CS70: Jean Walrand: Lecture 16.

Events, Conditional Probability, Independence, Bayes' Rule

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1. Probability Basics Review
2. Events
3. Conditional Probability
4. Independence of Events
5. Bayes' Rule

Probability Basics Review

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- Random Experiment.

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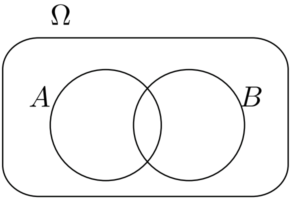

Figure: Two events

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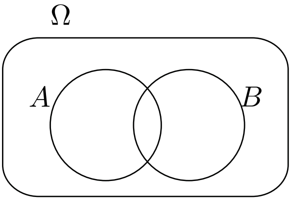

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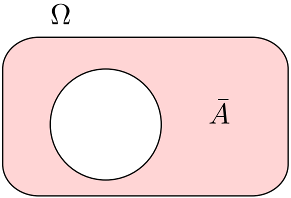

Figure: Complement (not)

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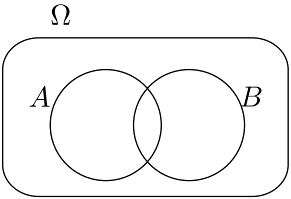

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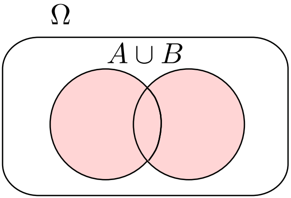

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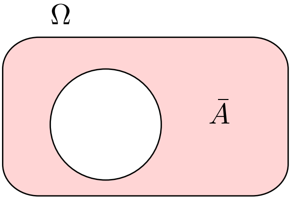

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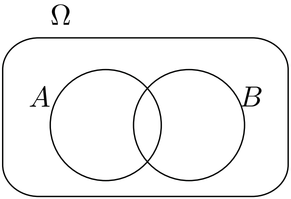

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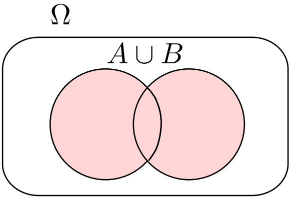

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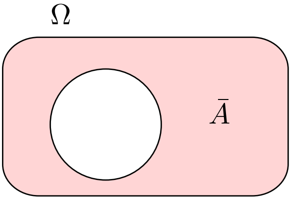

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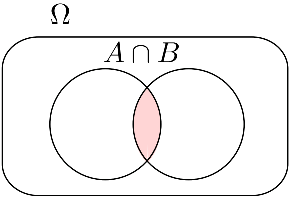

Figure: Intersection (and)

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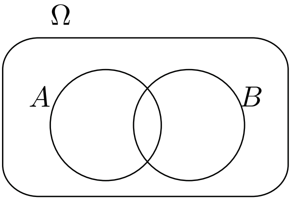

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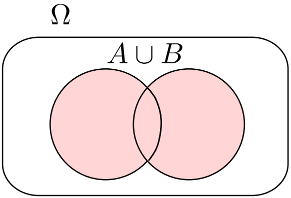

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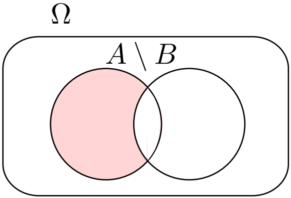

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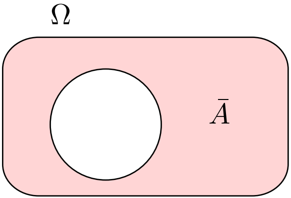

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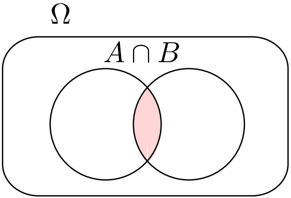

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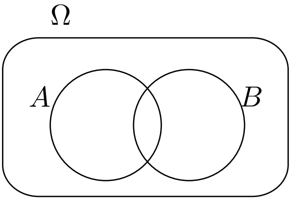

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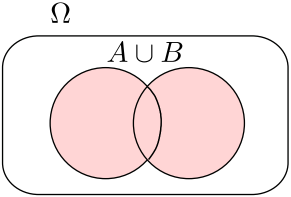

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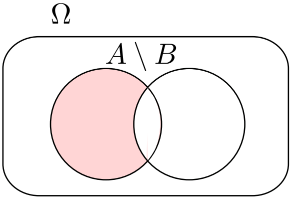

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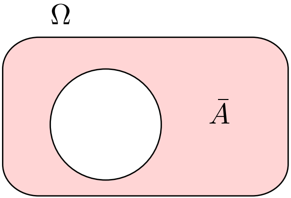

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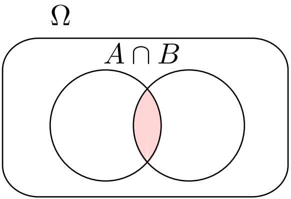

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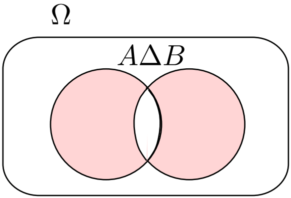

Figure: Symmetric difference (only one)

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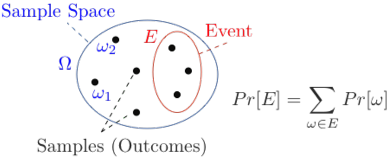

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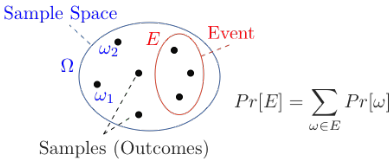

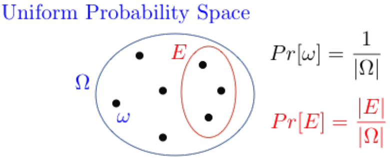

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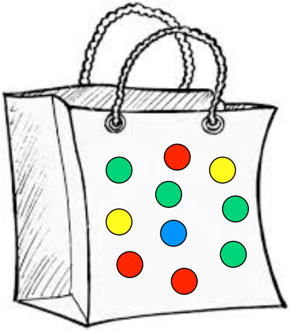
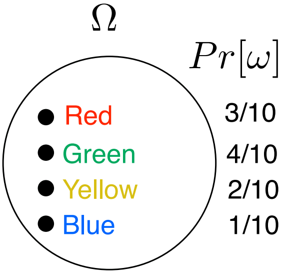


Physical experiment



Probability model

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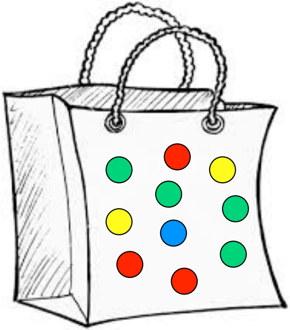

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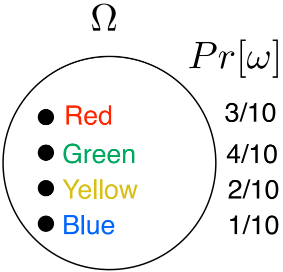

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$\Omega = \{$Red, Green, Yellow, Blue$\}$

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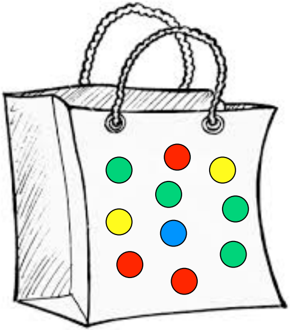

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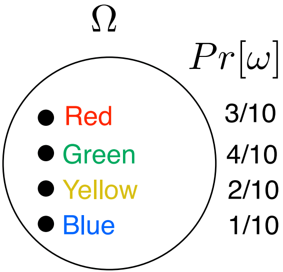

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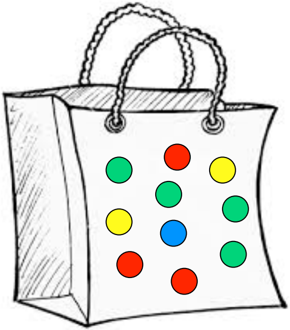

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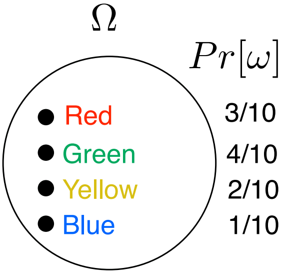

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$$Pr[\text{Red}] = \frac{3}{10},$$

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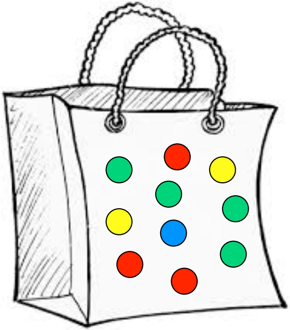

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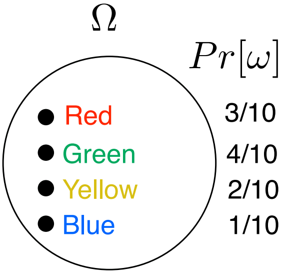

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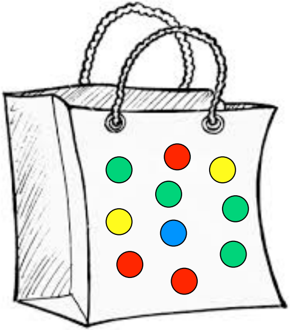

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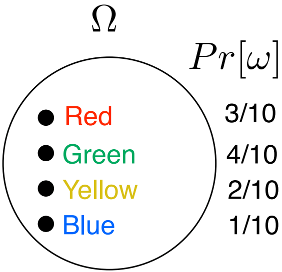

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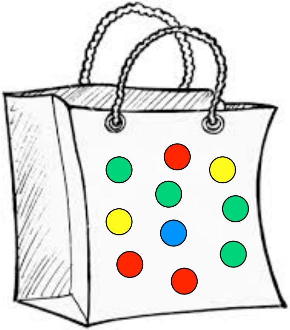

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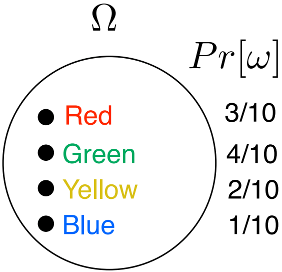

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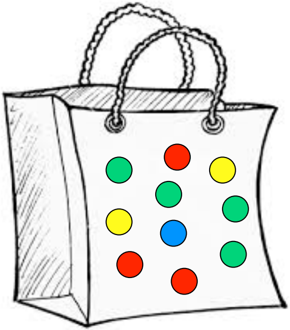

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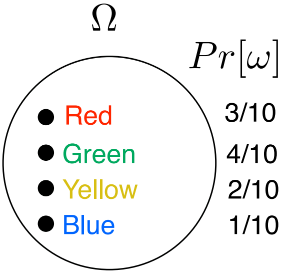

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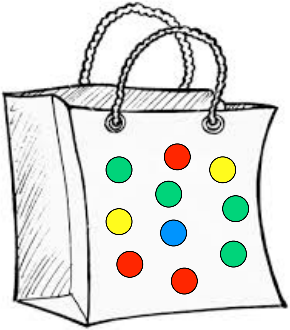

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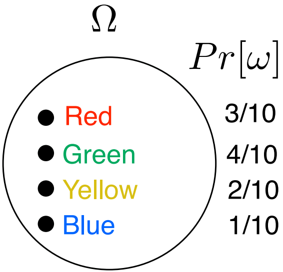

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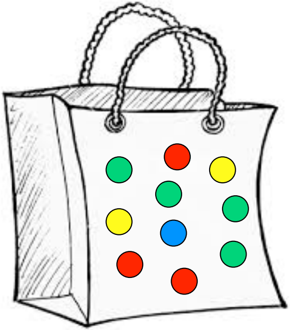

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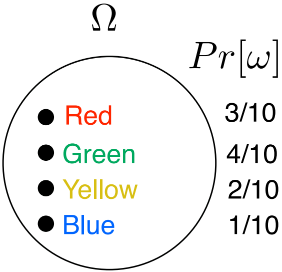

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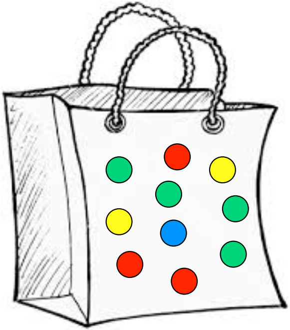

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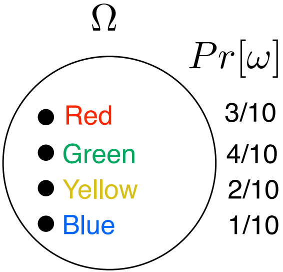

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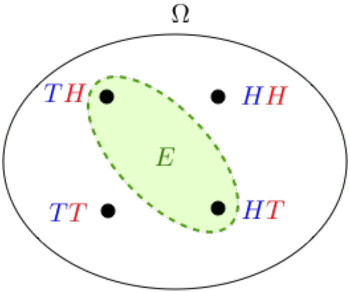

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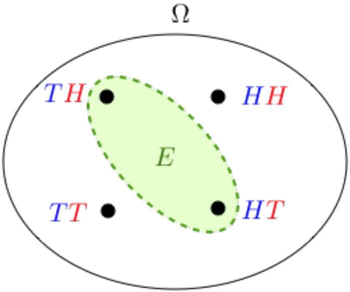

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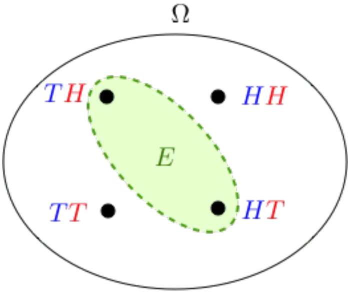

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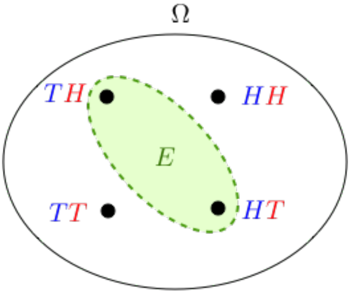

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Probability of exactly one heads in two coin flips?

Sample Space, $\Omega = \{HH, HT, TH, TT\}$.

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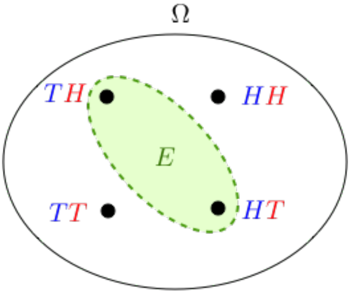

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20 coin tosses

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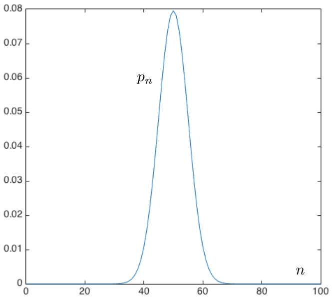

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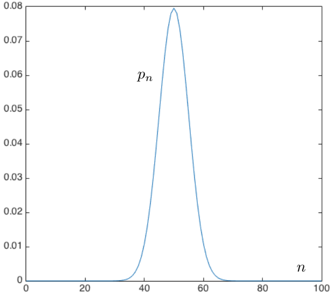

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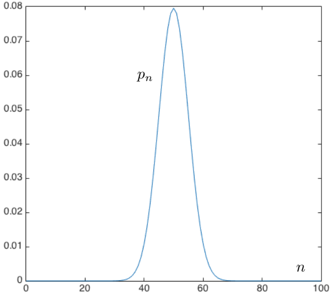

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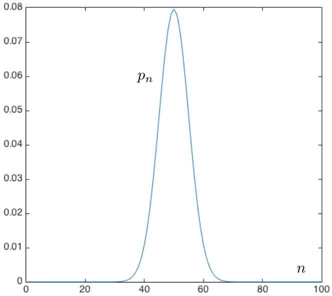

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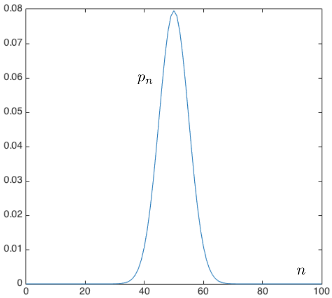

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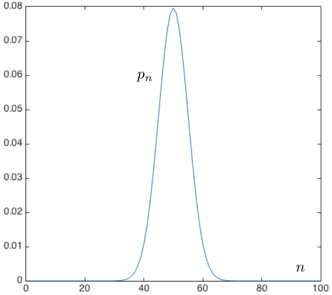

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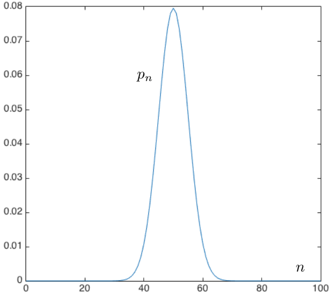

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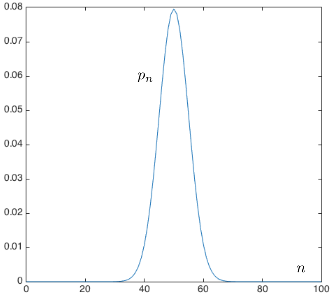

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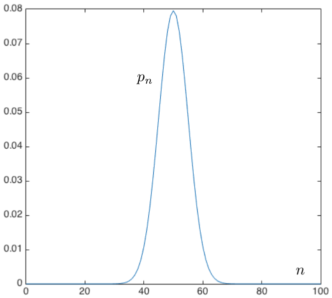

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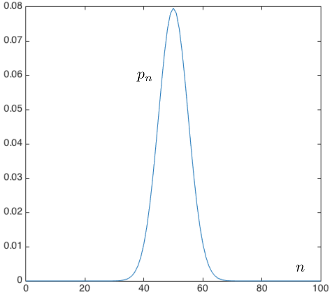

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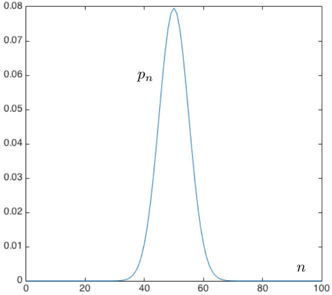

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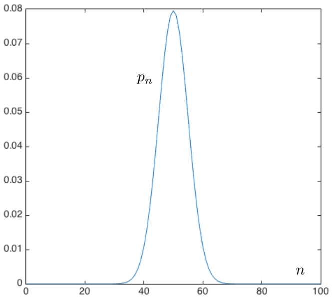

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Roll a red and a blue die.

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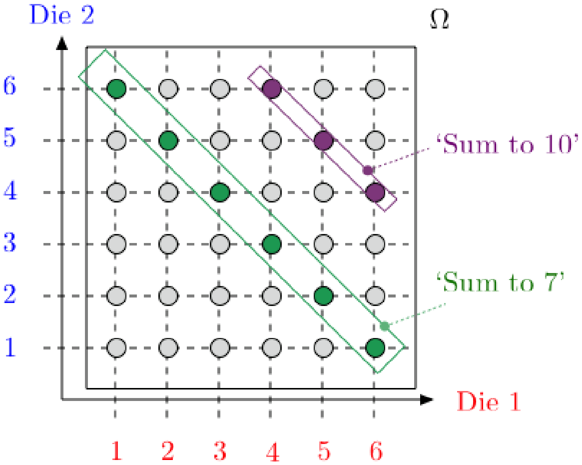

$Pr[\text{Sum to } 7] = \frac{6}{36}$ $\qquad$ $Pr[\text{Sum to } 10] = \frac{3}{36}$

Exactly 50 heads in 100 coin tosses.

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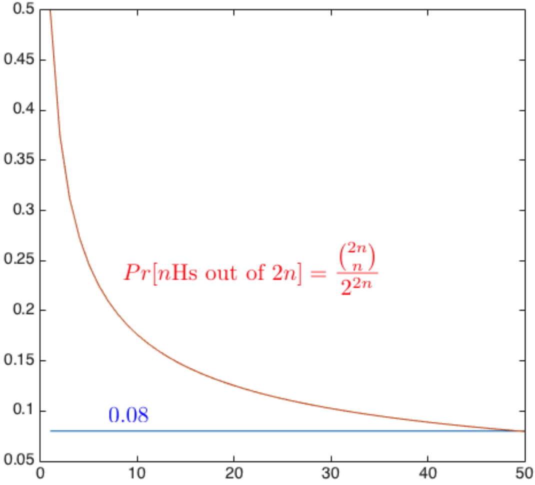
Exactly 50 heads in 100 coin tosses.



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Proof:

Obvious.

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See next two slides for (a) and (c).

Inclusion/Exclusion

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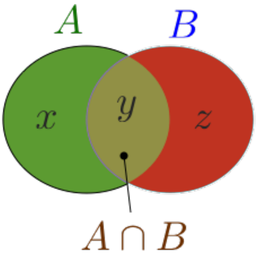
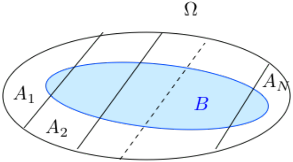


$$Pr[A] = x + y$$
$$Pr[B] = y + z$$
$$Pr[A \cap B] = y$$
$$Pr[A \cup B] = x + y + z$$

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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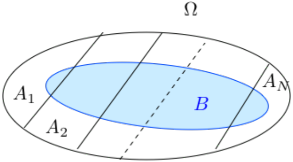

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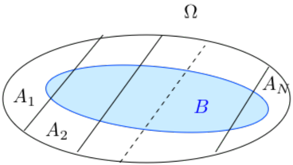

Then,

$$Pr[B] = Pr[A_1 \cap B] + \cdots + Pr[A_N \cap B].$$

Indeed, B is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$.

Roll a Red and a Blue Die.

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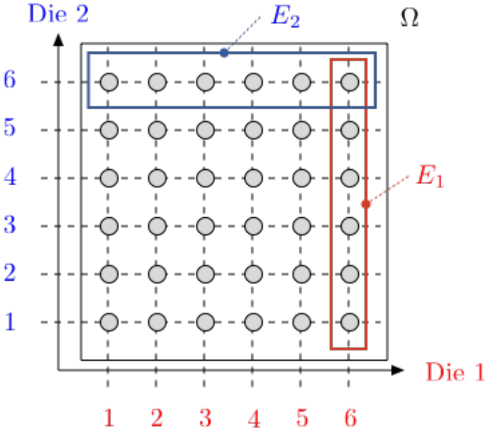

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

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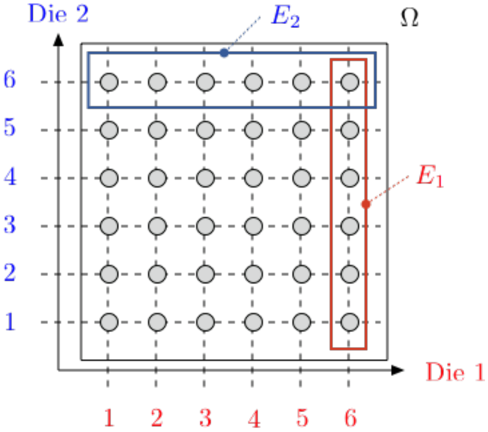

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

$E_1 =$ 'Red die shows 6';

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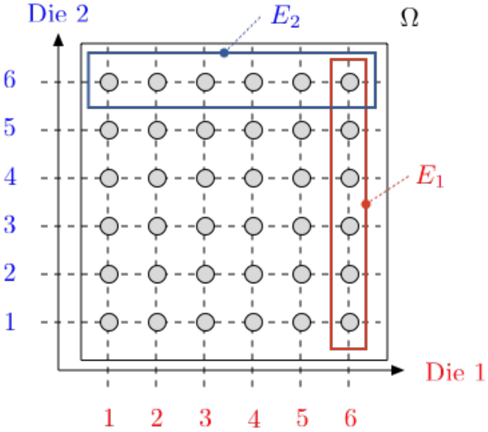

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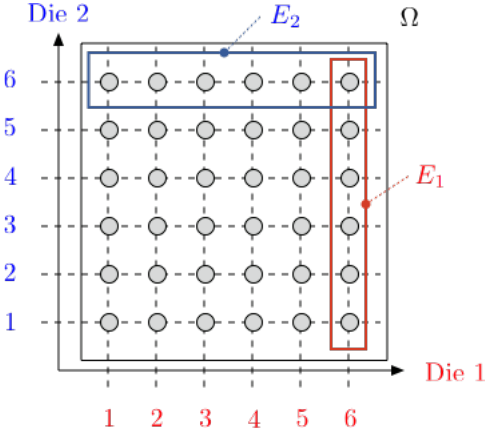

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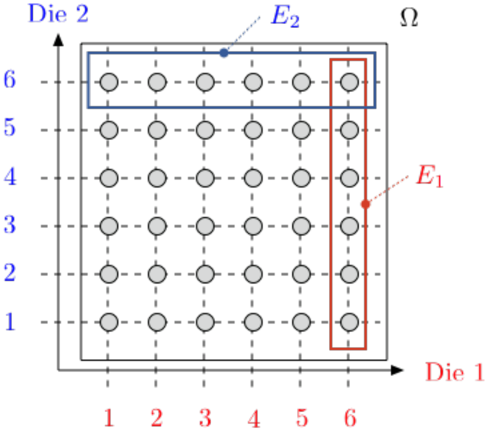

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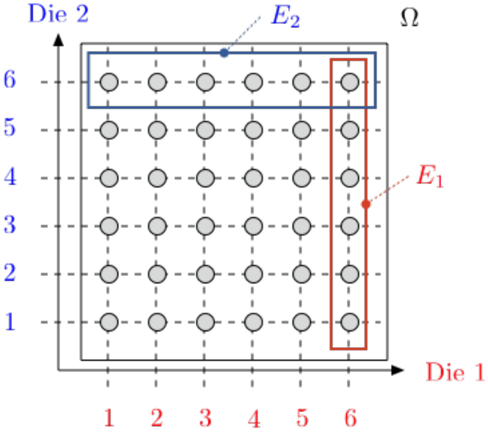
$E_1 =$ ‘Red die shows 6’; $E_2 =$ ‘Blue die shows 6’

$E_1 \cup E_2 =$ ‘At least one die shows 6’

$Pr[E_1] = \frac{6}{36},$

Roll a Red and a Blue Die.



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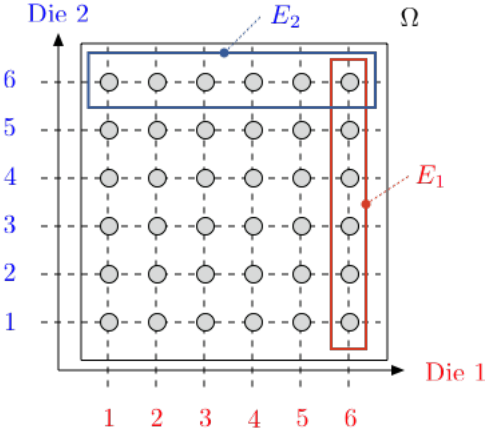

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$$Pr[E_1] = \frac{6}{36}, Pr[E_2] = \frac{6}{36}, Pr[E_1 \cup E_2] = \frac{11}{36}.$$

Conditional probability: example.

Two coin flips.

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Two coin flips. First flip is heads.

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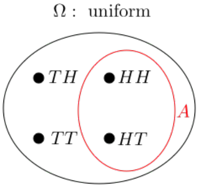

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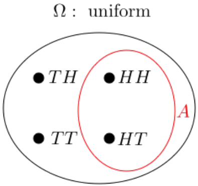

New sample space: A;

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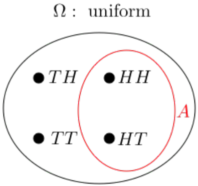

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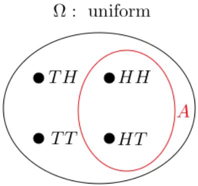

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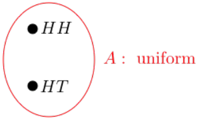

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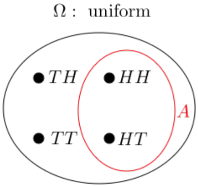

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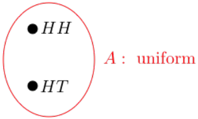

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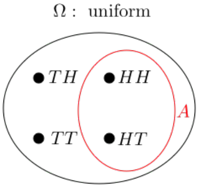

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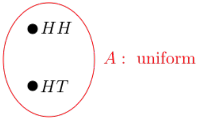

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The probability of two heads if the first flip is heads.

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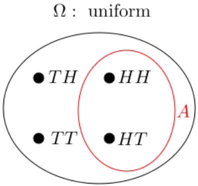

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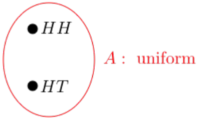

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The probability of two heads if the first flip is heads.
The probability of B given A

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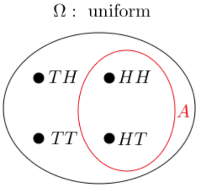

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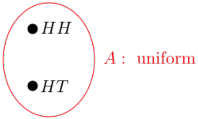

Event $B =$ two heads.

The probability of two heads if the first flip is heads.
The probability of B given A is $1/2$.

A similar example.

Two coin flips.

A similar example.

Two coin flips. At least one of the flips is heads.

A similar example.

Two coin flips. At least one of the flips is heads.
$\rightarrow$ Probability of two heads?

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$\rightarrow$ Probability of two heads?

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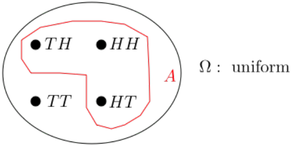

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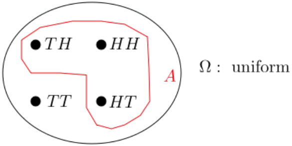

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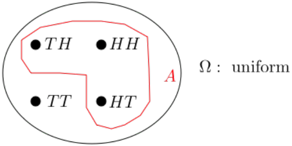

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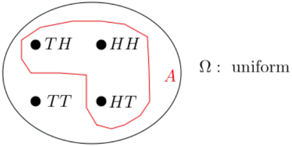

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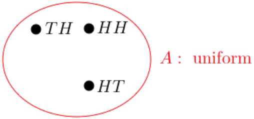

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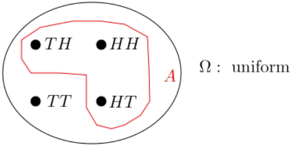

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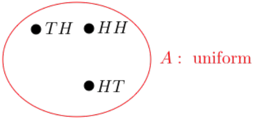

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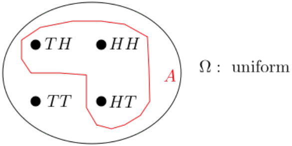

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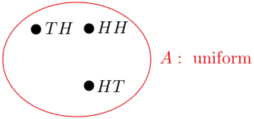

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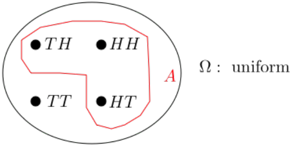

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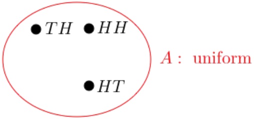

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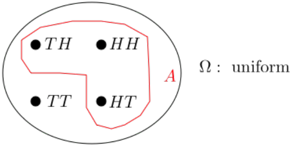

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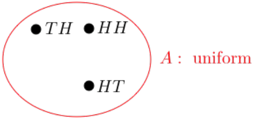

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The probability of two heads if at least one flip is heads.
The probability of B given A is $1/3$.

Conditional Probability: A non-uniform example

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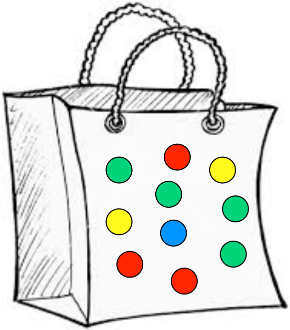
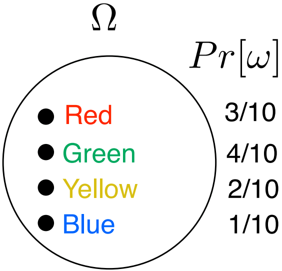


Physical experiment



Probability model

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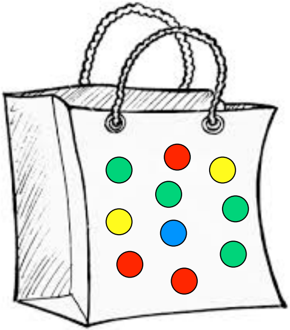

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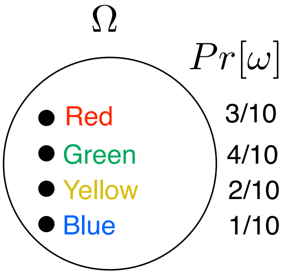

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$\Omega = \{$Red, Green, Yellow, Blue$\}$

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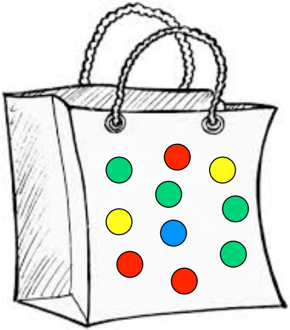

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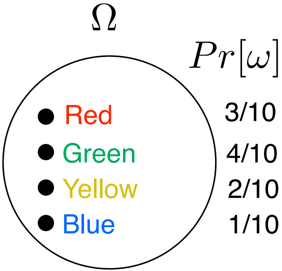

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$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$Pr[\text{Red}|\text{Red or Green}] =$

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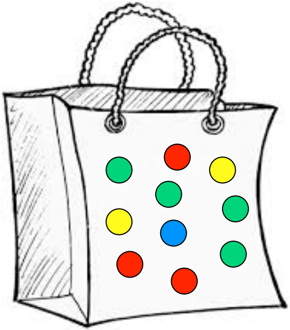

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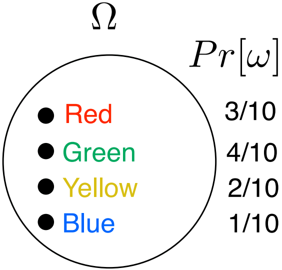

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$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

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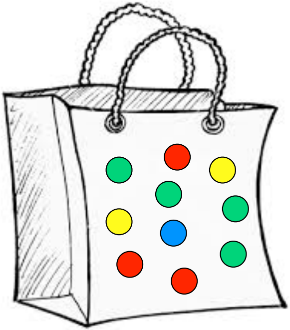
Conditional Probability: A non-uniform example



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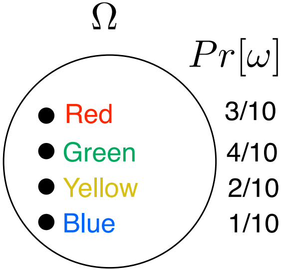

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$$\Omega = \{\text{Red, Green, Yellow, Blue}\}$$

$$Pr[\text{Red}|\text{Red or Green}] = \frac{3}{7} = \frac{Pr[\text{Red} \cap (\text{Red or Green})]}{Pr[\text{Red or Green}]}$$

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Consider $\Omega = \{1, 2, \ldots, N\}$ with $Pr[n] = p_n$.

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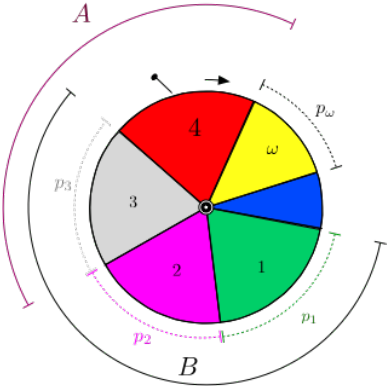

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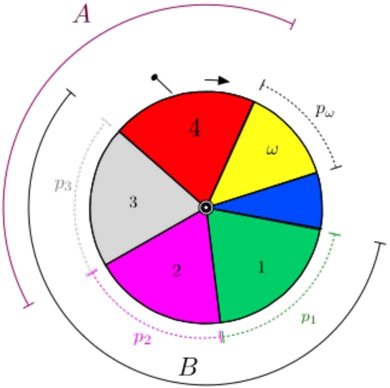

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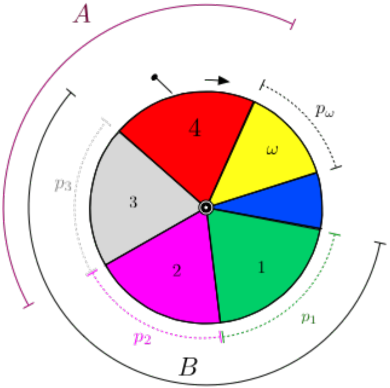

$$Pr[A|B] = \frac{p_3}{p_1 + p_2 + p_3} = \frac{Pr[A \cap B]}{Pr[B]}.$$

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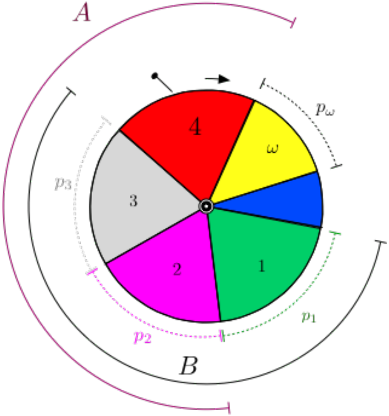

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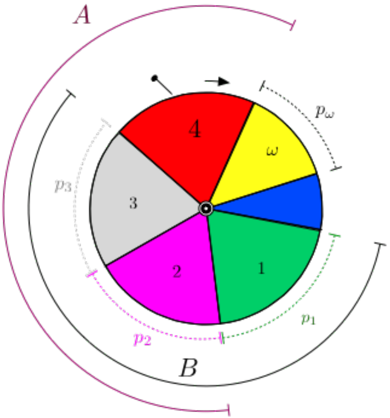

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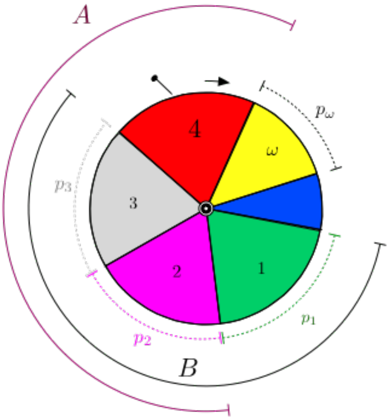

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Definition: The **conditional probability** of B given A is

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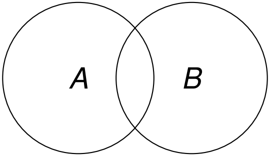

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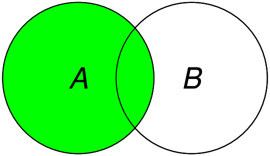

In A!

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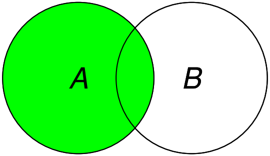

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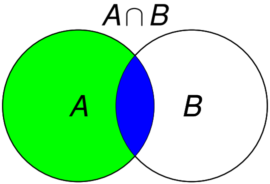

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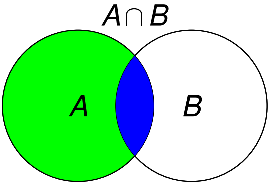

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More fun with conditional probability.

Toss a red and a blue die, sum is 4,

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Toss a red and a blue die, sum is 4,
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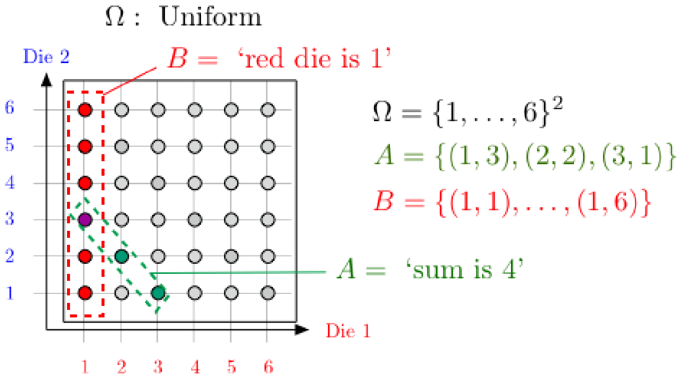

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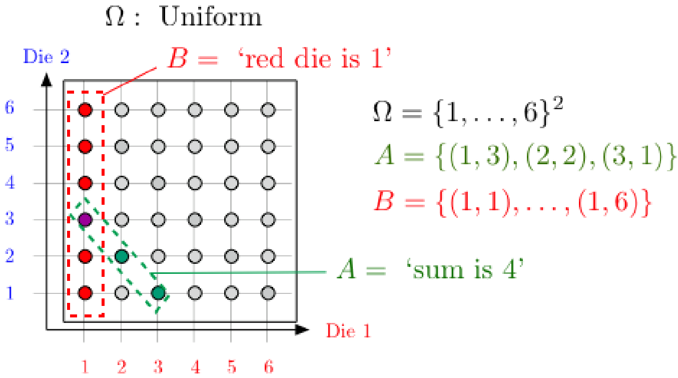

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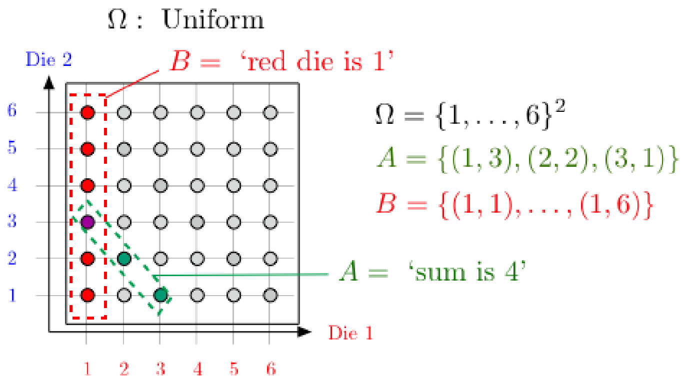

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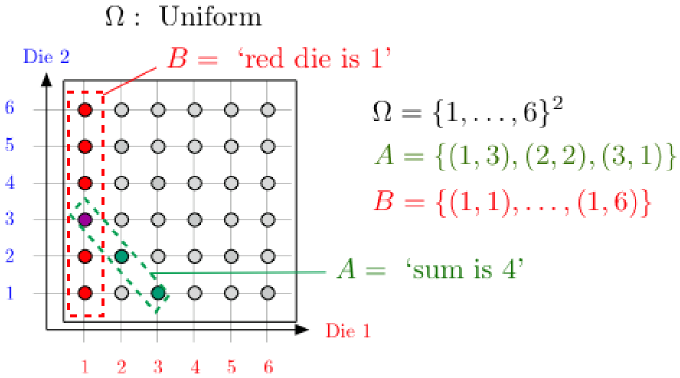

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B is more likely given *A*.

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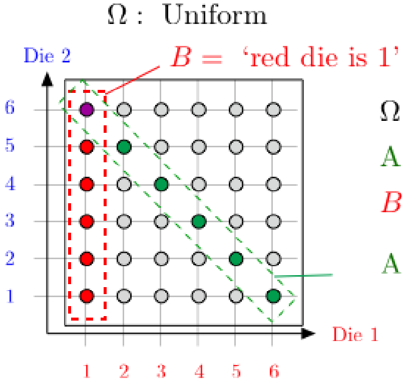

$\Omega = \{1, \ldots, 6\}^2$

$A = \{(1, 6), \ldots, (6,1)\}$

$B = \{(1,1), \ldots, (1,6)\}$

A = 'sum is 7'

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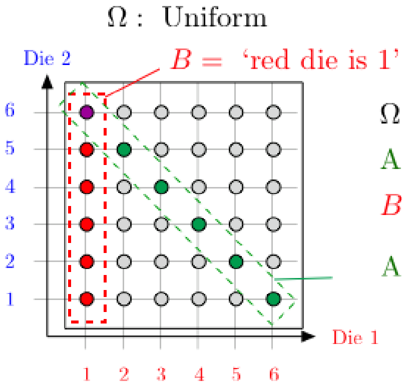

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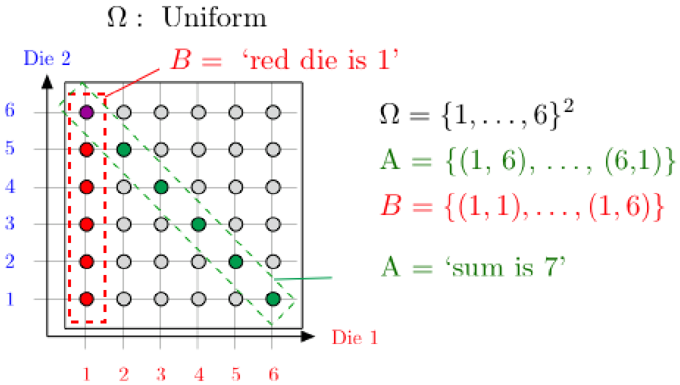

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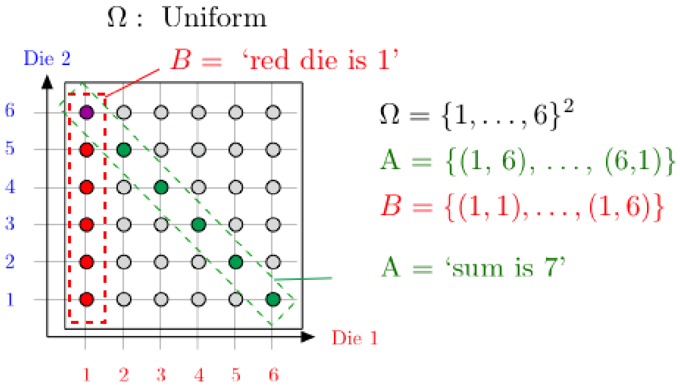

$Pr[B|A] = \frac{|B \cap A|}{|A|} = \frac{1}{6}$; versus $Pr[B] = \frac{1}{6}$.

Observing *A* does not change your mind about the likelihood of *B*.

Emptiness..

Suppose I toss 3 balls into 3 bins.

Emptiness..

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$A =$"1st bin empty";

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Suppose I toss 3 balls into 3 bins.
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Suppose I toss 3 balls into 3 bins.
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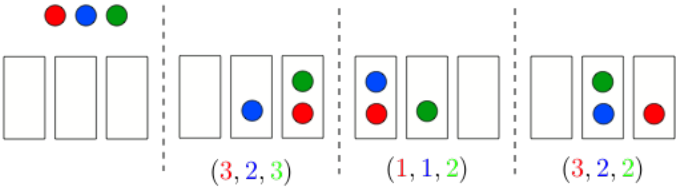
$\Omega = \{1, 2, 3\}^3$



$\omega =$ (bin of red ball, bin of blue ball, bin of green ball)

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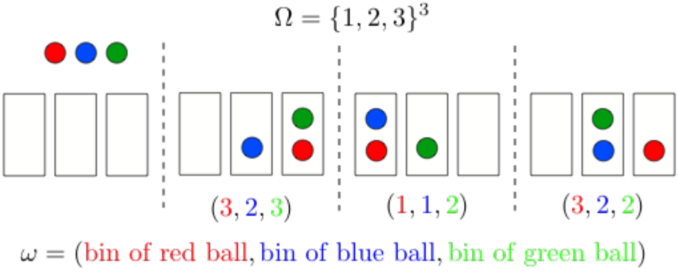

$Pr[B]$

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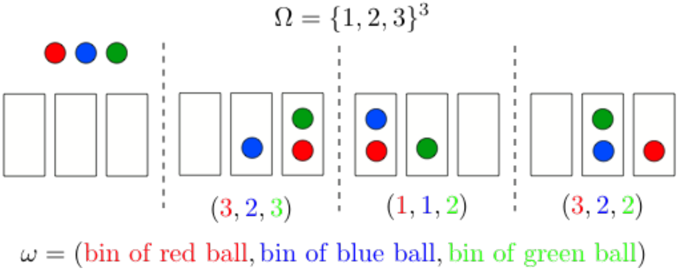

$Pr[B] = Pr[\{(a,b,c) \mid a,b,c \in \{1,3\}] =$

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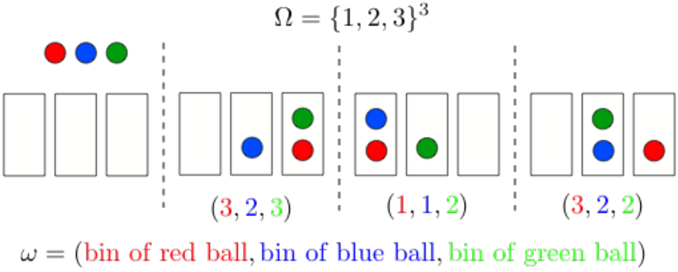

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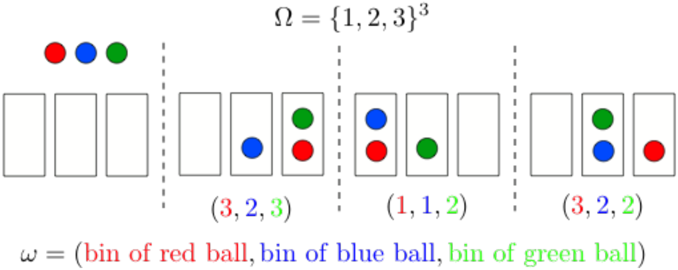

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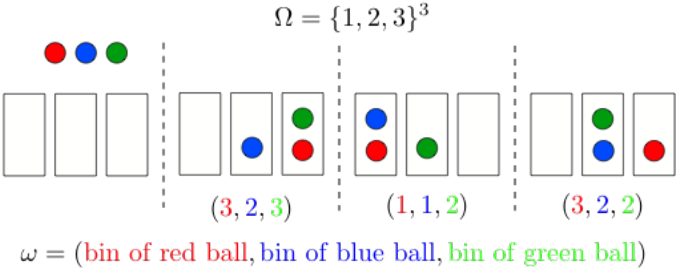

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$Pr[A \cap B]$

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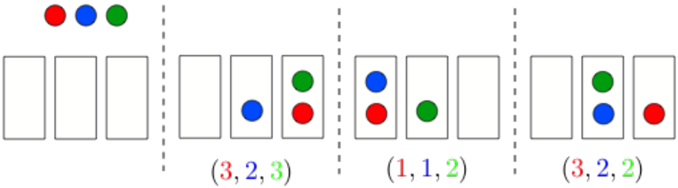
$$\Omega = \{1,2,3\}^3$$



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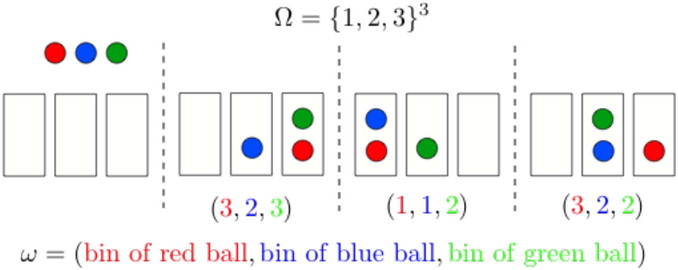

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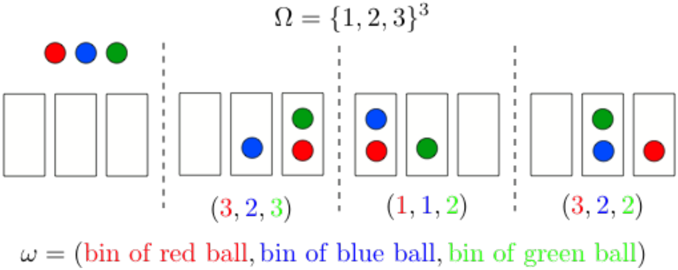

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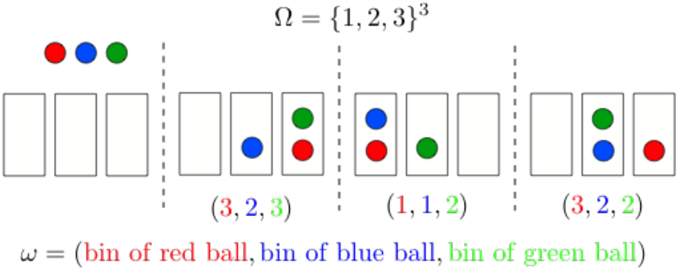

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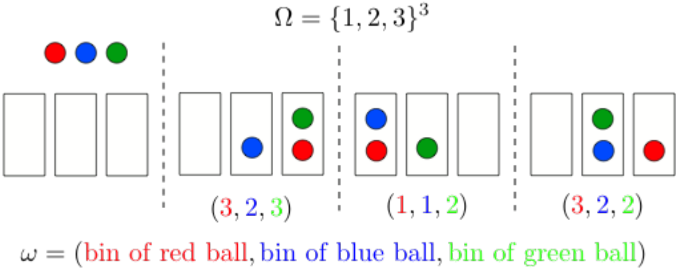

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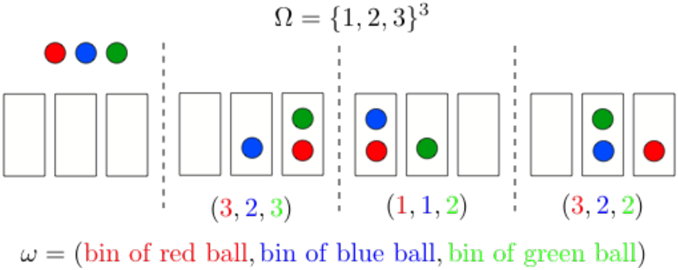

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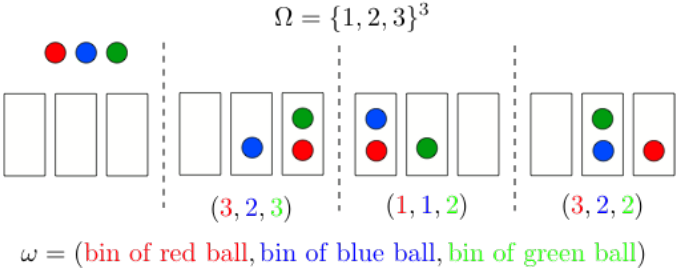

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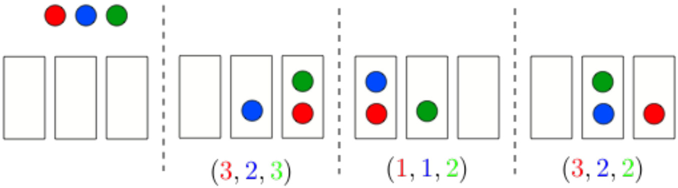

$$\omega = (\text{bin of red ball}, \text{bin of blue ball}, \text{bin of green ball})$$

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The likelihood of 51st heads does not depend on the previous flips.

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so that the result holds for $n + 1$. $\square$

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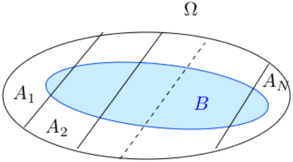
More about such questions later. For fun, check "N. Taleb: Fooled by randomness."

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



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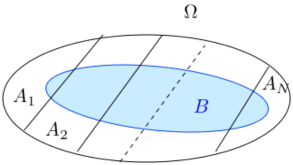

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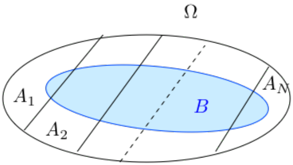

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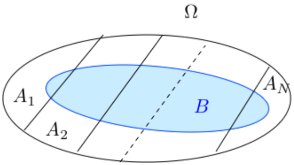

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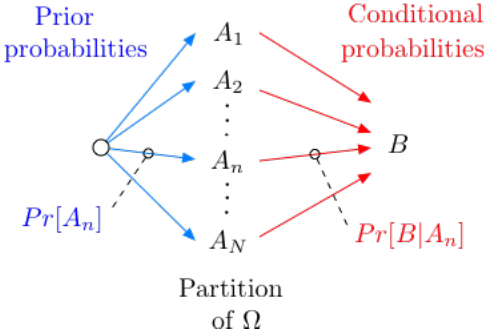
Indeed, B is the union of the disjoint sets $A_n \cap B$ for $n = 1, \ldots, N$.
Thus,

$$Pr[B] = Pr[A_1]Pr[B|A_1] + \cdots + Pr[A_N]Pr[B|A_N].$$

Total probability

Assume that Ω is the union of the disjoint sets $A_1, \ldots, A_N$.



$$Pr[B] = Pr[A_1]Pr[B|A_1] + \cdots + Pr[A_N]Pr[B|A_N].$$

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Thus,

$$Pr[A|B] = \frac{Pr[A]Pr[B|A]}{Pr[B]} = \frac{(1/2)(1/2)}{(1/2)(1/2) + (1/2)0.6} \approx 0.45.$$

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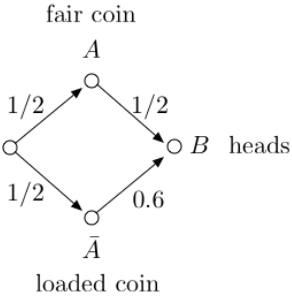

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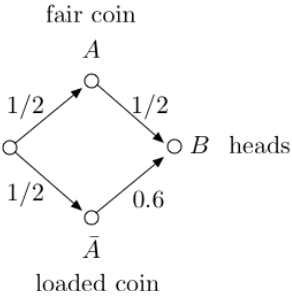

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$m := 100(1/2)(1/2)$ are such that A and B occur and
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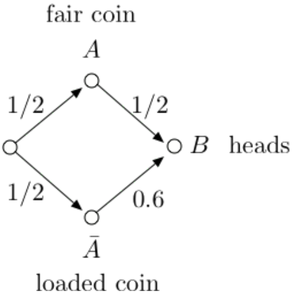

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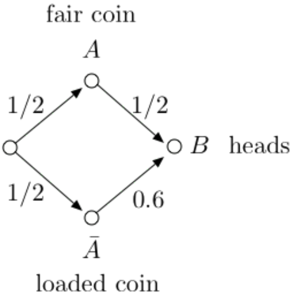

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Another picture: We imagine that there are N possible causes $A_1, \ldots, A_N$.

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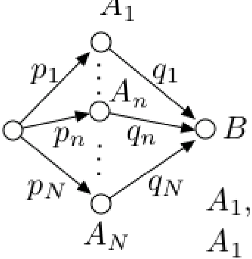

$$p_n = Pr[A_n]$$

$$q_n = Pr[B|A_n]$$

$A_1, \ldots, A_N$ disjoint

$A_1 \cup \cdots \cup A_N = \Omega$

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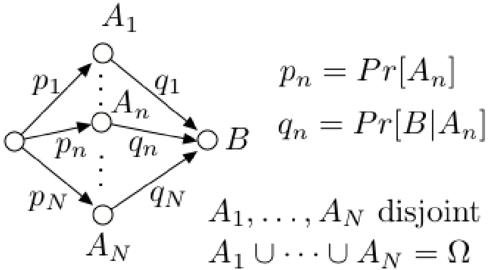

Imagine 100 situations, among which $100 p_n q_n$ are such that A_n and B occur, for $n = 1, \ldots, N$.
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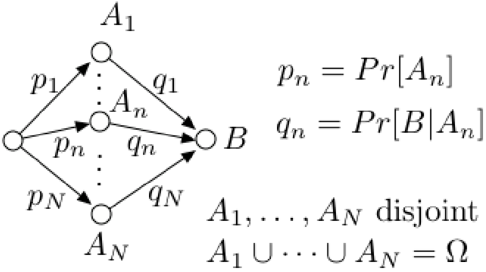

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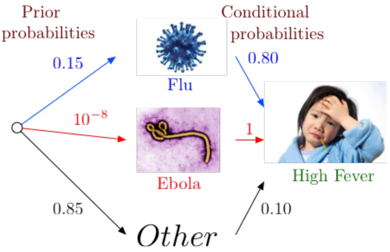
Hence,

$$Pr[A_n|B] = \frac{p_n q_n}{\sum_m p_m q_m}.$$

Why do you have a fever?



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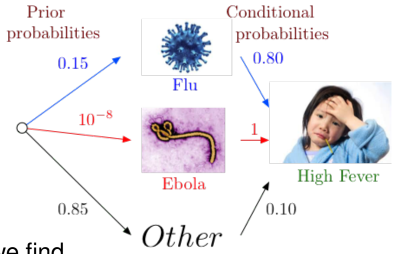

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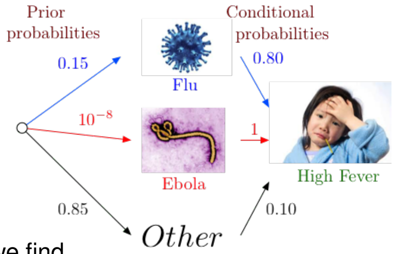

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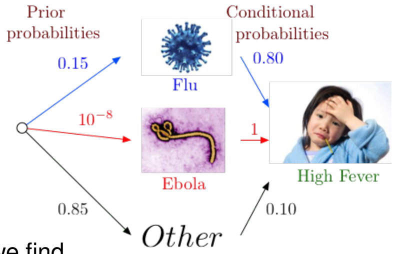

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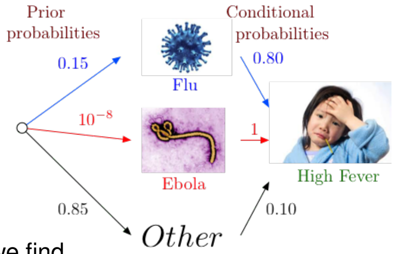

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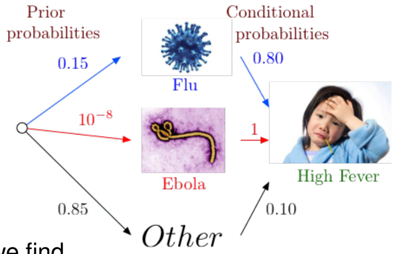

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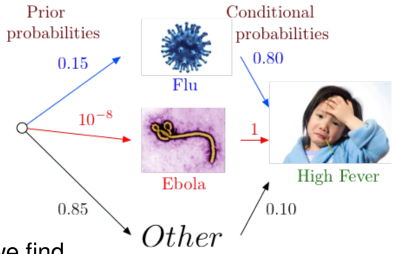

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These are the posterior probabilities. One says that 'Flu' is the Most Likely a Posteriori (MAP) cause of the high fever.

Bayes’ Rule Operations

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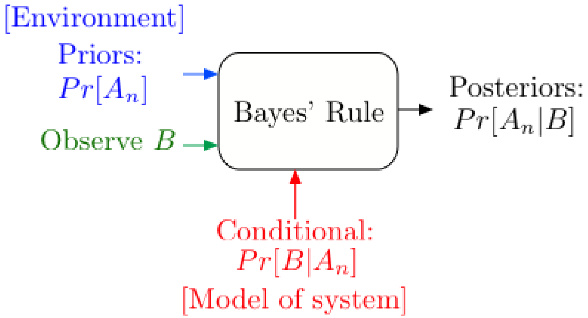

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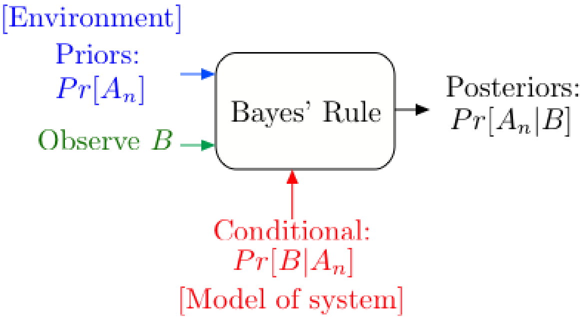

Bayes' Rule is the canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A

FIG. 3. *Joshua Bayes (1671–1746).*

$Pr[B|A]$ →

Max over B

B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Testing for disease.

Let’s watch TV!!

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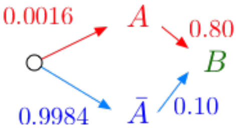
$$Pr[A|B]???$$

Bayes Rule.



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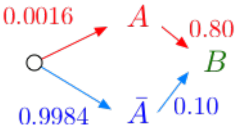

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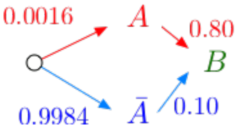

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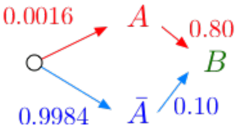

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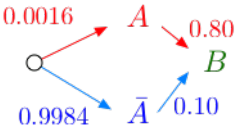

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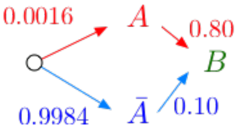

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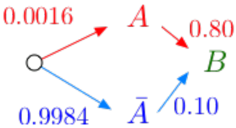

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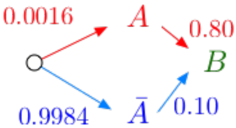

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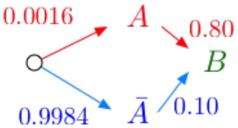

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Events, Conditional Probability, Independence, Bayes' Rule

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- All these are possible:

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