CS70: Jean Walrand: Lecture 18.

Random Variables & Midterm 2 Probability Review

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- Random Variables
- M2 Probability Review
- M2 Discrete Math Review: See Video (link given on Piazza)

Random Variables

1. Random Variables.
2. Distributions.
3. Combining random variables.
4. Expectation

Questions about outcomes ...

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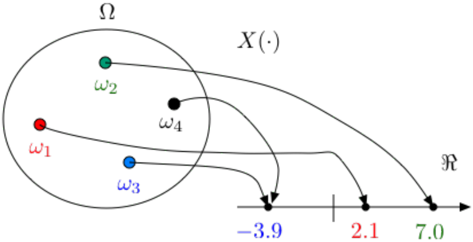

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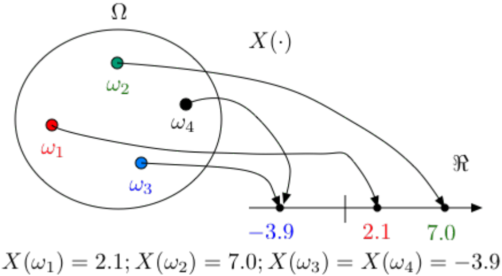

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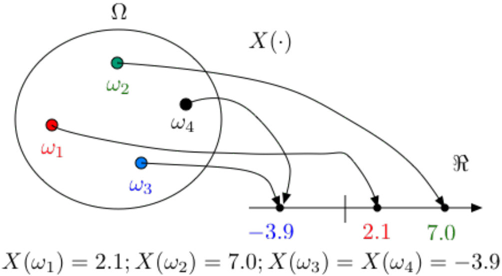

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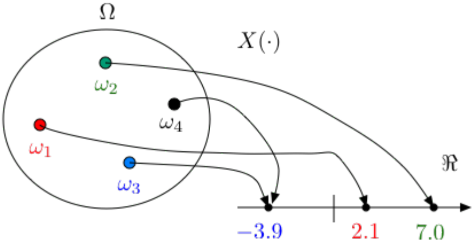

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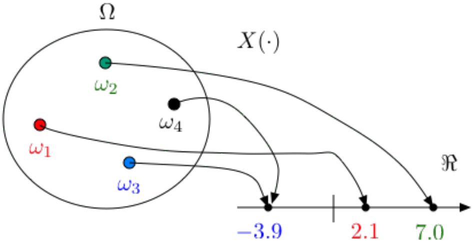

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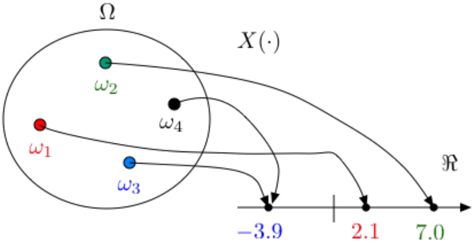

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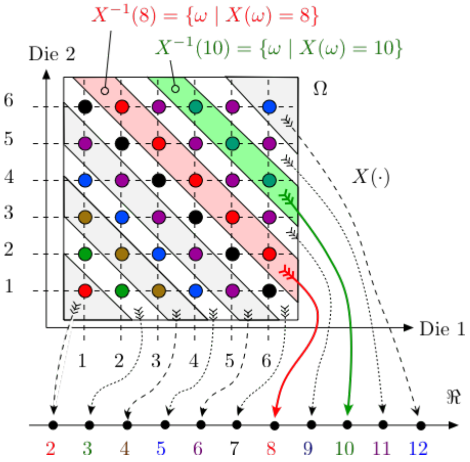

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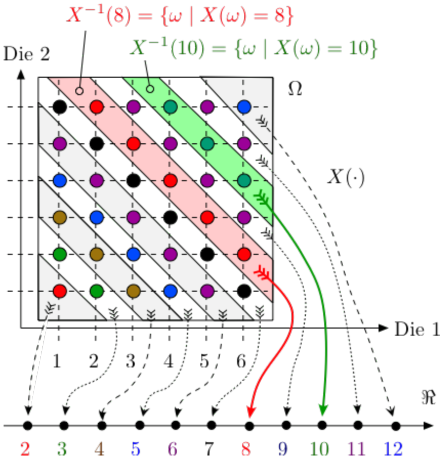

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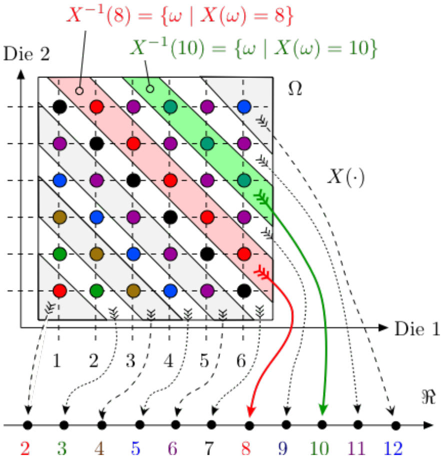

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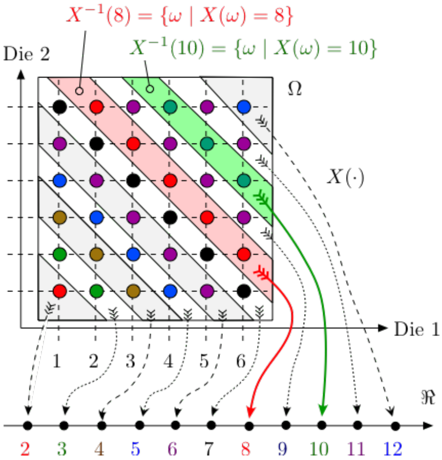

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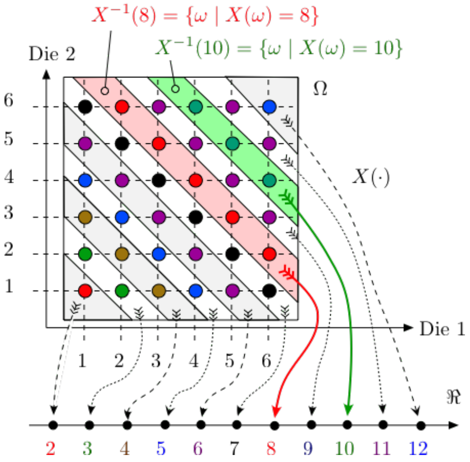

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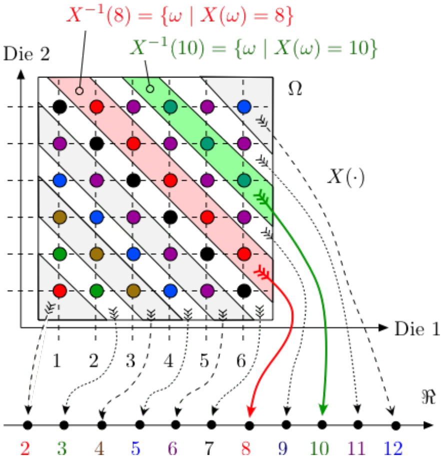

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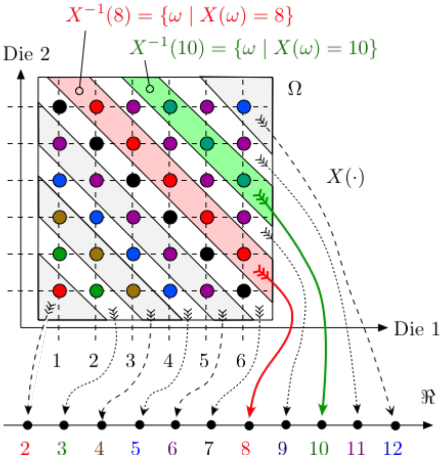

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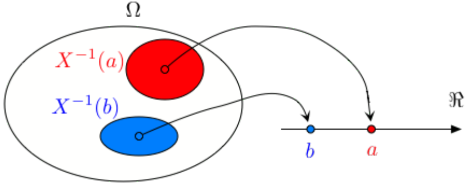

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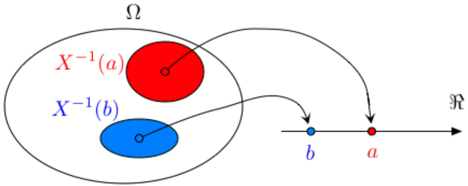

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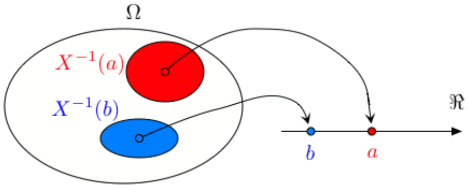

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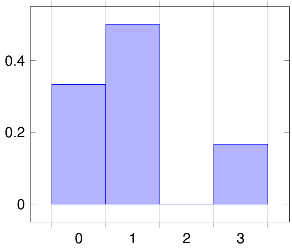

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Flip three coins

Experiment: flip three coins
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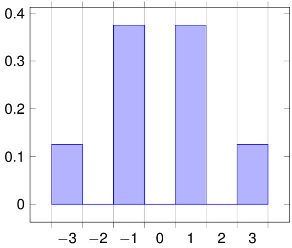

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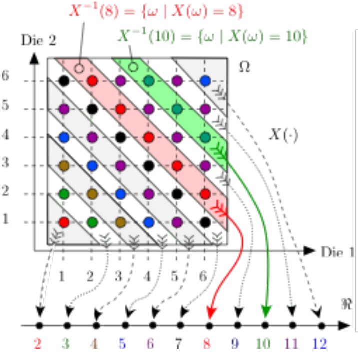

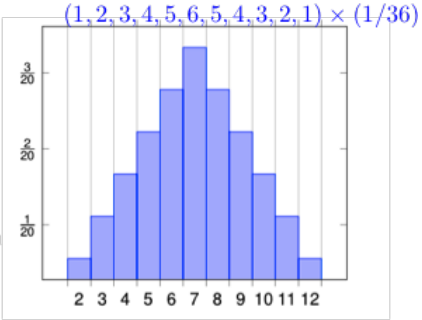

The binomial distribution.

Flip n coins with heads probability p.

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Probability of "$X = i$" is sum of $Pr[\omega]$, $\omega \in$ "$X = i$".

$$Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}, i = 0, 1, \ldots, n : B(n,p) \text{ distribution}$$

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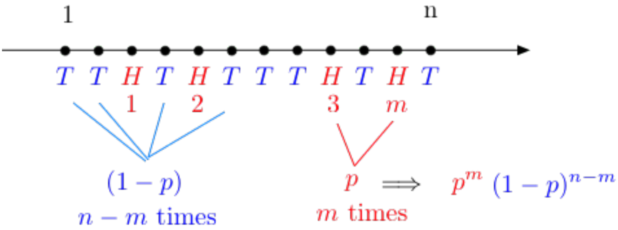

$\binom{n}{m}$ outcomes with m Hs and $n - m$ Ts

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Confidence in polling, experiments, etc.

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Summary of distribution?

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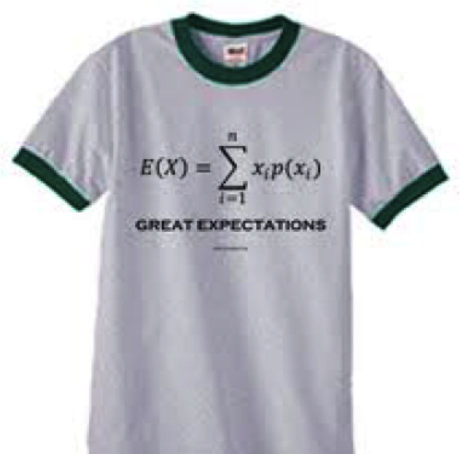

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We use this frequentist interpretation as a definition.

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The subjectivist interpretation of $E[X]$ is less obvious.

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Also,

$$\sum_{a} a \times Pr[X = a] = 3 \times \frac{1}{8} + 2 \times \frac{3}{8} + 1 \times \frac{3}{8} + 0 \times \frac{1}{8}.$$

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$X(m)$ = score of student m, for $m = 1, 2, \ldots, n$.

"Average score" of the n students: add scores and divide by n:

$$\text{Average} = \frac{X(1) + X(1) + \cdots + X(n)}{n}.$$

Experiment: choose a student uniformly at random.
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This holds for a uniform probability space.

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Apparently: expected value is not a common value, by any means.

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The random variable X is sometimes written as

$$1\{\omega \in A\} \text{ or } 1_A(\omega).$$

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Using Linearity - 2: Fixed point.

Hand out assignments at random to n students.

$X =$ number of students that get their own assignment back.

$X = X_1 + \cdots + X_n$ where
$X_m = 1\{\text{student } m \text{ gets his/her own assignment back}\}$.

One has

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\begin{aligned}
E[X] &= E[X_1 + \cdots + X_n] \\
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&= nPr[X_1 = 1], \text{ because } X_1 \text{ is an indicator} \\
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Note that linearity holds even though the X_m are not independent (whatever that means).

Using Linearity - 3: Binomial Distribution.

Flip n coins with heads probability p.

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Flip n coins with heads probability p. X - number of heads

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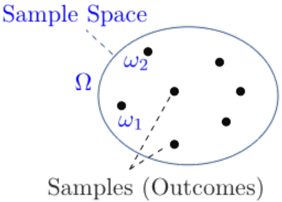
- Expectation is Linear.
- $B(n, p)$.

Probability: Midterm 2 Review.

- Framework:
  - Probability Space
  - Conditional Probability & Bayes' Rule
  - Independence
  - Mutual Independence
- Collisions & Collecting
- Random Variables

See Note 25: 1, 2, 3, 4 (paragraphs 1, 2, 3; examples 1 through 8)

Probability Space



$$0 \le Pr[\omega] \le 1$$

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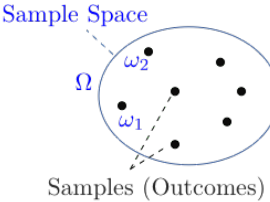

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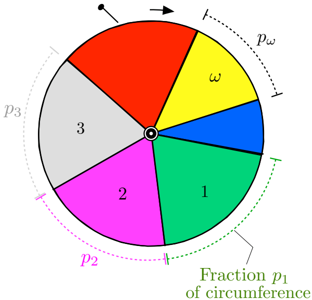

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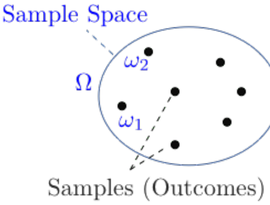

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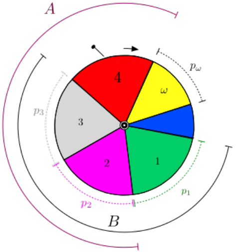
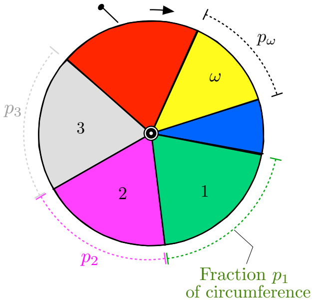


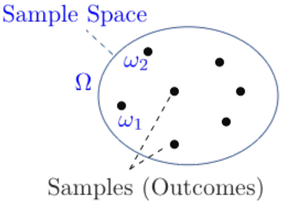


$Pr[A|B] = Pr[A \cap B]/Pr[B]$.

$Pr[A \cap B \cap C]$

$\quad = Pr[A]Pr[B|A]Pr[C|A \cap B]$.

Probability Space



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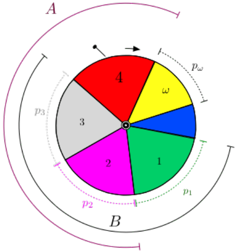

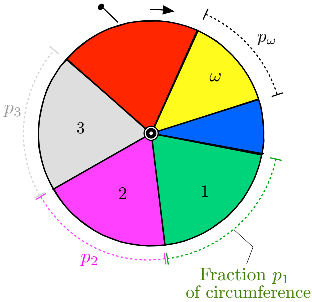

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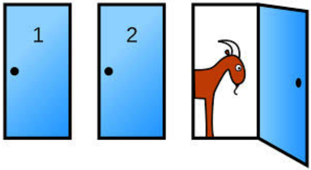

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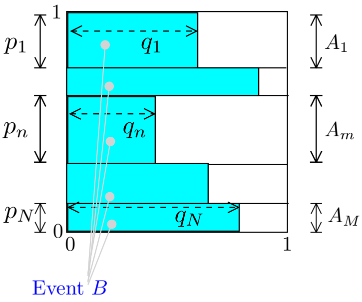

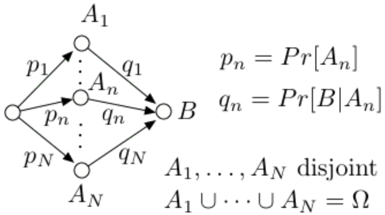

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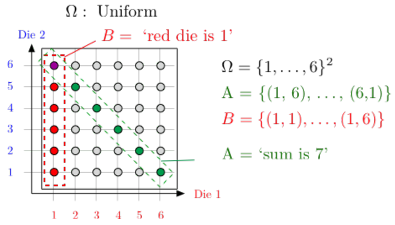

"First coin yields 1" and "Sum is 7" are independent

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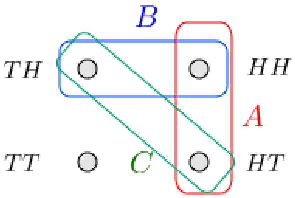
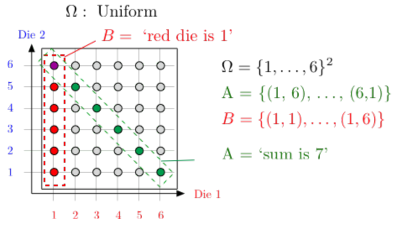


Pairwise, but not mutually



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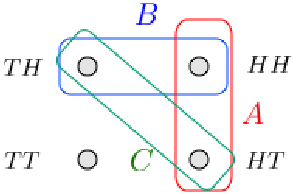

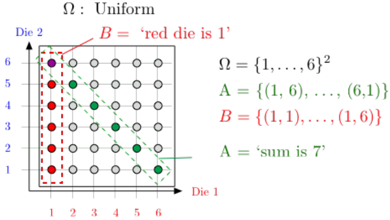

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Pairwise, but not mutually

If $\{A_j, i \in J\}$ are mutually independent, then $[A_1 \cap \bar{A}_2] \Delta A_3$ and $A_4 \setminus A_5$ are independent.

Our intuitive meaning of "independent events" is mutual independence.

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- $\{A_j, j \in J\}$ are mutually independent if
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- A and B are independent if $Pr[A \cap B] = Pr[A]Pr[B]$.
- $\{A_j, j \in J\}$ are mutually independent if
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Thus, A, B, C, D are mutually independent if there are

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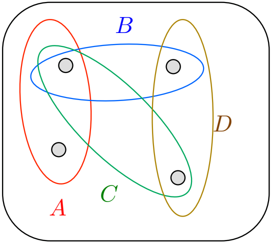

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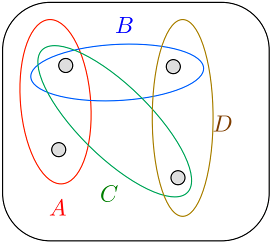

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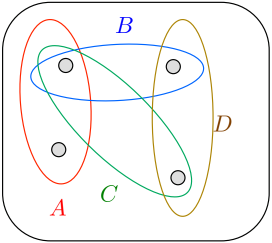

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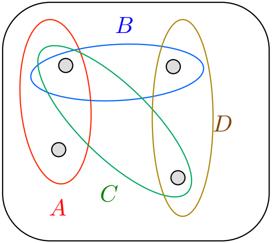

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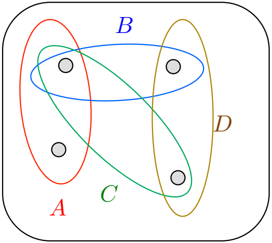

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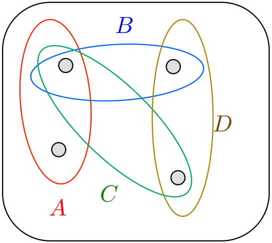

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No: We would need an outcome with probability $1/8$.

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This is not possible since $a, b < p$.

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  $Pr[\text{first} > \text{second}] = \frac{24}{51}$.

Summary

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And good time was had by all ....

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Enjoy spring break

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Enjoy spring break and the midterm.