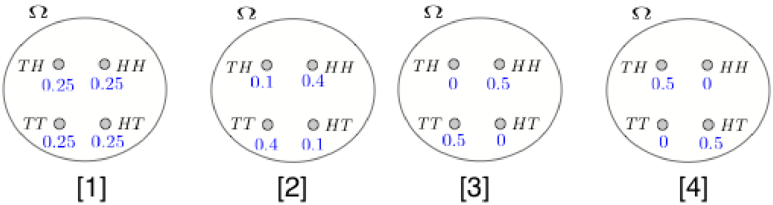

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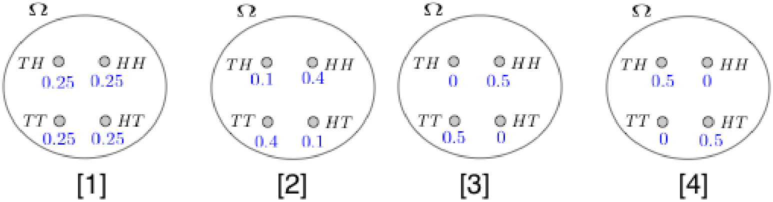

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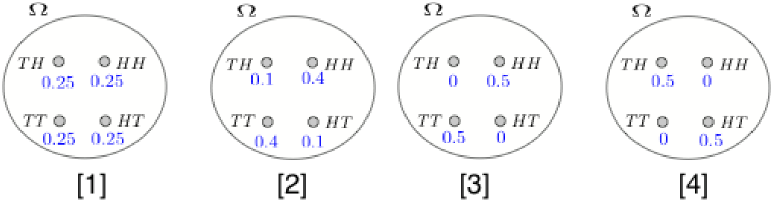

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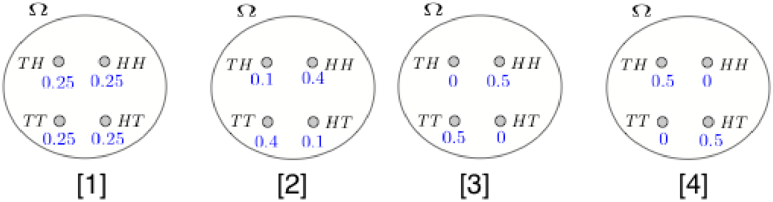

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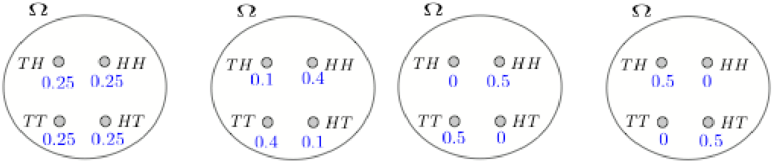

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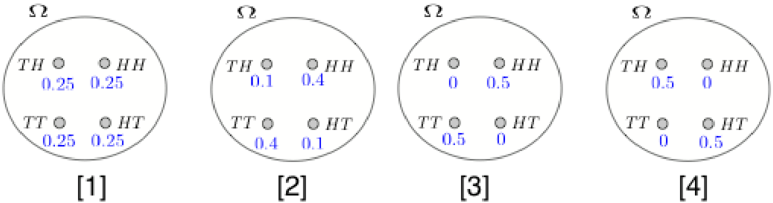

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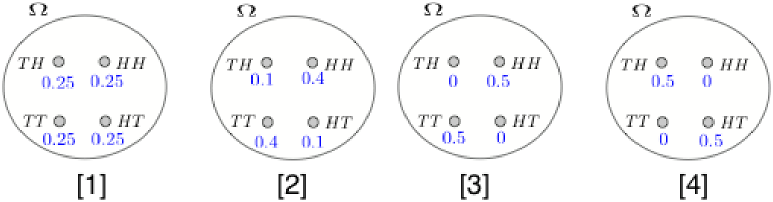

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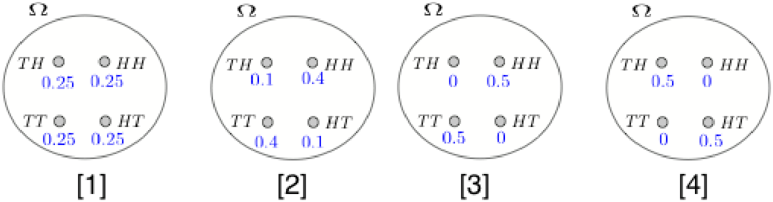

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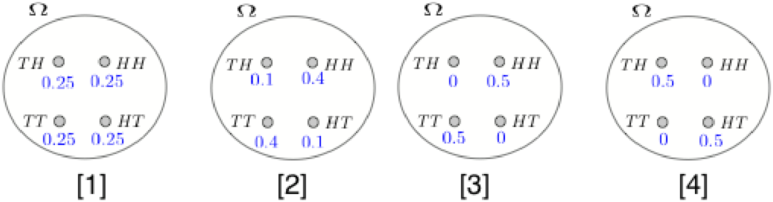

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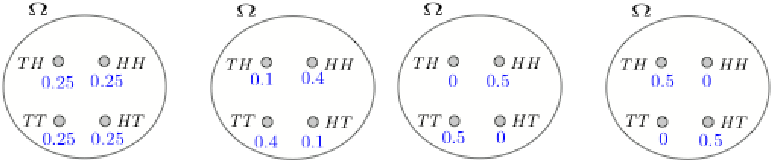

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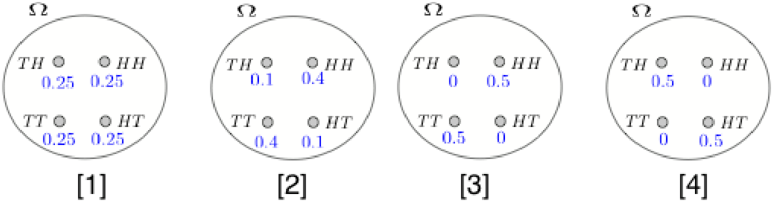

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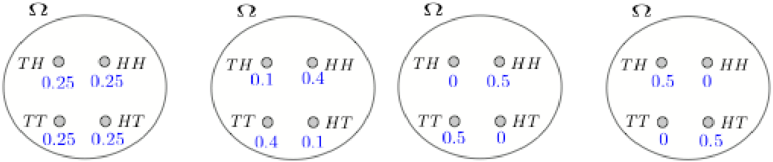

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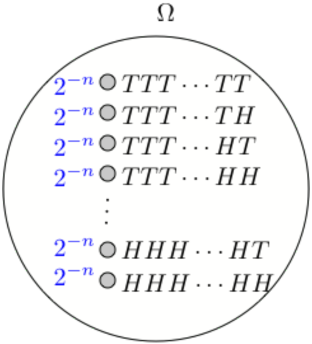

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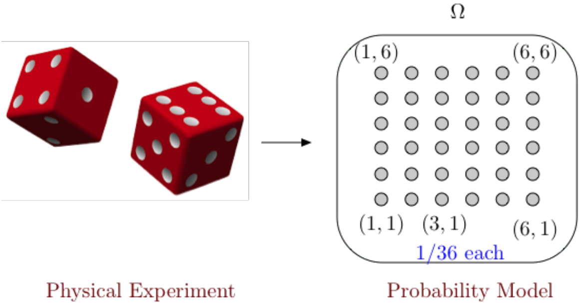

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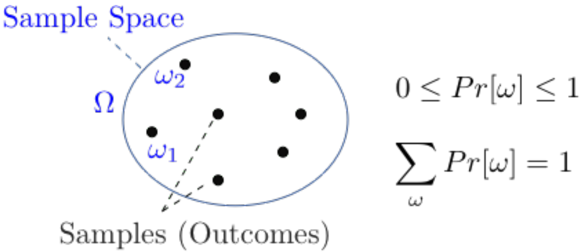

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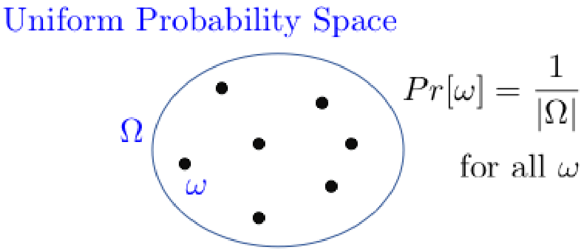

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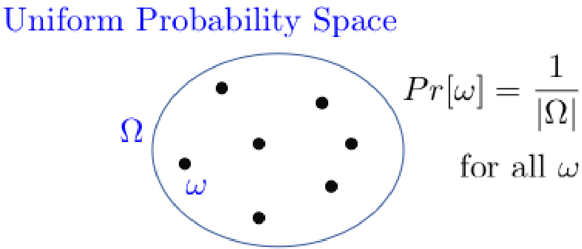

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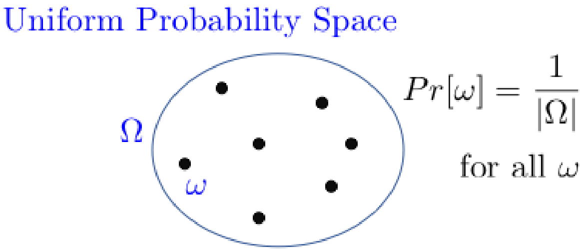

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Simplest physical model of a uniform probability space:

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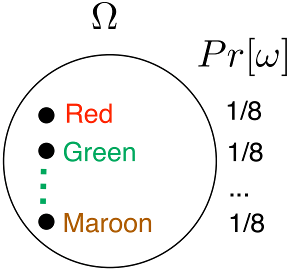

Probability model

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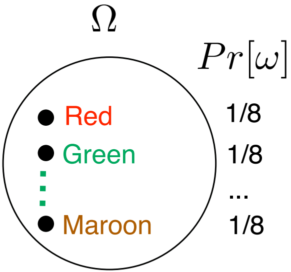

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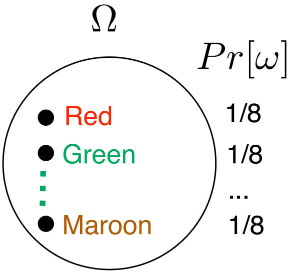

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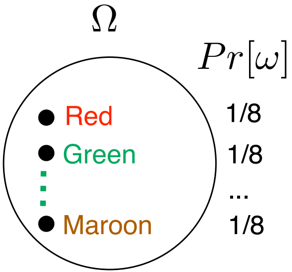

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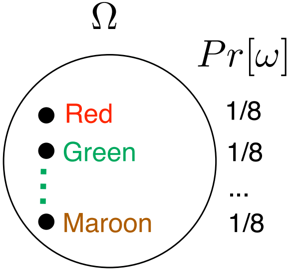

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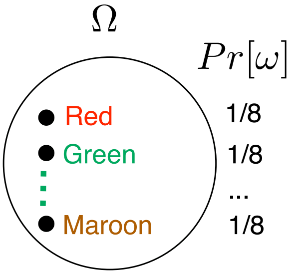

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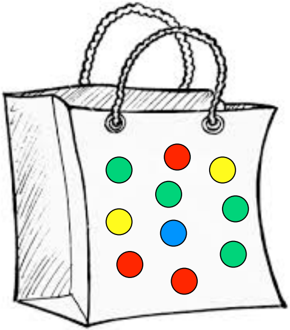

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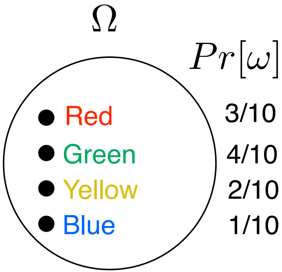

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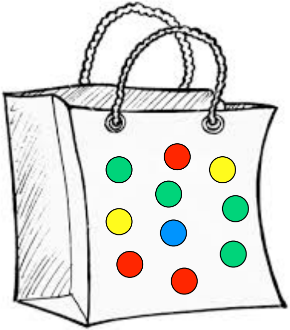

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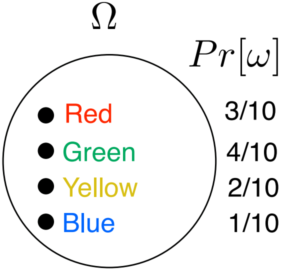
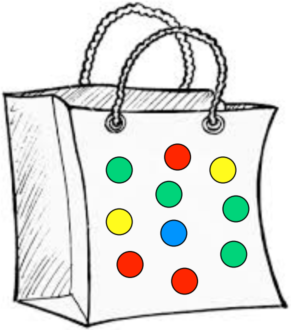


Probability model

$\Omega = \{$Red, Green, Yellow, Blue$\}$

Probability Space: Formalism

Simplest physical model of a non-uniform probability space:



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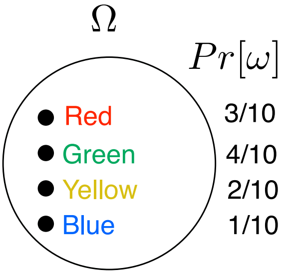

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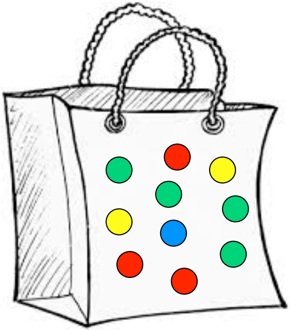

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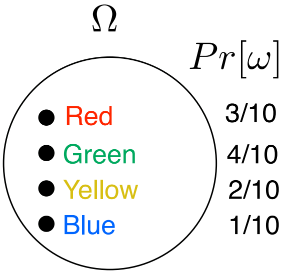

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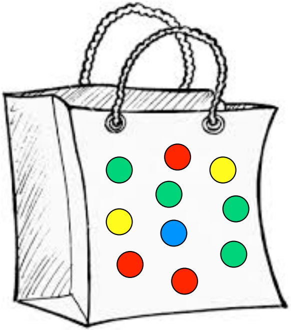

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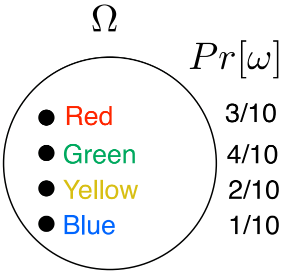

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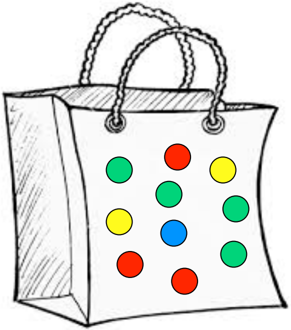

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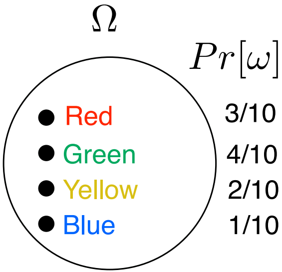

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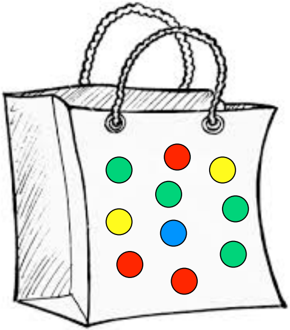

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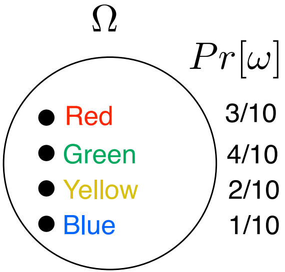

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Note: Probabilities are restricted to rational numbers: $\frac{N_k}{N}$.

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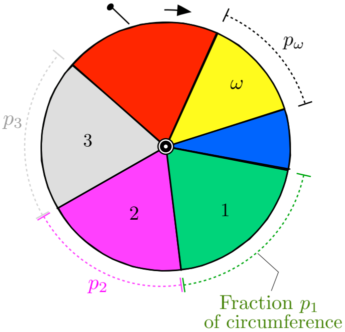

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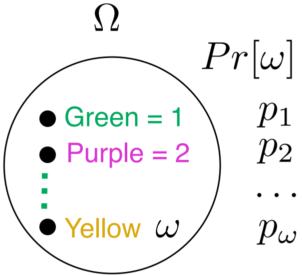

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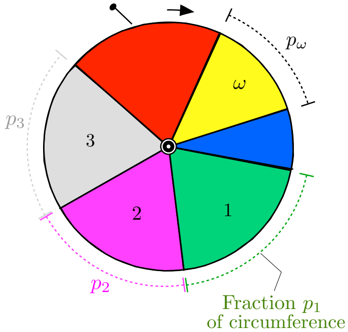

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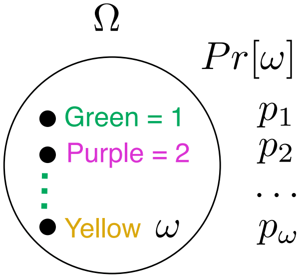

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The roulette wheel stops in sector ω with probability p_ω.

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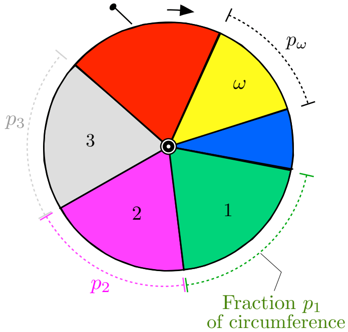

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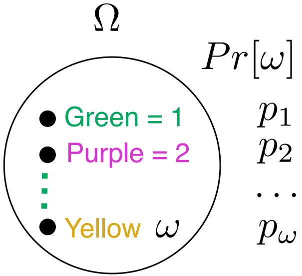

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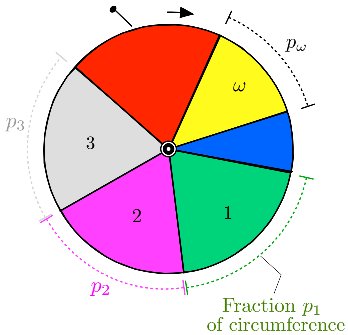

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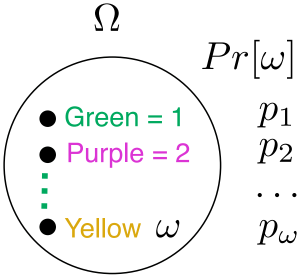

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Lecture 15: Summary

Modeling Uncertainty: Probability Space

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