CS70: Jean Walrand: Lecture 22.

Confidence Intervals; Linear Regression

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Confidence Intervals; Linear Regression

1. Review
2. Confidence Intervals
3. Motivation for LR
4. History of LR
5. Linear Regression
6. Derivation
7. More examples

Review: Probability Ideas Map

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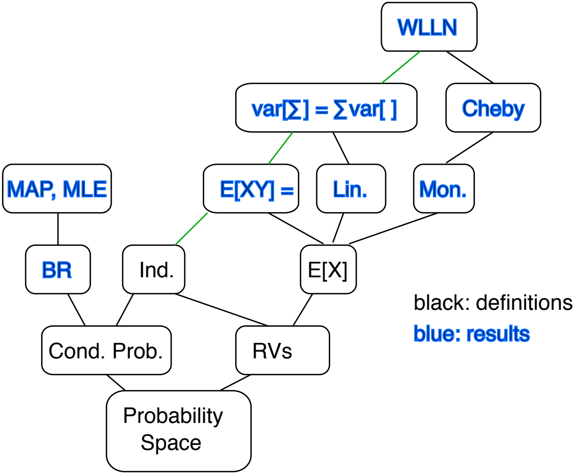

Review: Probability Ideas Map - Details

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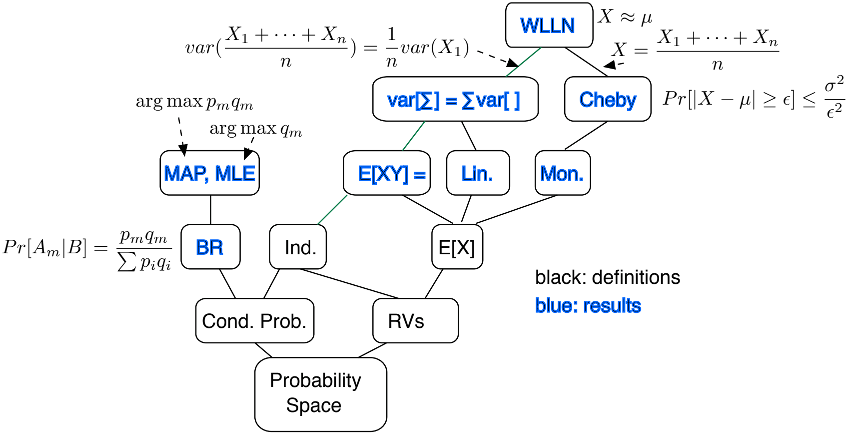

Review: Probability Ideas Map - Today

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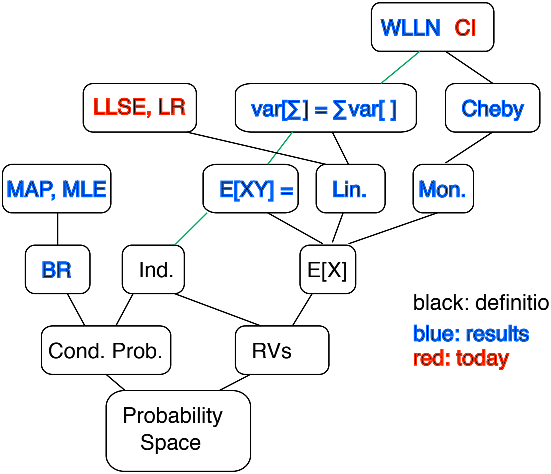

black: definitions

blue: results

red: today

Confidence Intervals: Example

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Hence, $[A_n - 4.5\frac{1/2}{\sqrt{n}}, A_n + 4.5\frac{1/2}{\sqrt{n}}]]$ is a 95%-CI for p.

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Hence,

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Hence, $E[(Y-a)^2] \geq E[(Y-E[Y])^2], \forall a$. □

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Here is a picture that summarizes the calculation.

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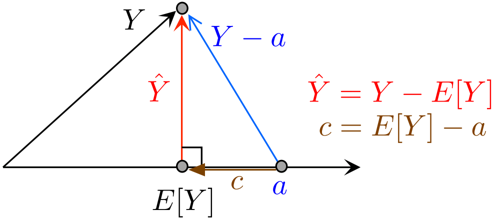

$$E[\hat{Y}c] = 0 \Leftrightarrow \hat{Y} \perp c$$

$$\begin{aligned} E[(Y-a)^2] &= E[(\hat{Y}+c)^2] \\ &= E[\hat{Y}^2 + 2c\hat{Y} + c^2] \\ &= E[\hat{Y}^2] + c^2 \end{aligned}$$

(Pythagoras)

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A bit later, we will consider a general function $g(X)$.

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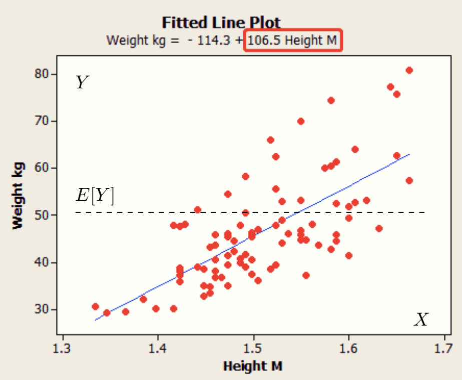

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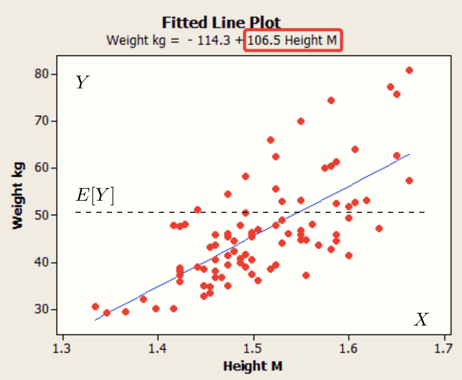

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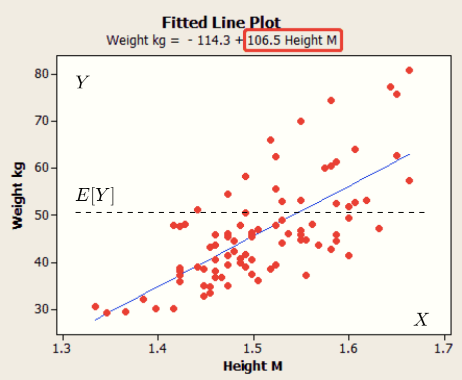

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Best linear fit: Linear Regression.

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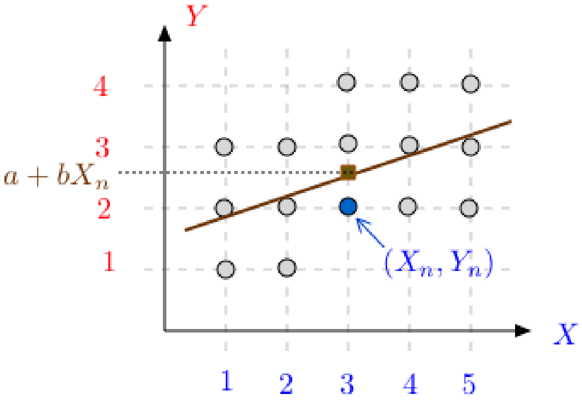

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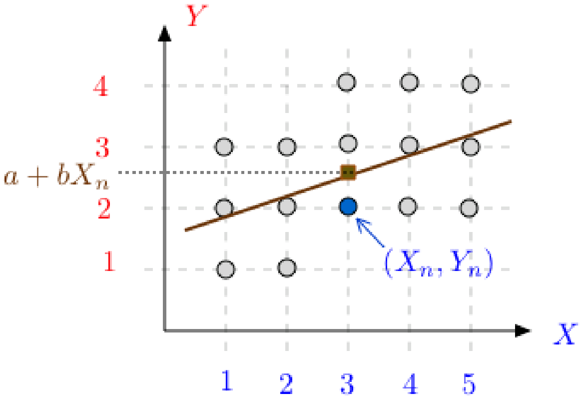
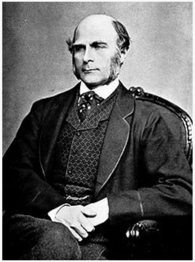


The line $Y = a + bX$ is the linear regression.

History

Sir Francis Galton



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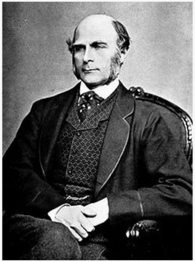

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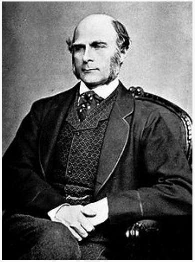

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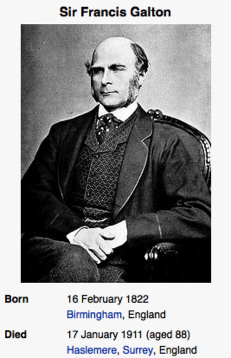

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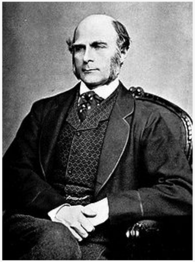

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The lesson is that smart people can also be stupid.

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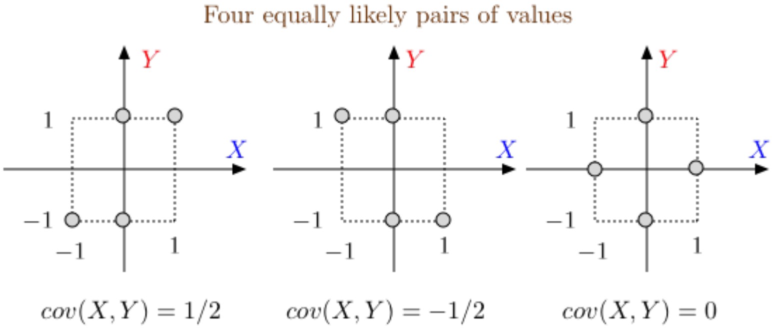

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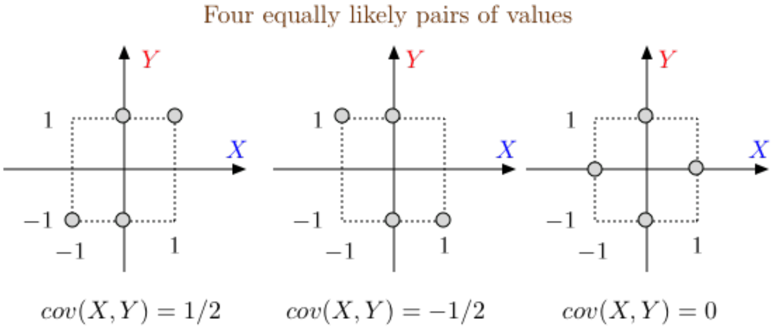

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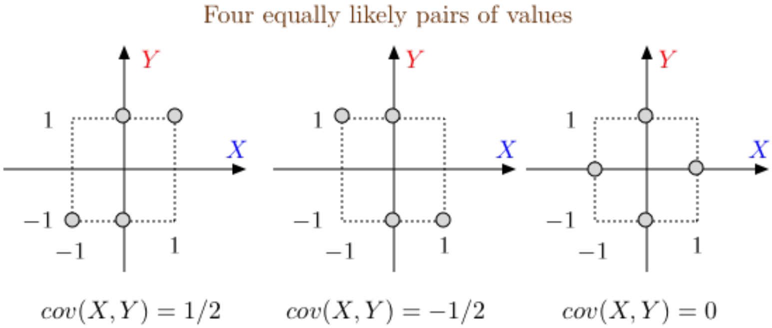

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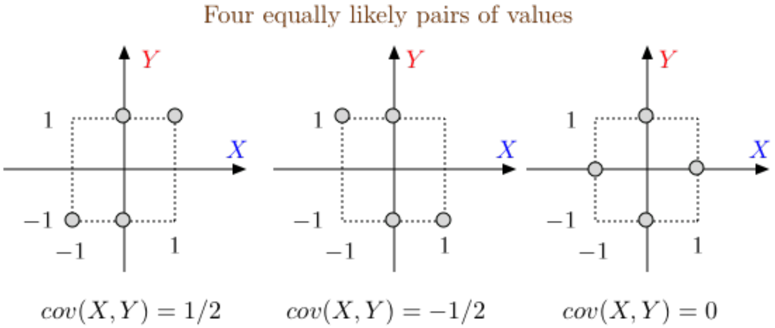

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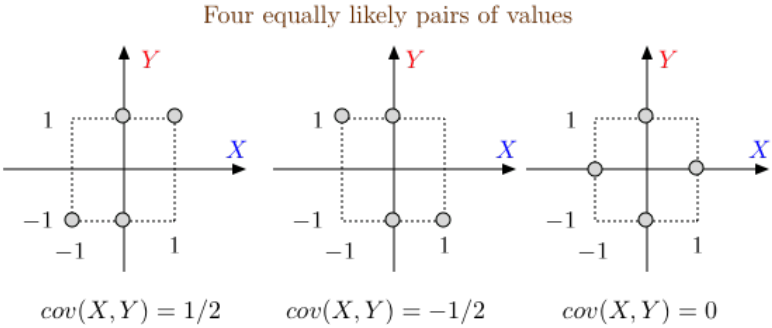

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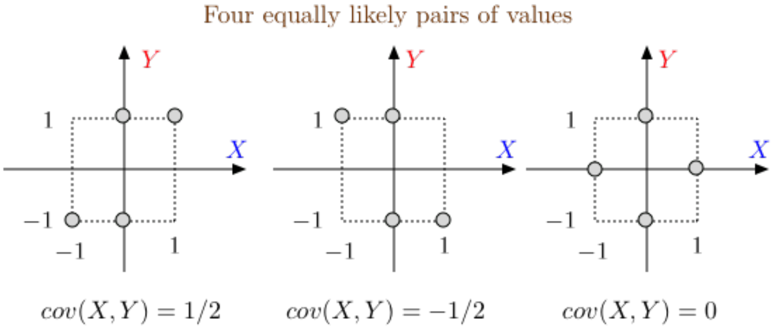

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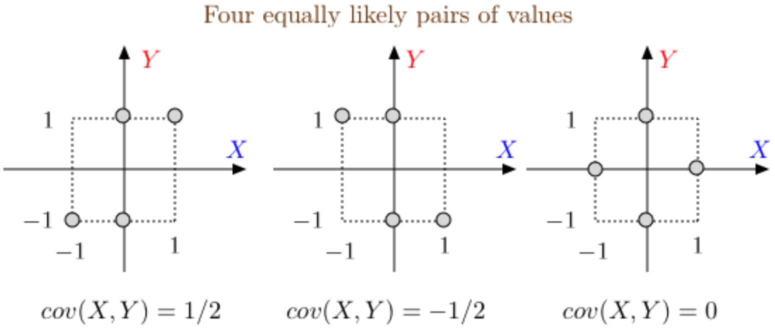

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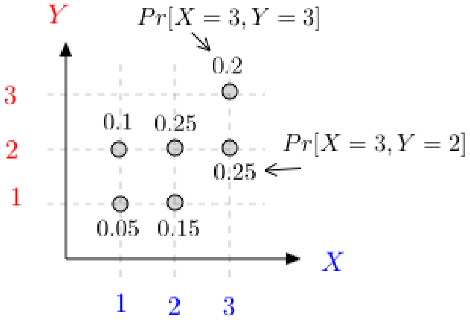

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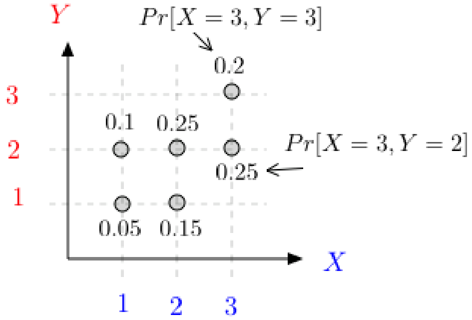

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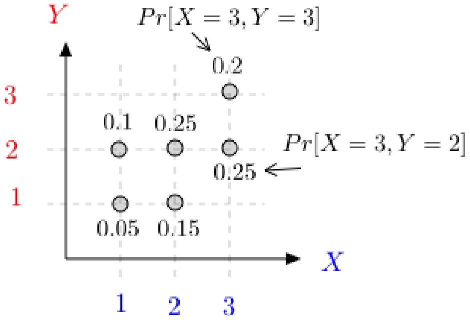

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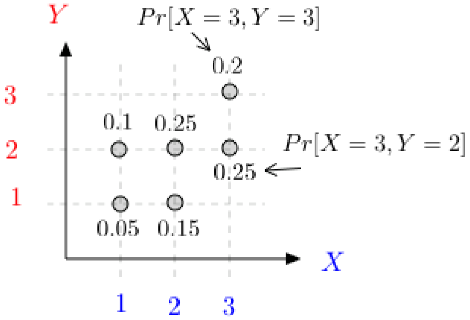

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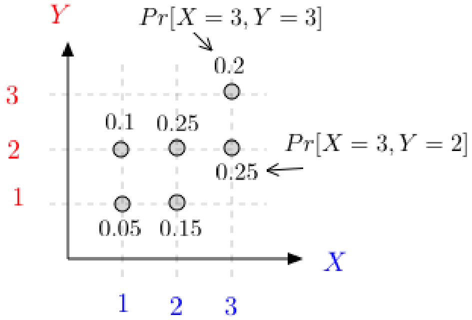

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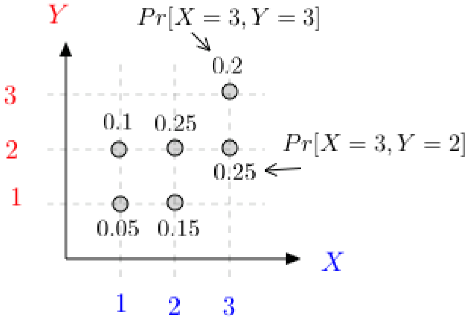

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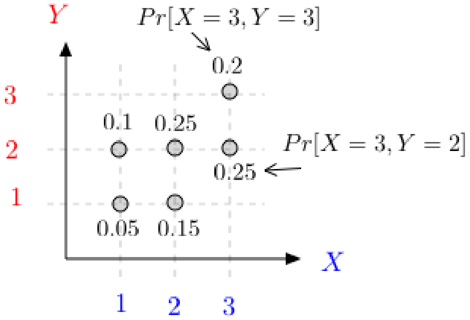

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Note: This is a non-Bayesian formulation: there is no prior.

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Given two RVs X and Y with known distribution $Pr[X = x, Y = y]$, the Linear Least Squares Estimate of Y given X is

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Note: This is a Bayesian formulation: there is a prior.

LR: Non-Bayesian or Uniform?

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However, the interpretations are different!

LLSE

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Thus $\hat{Y}$ is the LLSE. □

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$^{(*)}$ Recall that $cov(X,Y) = E[(X - E[X])(Y - E[Y])]$ and $var[X] = E[(X - E[X])^2]$.

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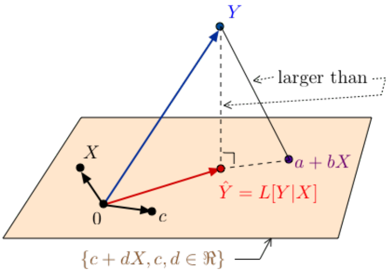

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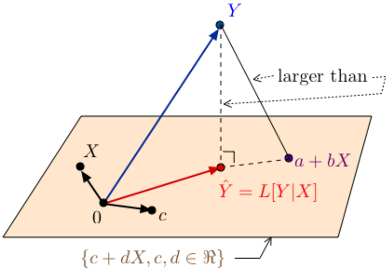

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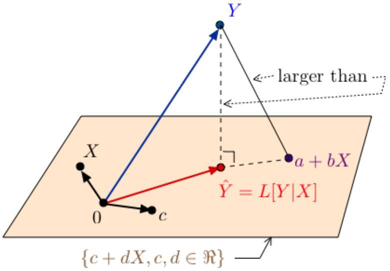

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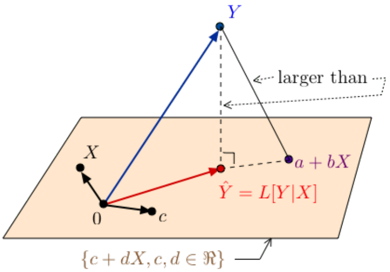

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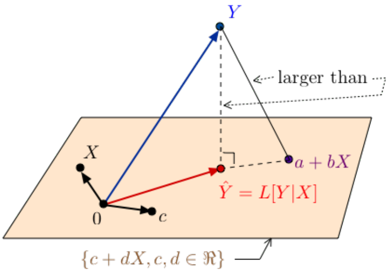

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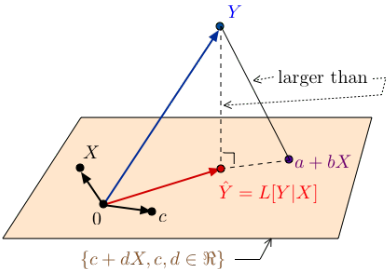

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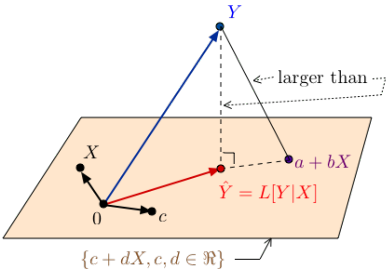

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That is, $\hat{Y}$ is the projection of Y onto the plane.

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□

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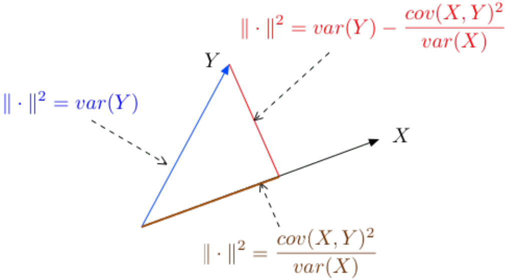

Linear Regression Examples

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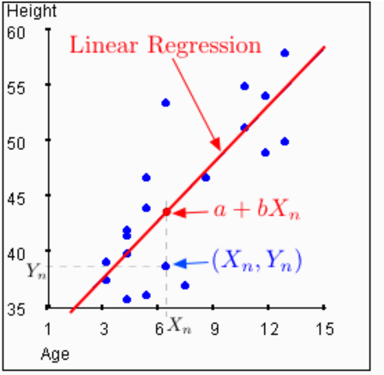

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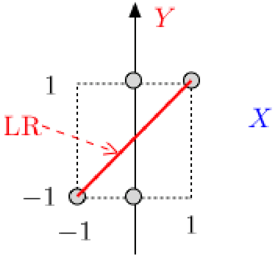

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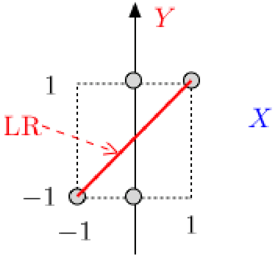

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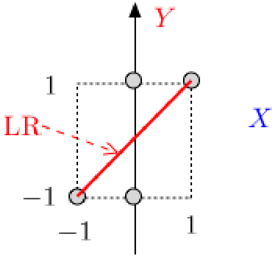

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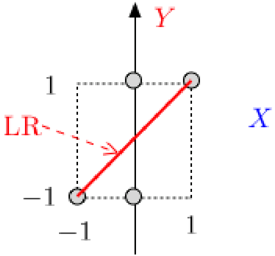

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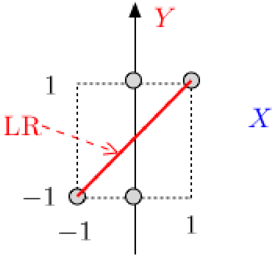

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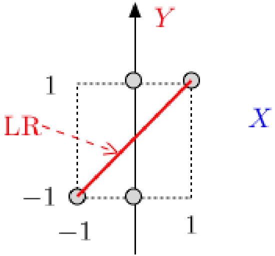

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Linear Regression Examples

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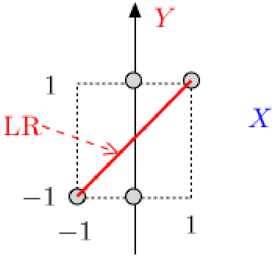

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Linear Regression Examples

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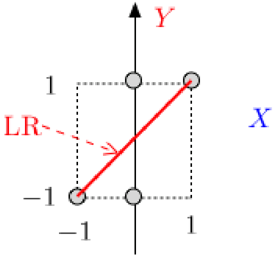

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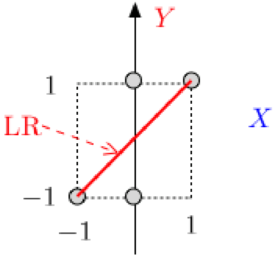

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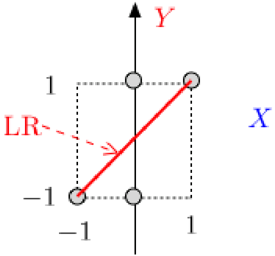

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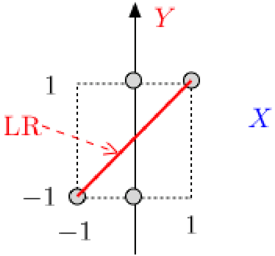

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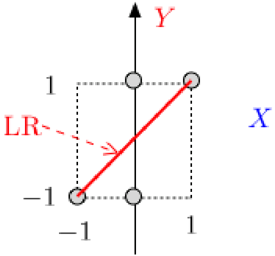

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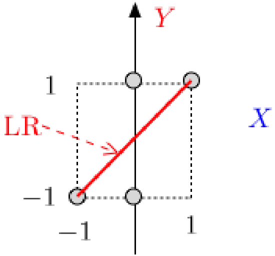

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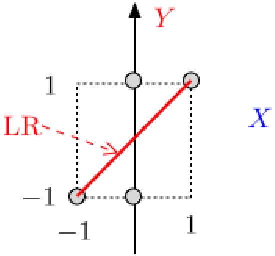

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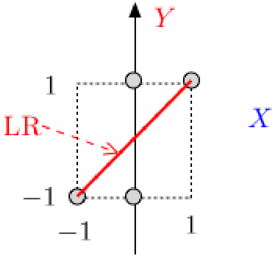

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Linear Regression Examples

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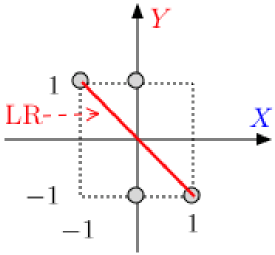

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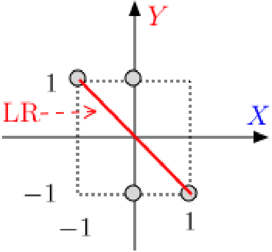

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$E[X] =$

Linear Regression Examples

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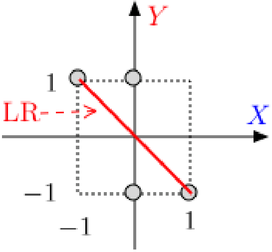

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Linear Regression Examples

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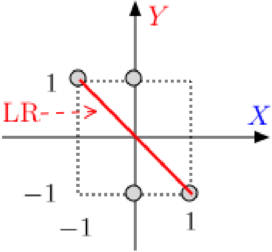

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Linear Regression Examples

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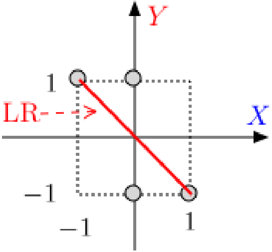

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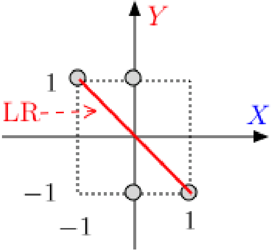

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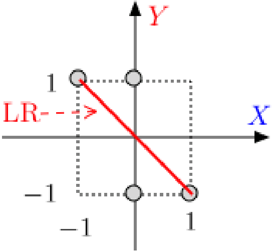

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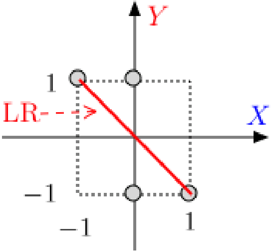

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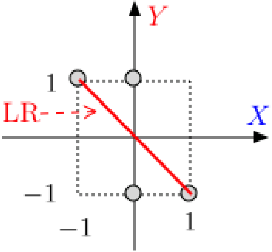

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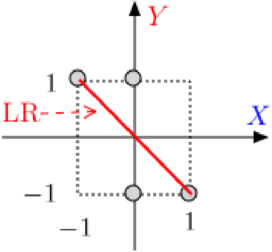

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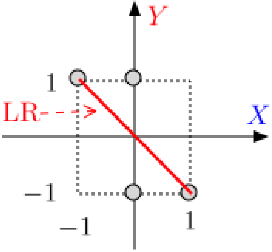

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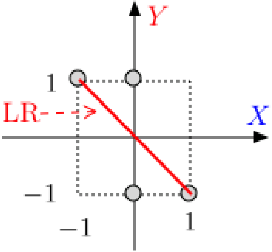

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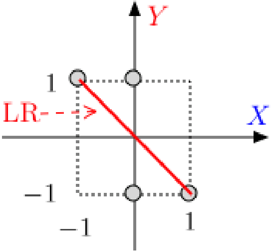

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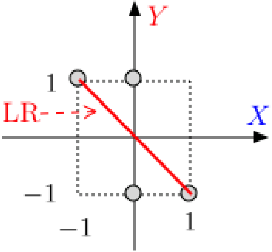

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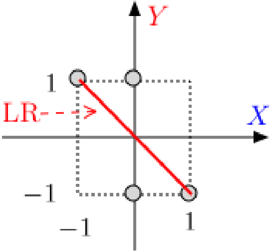

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Linear Regression Examples

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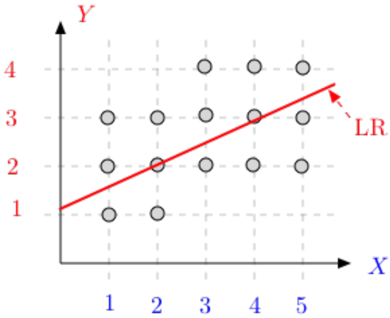

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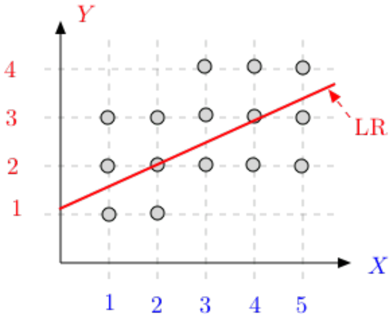

We find:

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Linear Regression Examples

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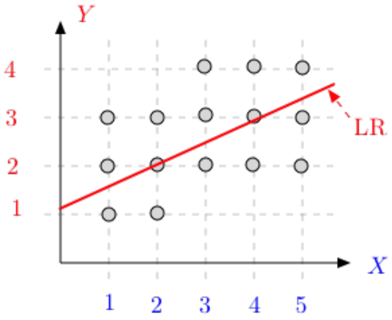

We find:

$E[X] = 3;$

Linear Regression Examples

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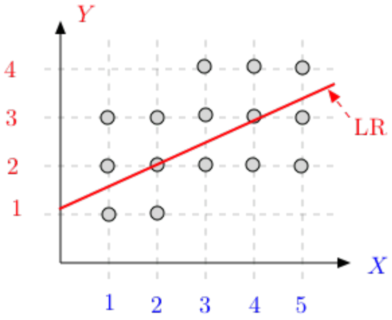

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Linear Regression Examples

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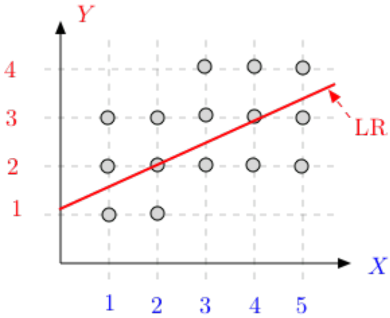

We find:

$$E[X] = 3; E[Y] = 2.5;$$

Linear Regression Examples

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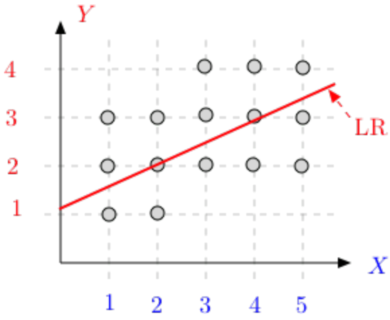

We find:

$$E[X] = 3; E[Y] = 2.5; E[X^2] = (3/15)(1 + 2^2 + 3^2 + 4^2 + 5^2) = 11;$$

Linear Regression Examples

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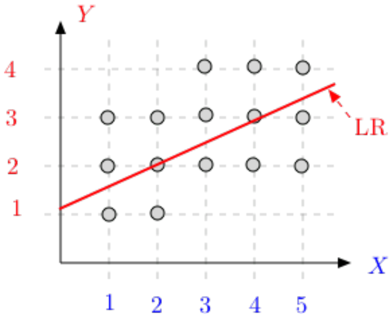

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Linear Regression Examples

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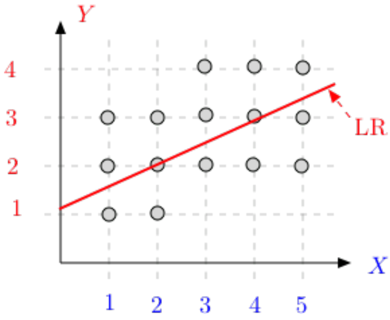

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Linear Regression Examples

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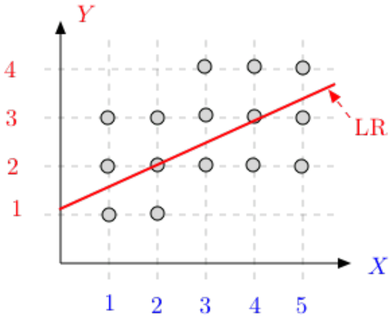

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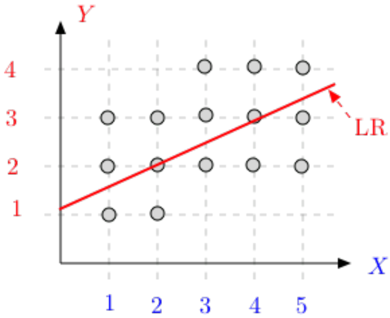

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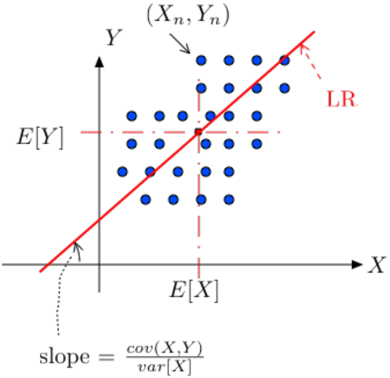
$$var[X] = 11 - 9 = 2; cov(X, Y) = 8.4 - 3 \times 2.5 = 0.9;$$
$$\text{LR: } \hat{Y} = 2.5 + \frac{0.9}{2}(X - 3) = 1.15 + 0.45X.$$

LR: Another Figure



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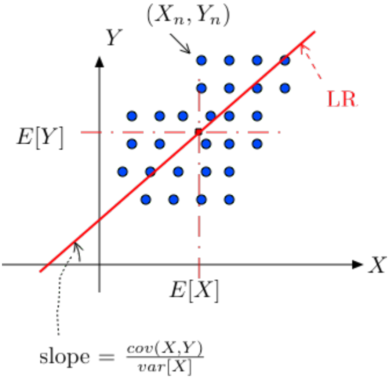

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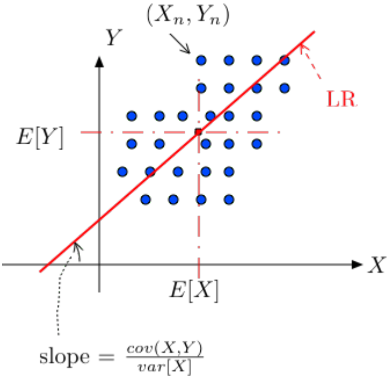

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