CS70: Jean Walrand: Lecture 19.

Random Variables: Expectation

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Random Variables: Expectation

1. Random Variables: Brief Review
2. Expectation
3. Important Distributions

Random Variables: Definitions

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- $(X - a)^2$
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- $X \cos(2\pi Y + Z)$.

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Also,

$$\sum_{a} a \times Pr[X = a] = 3 \times \frac{1}{8} + 2 \times \frac{3}{8} + 1 \times \frac{3}{8} + 0 \times \frac{1}{8}.$$

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the Law of Large Numbers. (See later.)

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An Illustration: Rolling Dice

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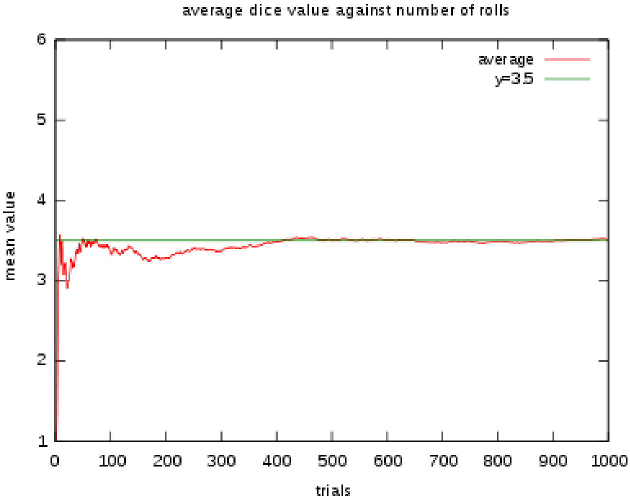

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Thus, we will write $X = 1_A$.

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Note: If we had defined $Y = a_1X_1 + \cdots + a_nX_n$ has had tried to compute $E[Y] = \sum_y yPr[Y = y]$, we would have been in trouble!

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Note: Computing $\sum_x xPr[X = x]$ directly is not easy!

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Thus, $E[X + b] = E[X] + b$.

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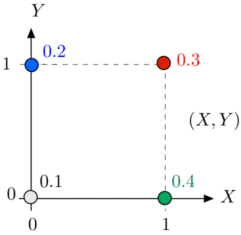

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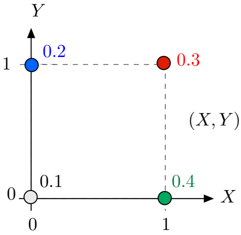

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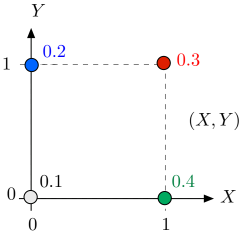

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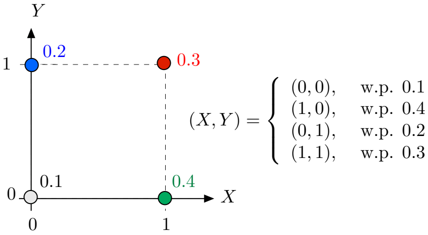

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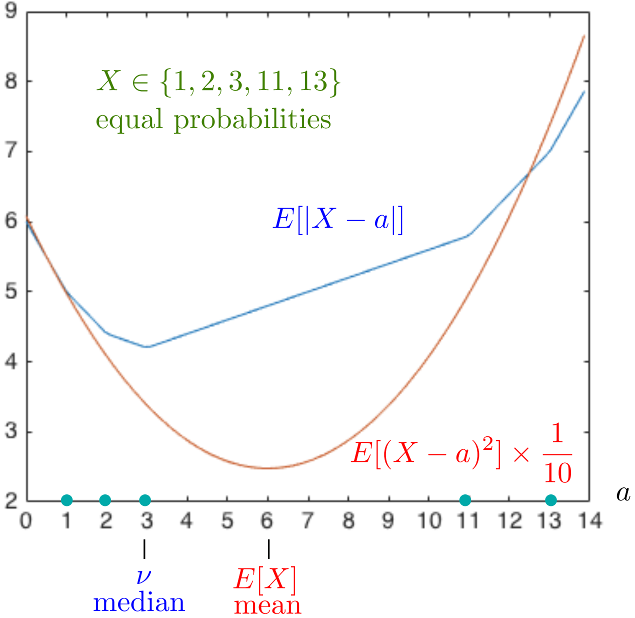

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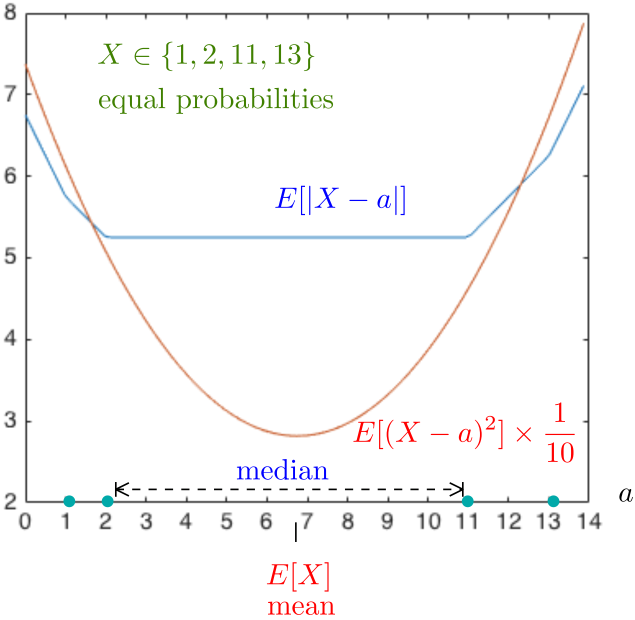

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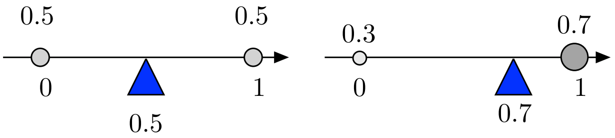
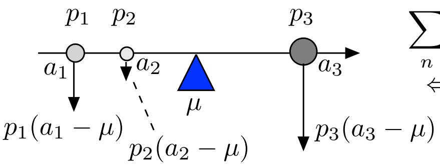
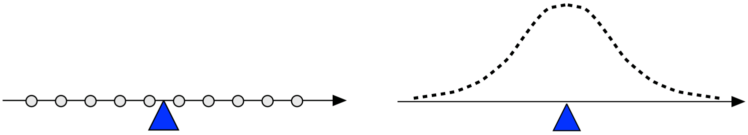




$$\sum_n p_n(a_n - \mu) = 0$$

$$\Leftrightarrow \mu = \sum_n a_n p_n = E[X]$$



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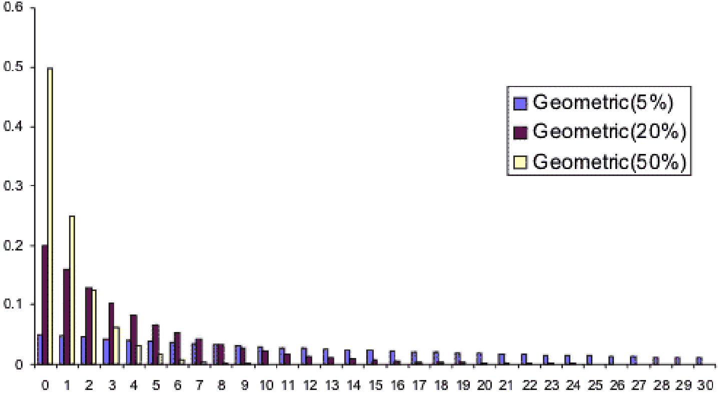

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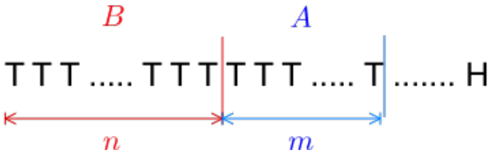

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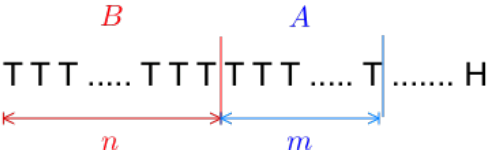

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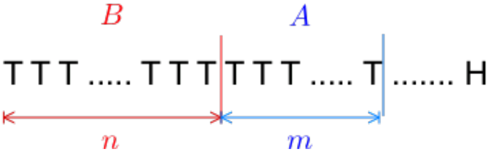

$$Pr[X > n+m | X > n] = Pr[A|B] = Pr[A] = Pr[X > m].$$

The coin is memoryless, therefore, so is X.

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Theorem: For a r.v. X that takes the values $\{0, 1, 2, \ldots\}$, one has

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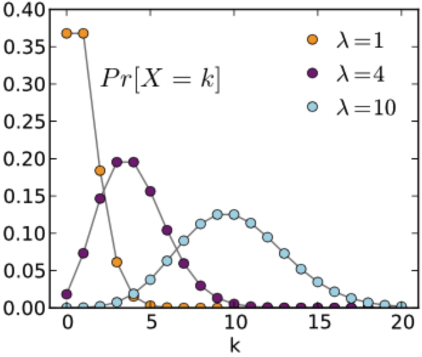

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For (1) we used $m \ll n$; for (2) we used $(1 - a/n)^n \approx e^{-a}$.

Poisson Distribution: Definition and Mean

Definition Poisson Distribution with parameter $\lambda > 0$

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$$\begin{aligned} E[X] &= \sum_{m=1}^{\infty} m \times \frac{\lambda^m}{m!} e^{-\lambda} = e^{-\lambda} \sum_{m=1}^{\infty} \frac{\lambda^m}{(m-1)!} \\ &= e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{m!} = e^{-\lambda} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \\ &= e^{-\lambda} \lambda e^{\lambda} \end{aligned}$$

Poisson Distribution: Definition and Mean

Definition Poisson Distribution with parameter $\lambda > 0$

$$X = P(\lambda) \Leftrightarrow Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0.$$

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$\square$

Simeon Poisson

The Poisson distribution is named after:

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Siméon Poisson

Siméon Denis Poisson (1781–1840)

Equal Time: B. Geometric

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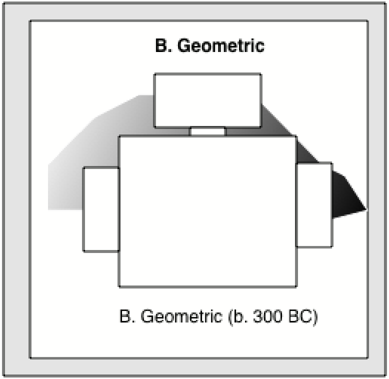

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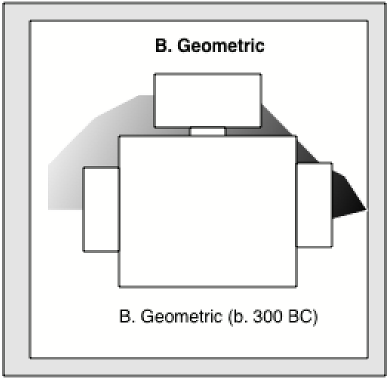

I could not find a picture of D. Binomial, sorry.

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- $B(n, p), U[1 : n], G(p), P(\lambda)$.