CS70: Jean Walrand: Lecture 17.

Bayes' Rule, Mutual Independence, Collisions and Collecting

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Bayes' Rule, Mutual Independence, Collisions and Collecting

1. Conditional Probability
2. Independence
3. Bayes' Rule
4. Balls and Bins
5. Coupons

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Conditional Probability: Pictures

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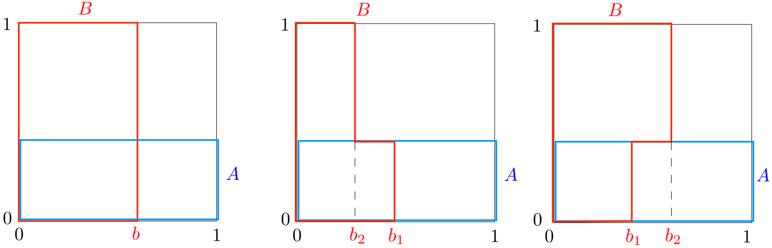
Illustrations: Pick a point uniformly in the unit square



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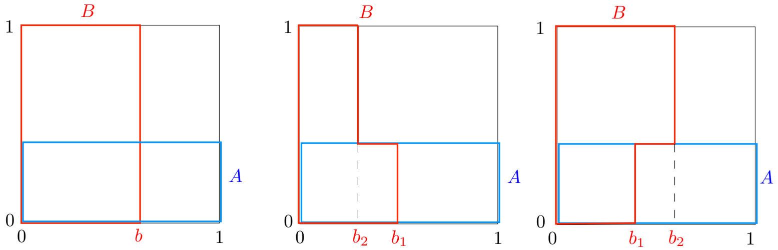

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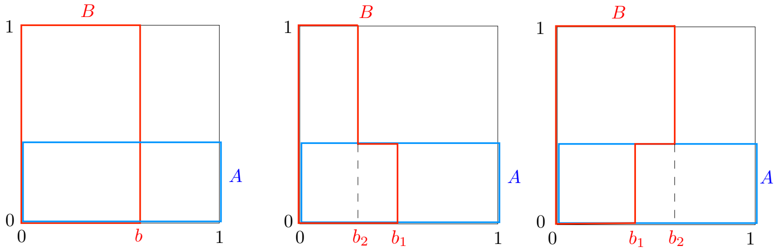

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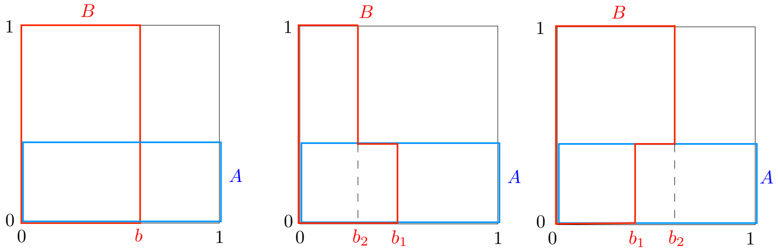

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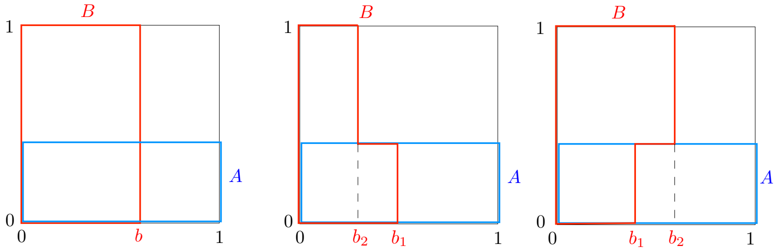

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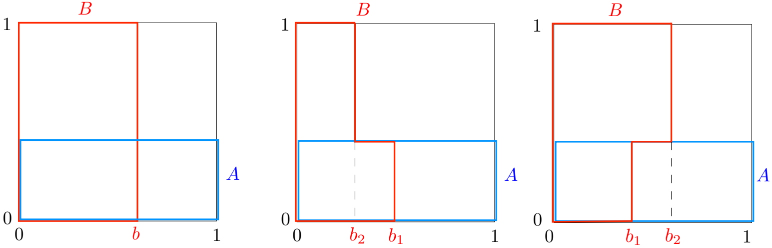

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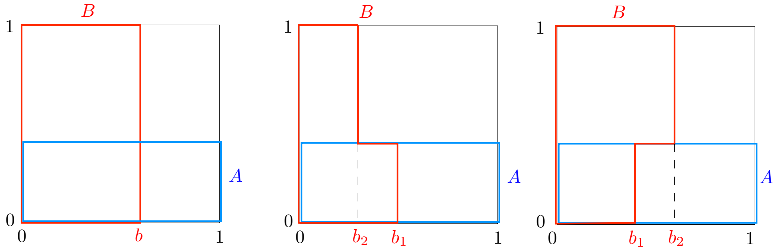

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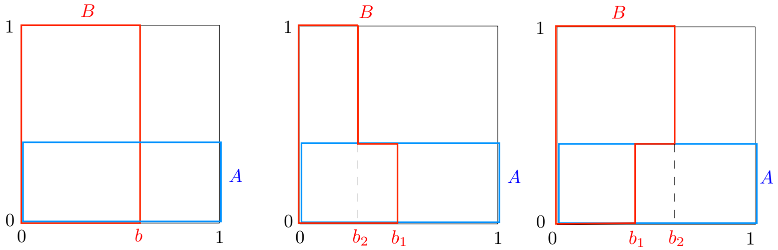

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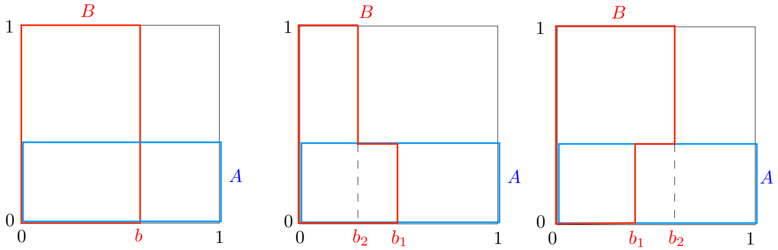

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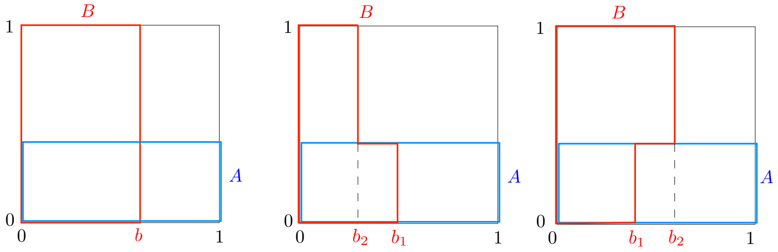

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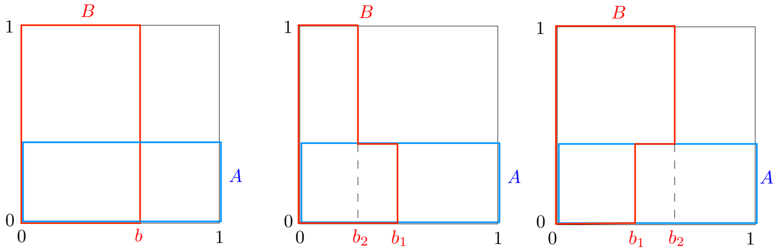

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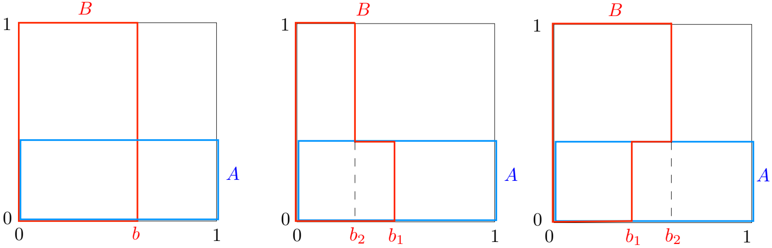

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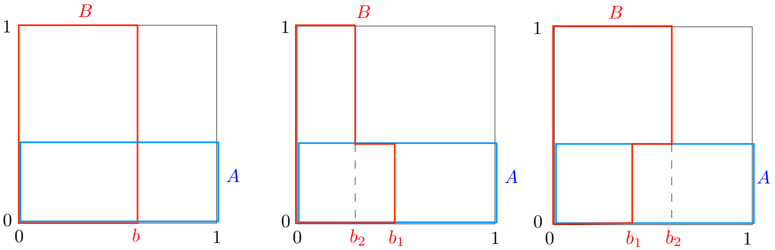

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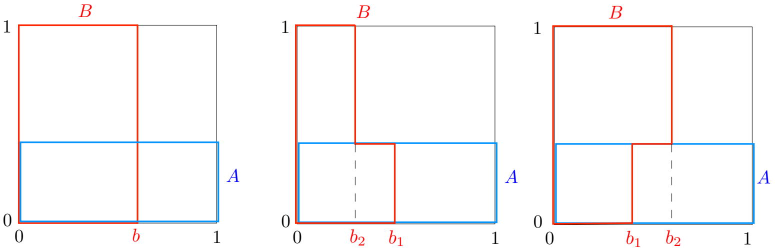

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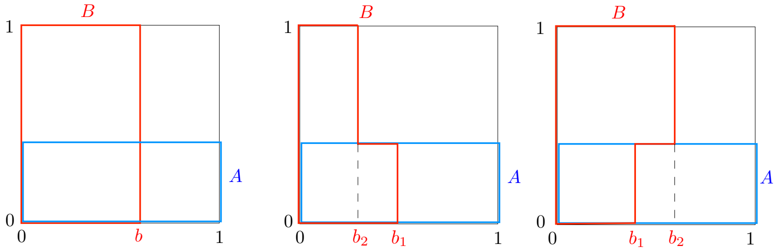

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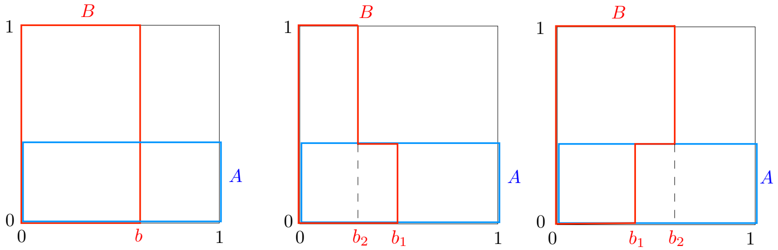

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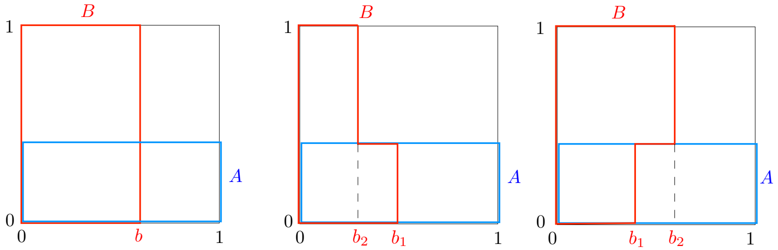

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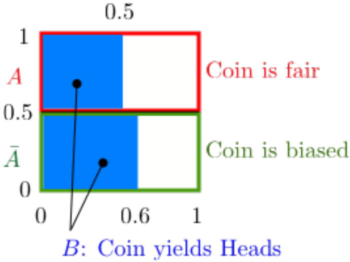

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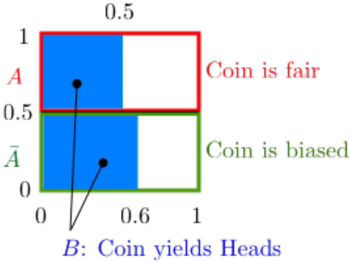

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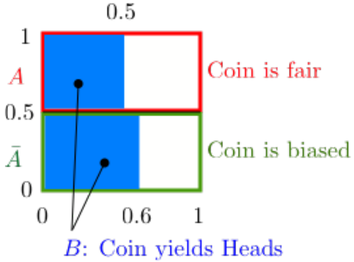

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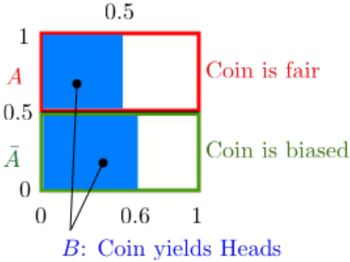

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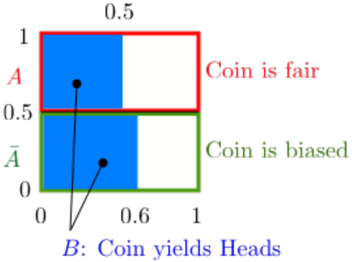

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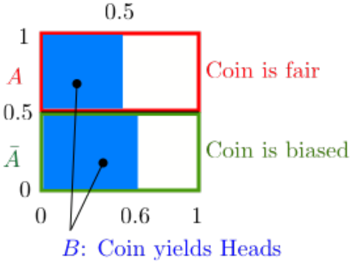

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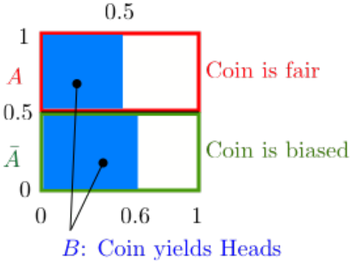

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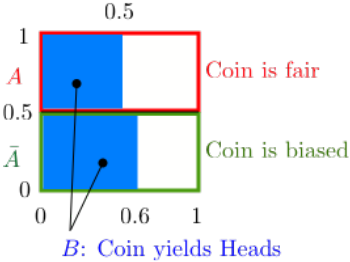

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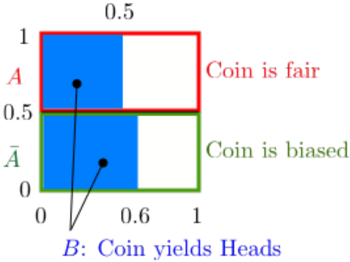

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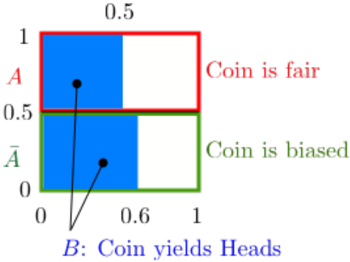

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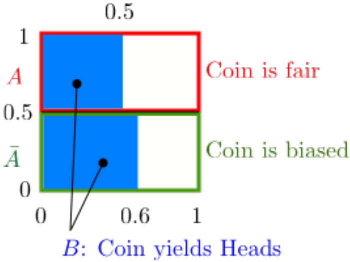

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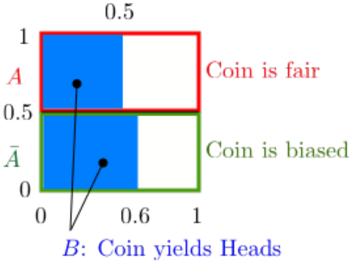

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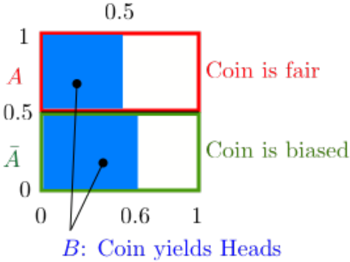

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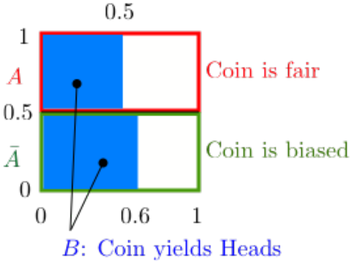

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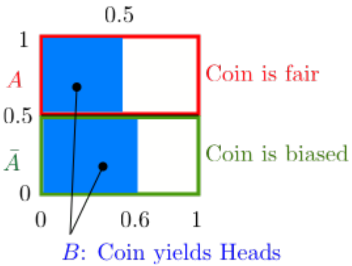
Bayes and Biased Coin



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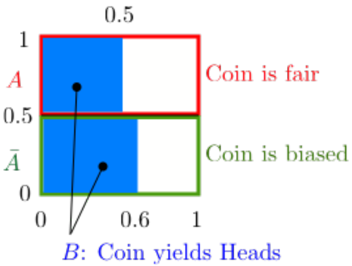

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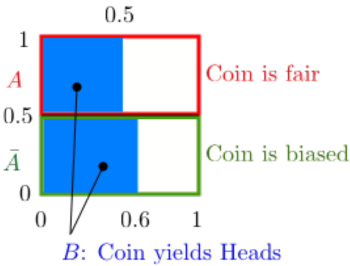

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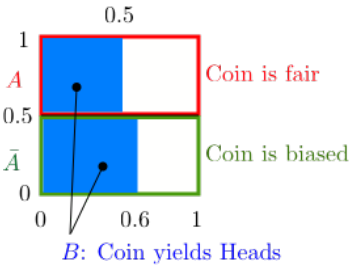

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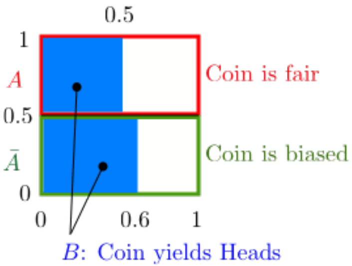
$$\approx 0.46$$

Bayes and Biased Coin



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$$\approx 0.46 = \text{fraction of } B \text{ that is inside } A$$

Bayes: General Case

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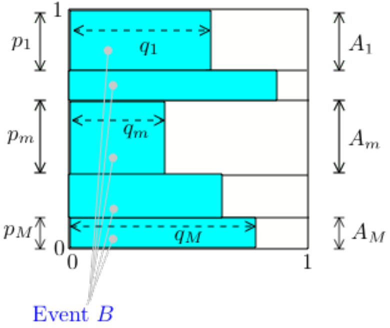

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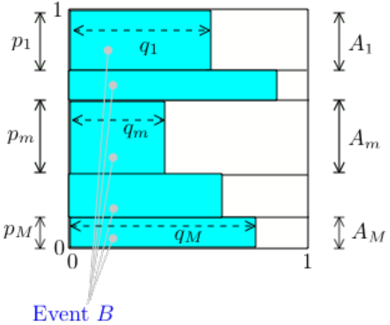

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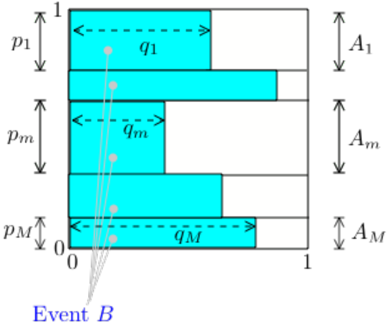

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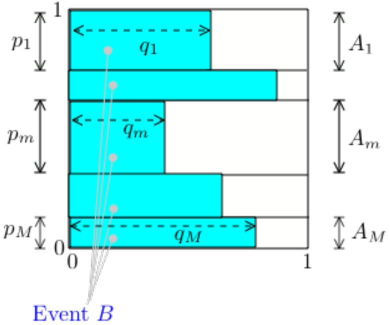

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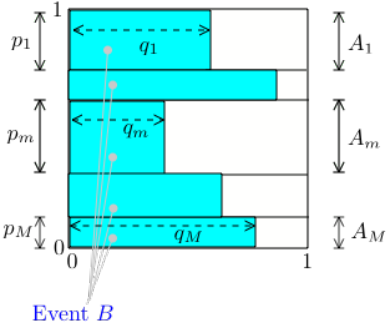

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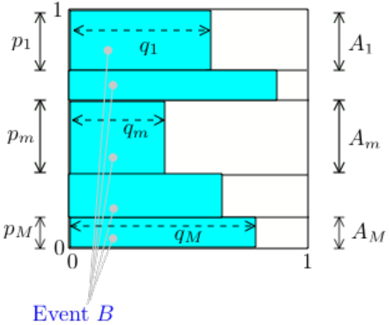

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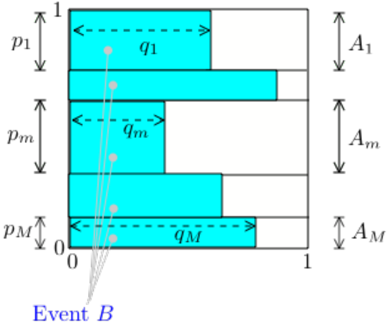

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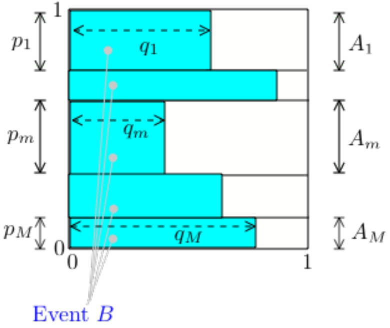

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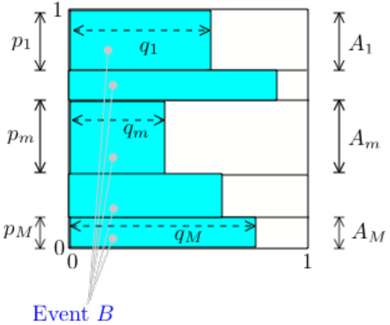

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Bayes Rule

Another picture:

Bayes Rule

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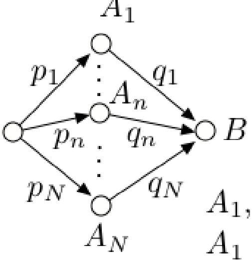

$$p_n = Pr[A_n]$$

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$A_1, \ldots, A_N$ disjoint

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Bayes Rule

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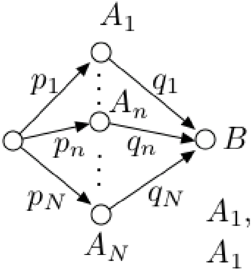

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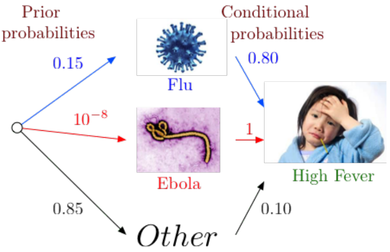
Why do you have a fever?



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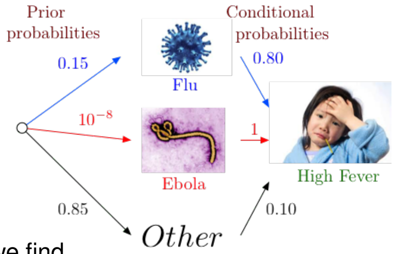

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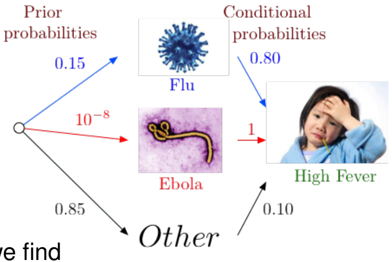

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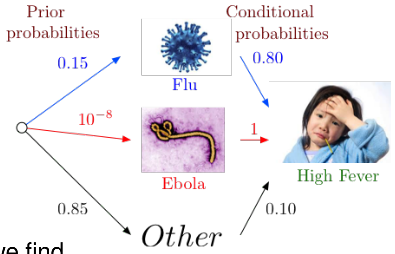

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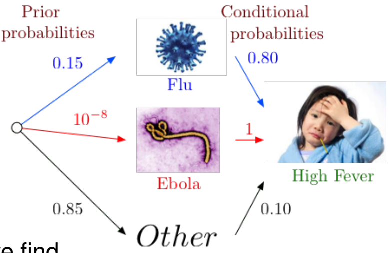

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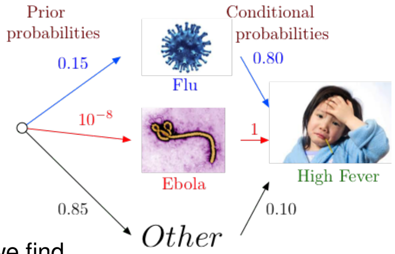

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The values $0.58, 5 \times 10^{-8}, 0.42$ are the posterior probabilities.

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Our “Bayes’ Square” picture:

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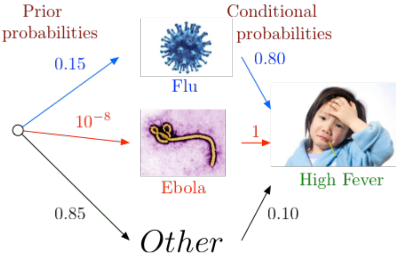

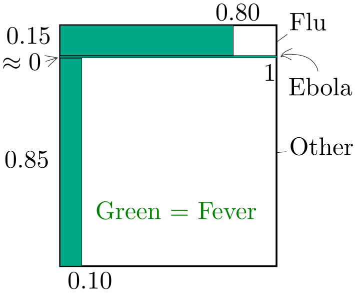

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$\approx$ 0% of Fever = Ebola

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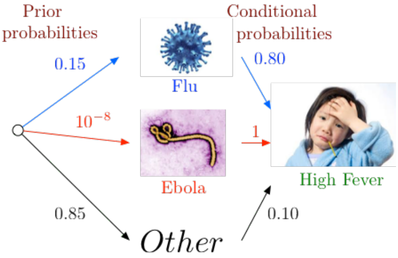

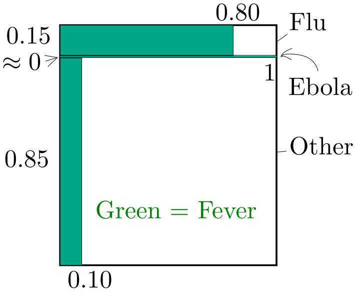

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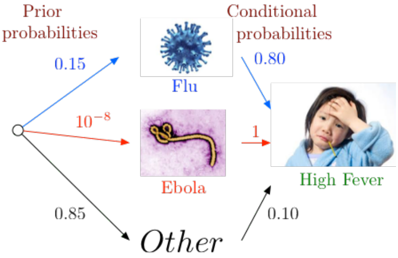

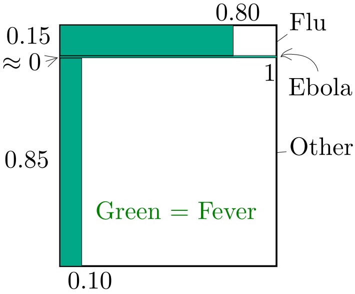

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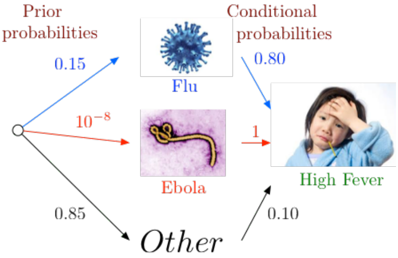

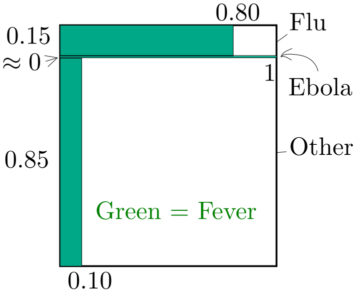

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This example shows the importance of the prior probabilities.

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Bayes' Rule Operations

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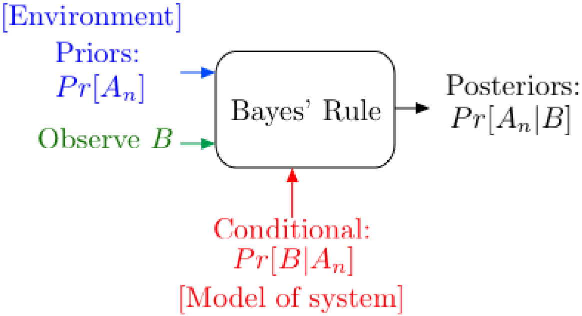
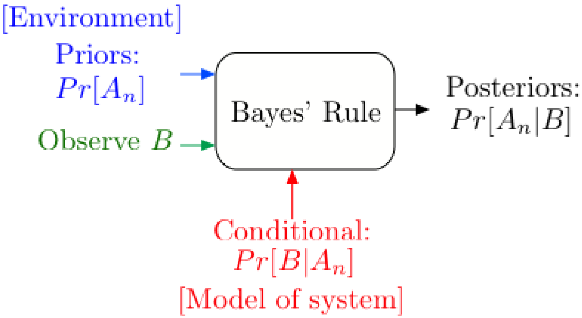


Bayes' Rule Operations



Bayes' Rule is the canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A

FIG. 3. *Joshua Bayes (1671–1746).*

$\xrightarrow{Pr[B|A]}$

Max over B

B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Independence

Recall :

A and B are independent

Independence

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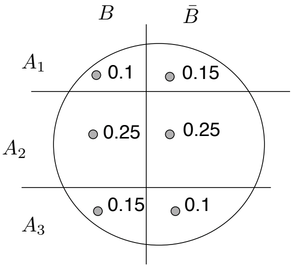

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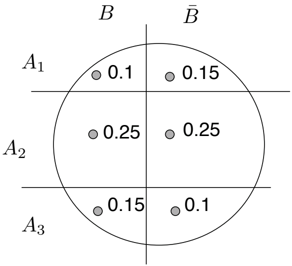

(A_2, B) are independent:

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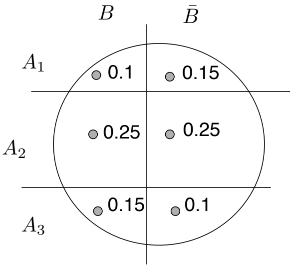

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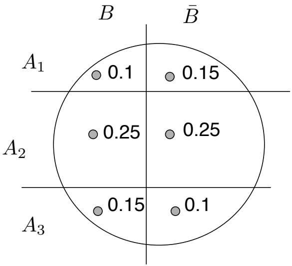

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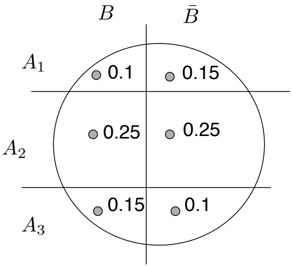

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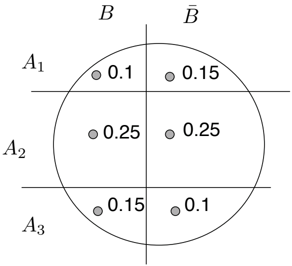

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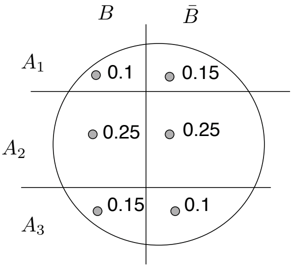

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Pairwise Independence

Flip two fair coins. Let

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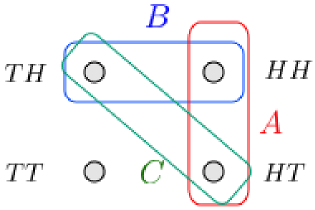

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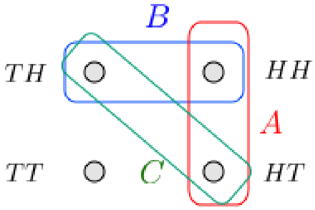

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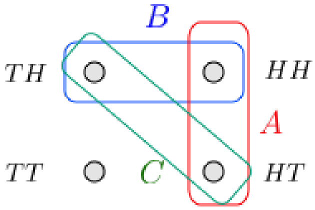

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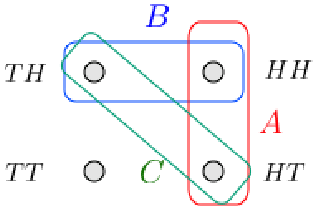

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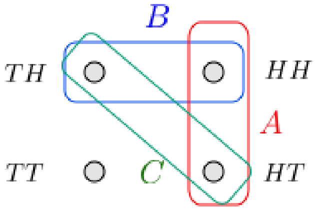

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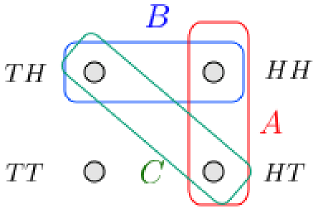

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Example: Flip a fair coin forever. Let A_n = 'coin n is H.' Then the events A_n are mutually independent.

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Proof:
See Notes 25, 2.7. □

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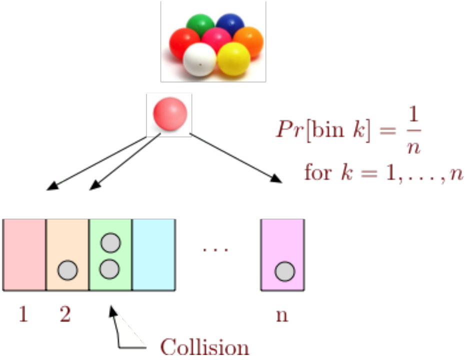

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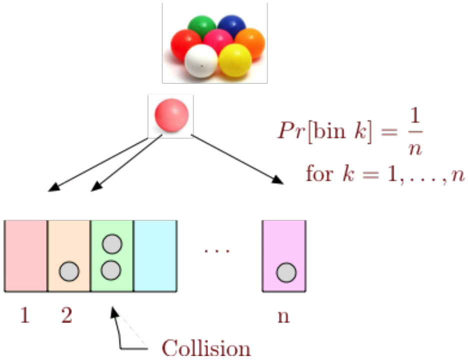

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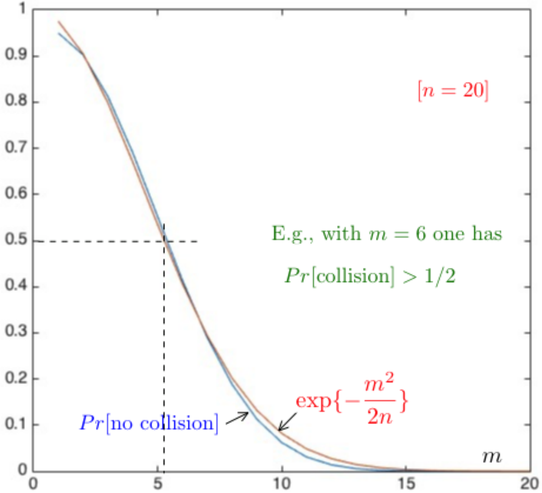

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(†) $1 + 2 + \cdots + m - 1 = (m-1)m/2$.

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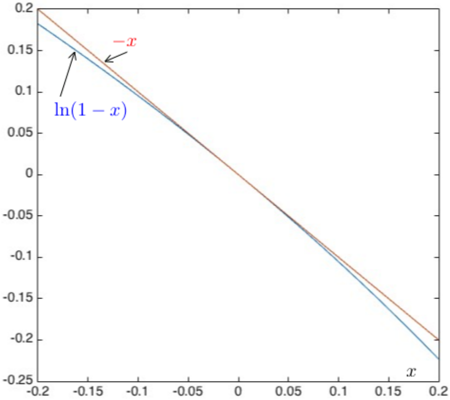

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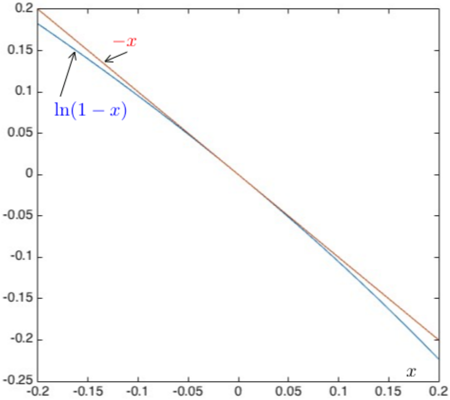

$$\exp\{-x\} = 1 - x + \frac{1}{2!}x^2 + \cdots \approx 1 - x, \text{ for } |x| \ll 1.$$

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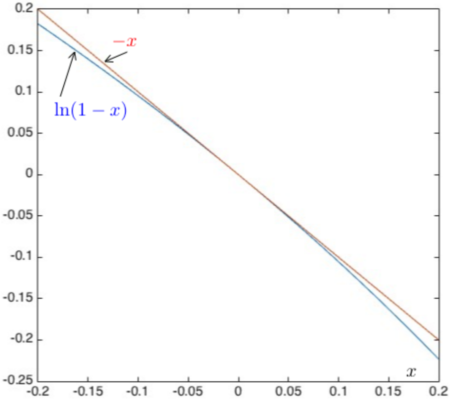

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Hence, $-x \approx \ln(1-x)$ for $|x| \ll 1$.

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Note: $\log_2(x) = \log_2(e)\ln(x) \approx 1.44\ln(x)$.

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For $p_m = \frac{1}{2}$, we need around $n \ln 2 \approx 0.69n$ boxes.

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Plug in and get

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Bayes' Rule, Mutual Independence, Collisions and Collecting

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