CS70: Jean Walrand: Lecture 23.

Conditional Expectation

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Conditional Expectation

1. Review: LR and LLSE
2. Conditional expectation
3. Applications: Diluting, Mixing, Rumors
4. CE = MMSE

Review: LLSE and LR

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Linear Regression Examples

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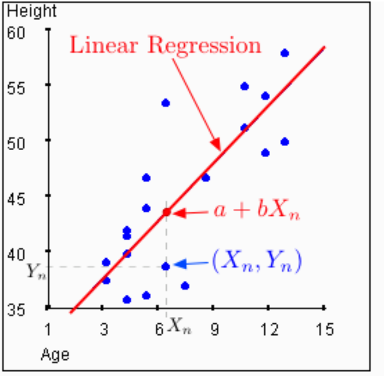

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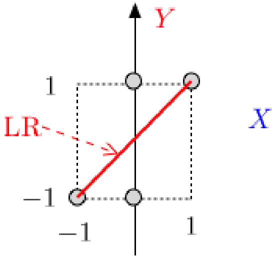

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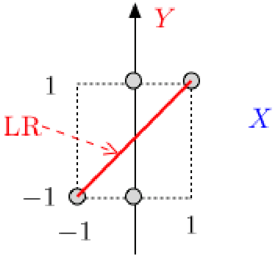

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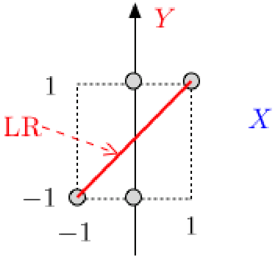

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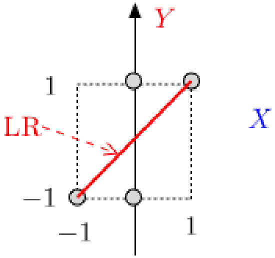

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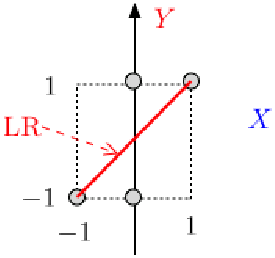

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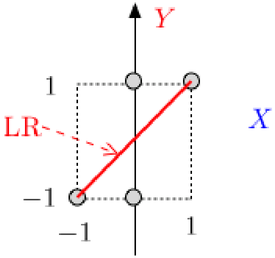

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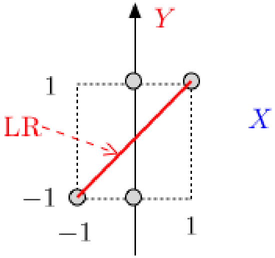

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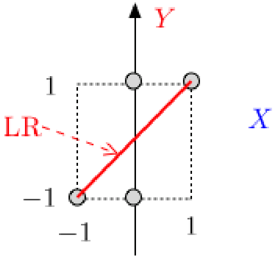

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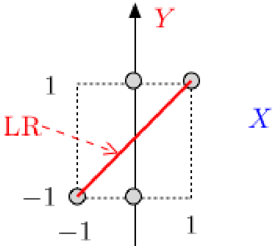

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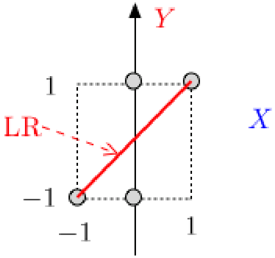

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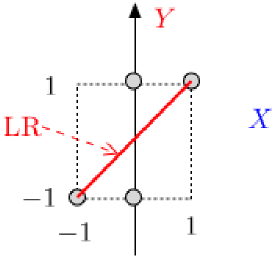

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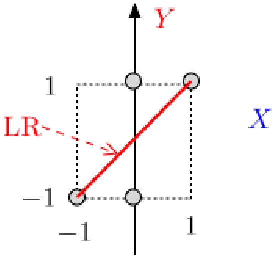

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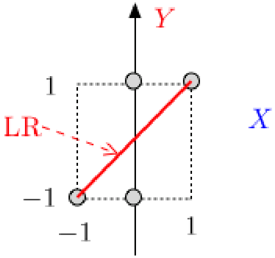

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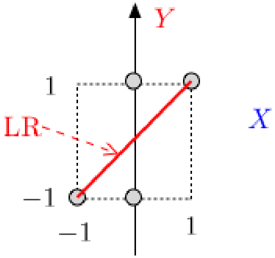

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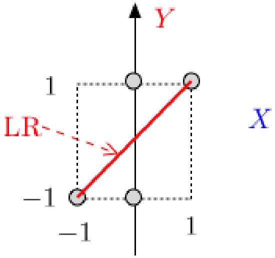

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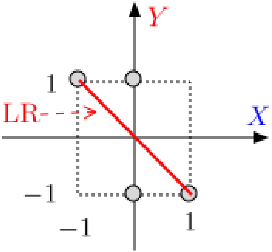

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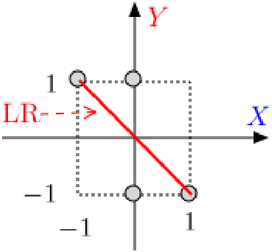

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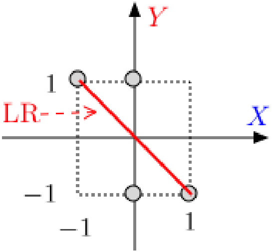

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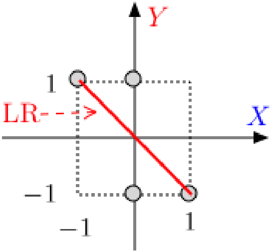

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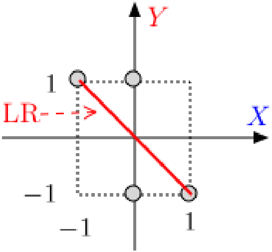

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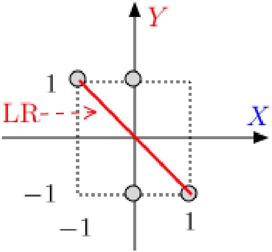

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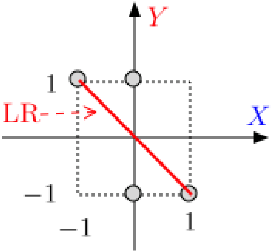

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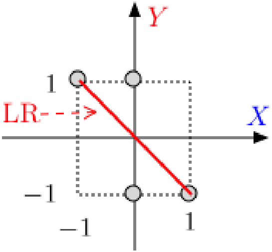

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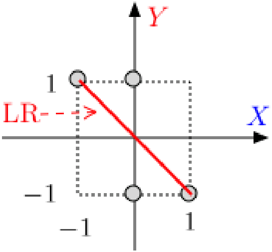

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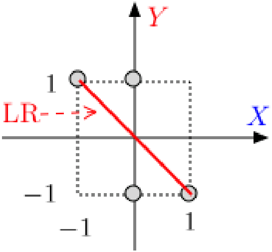

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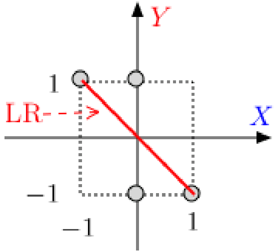

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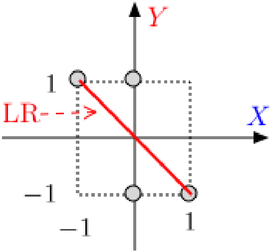

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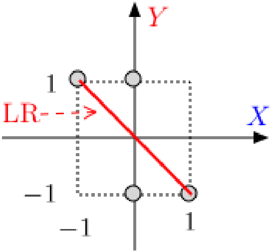

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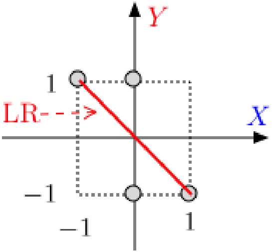

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Linear Regression Examples

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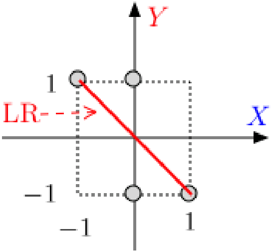

We find:

$$E[X] = 0; E[Y] = 0; E[X^2] = 1/2; E[XY] = -1/2;$$
$$var[X] = E[X^2] - E[X]^2 = 1/2; cov(X, Y) = E[XY] - E[X]E[Y] = -1/2;$$
$$\text{LR: } \hat{Y} = E[Y] + \frac{cov(X, Y)}{var[X]}(X - E[X]) = -X.$$

Linear Regression Examples

Example 4:

Linear Regression Examples

Example 4: Equally likely values of (X, Y),

Linear Regression Examples

Example 4: Equally likely values of (X, Y), or samples.

Linear Regression Examples

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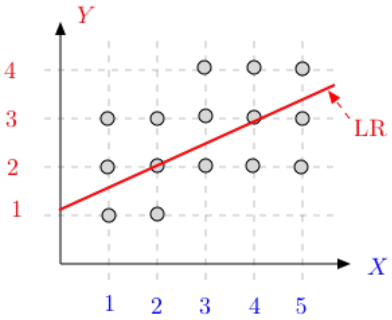
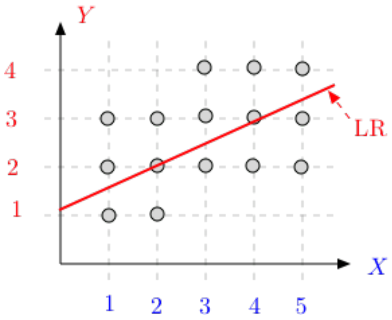


Linear Regression Examples

Example 4: Equally likely values of (X, Y), or samples.



We find:

$$E[X] =$$

Linear Regression Examples

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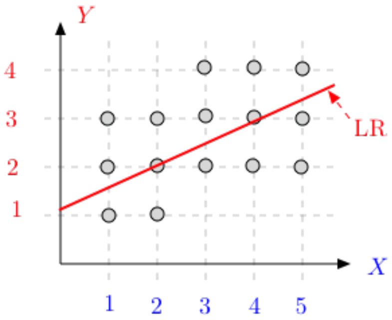

We find:

$E[X] = 3;$

Linear Regression Examples

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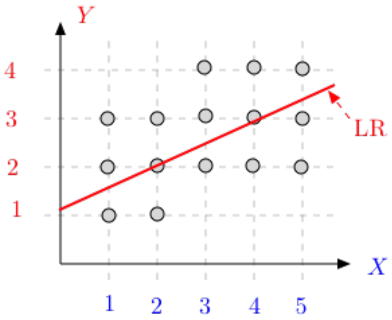

We find:

$$E[X] = 3; E[Y] =$$

Linear Regression Examples

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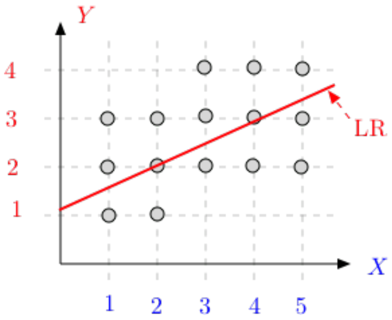
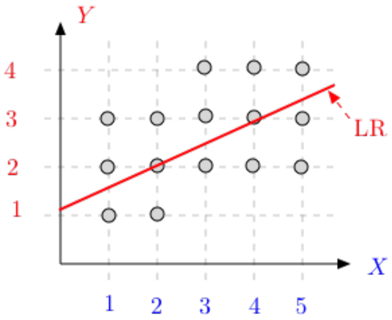


We find:

$$E[X] = 3; E[Y] = 2.5;$$

Linear Regression Examples

Example 4: Equally likely values of (X, Y), or samples.



We find:

$$E[X] = 3; E[Y] = 2.5; E[X^2] = (3/15)(1 + 2^2 + 3^2 + 4^2 + 5^2) = 11;$$

Linear Regression Examples

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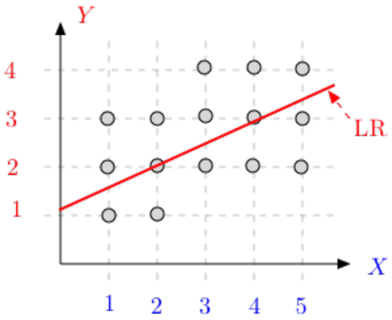

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Linear Regression Examples

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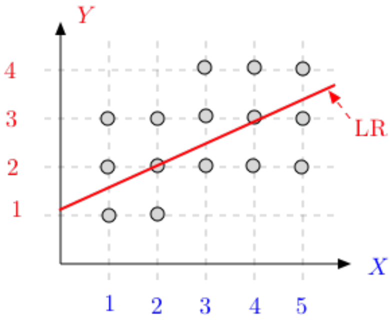

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Linear Regression Examples

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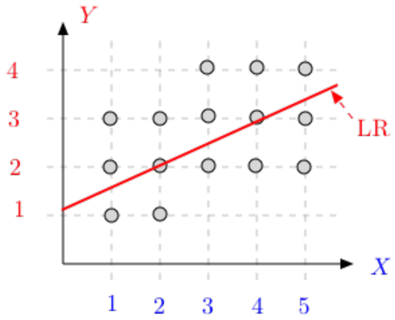

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Linear Regression Examples

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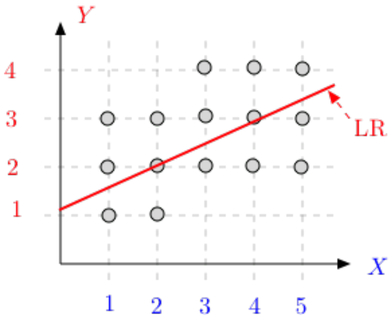

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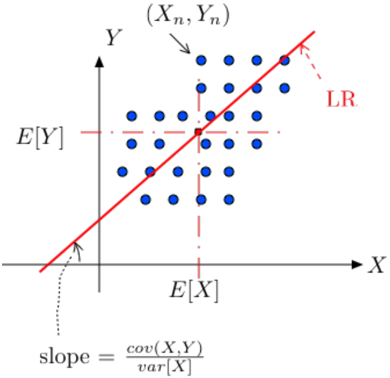
$$var[X] = 11 - 9 = 2; cov(X, Y) = 8.4 - 3 \times 2.5 = 0.9;$$
$$\text{LR: } \hat{Y} = 2.5 + \frac{0.9}{2}(X - 3) = 1.15 + 0.45X.$$

LR: Another Figure



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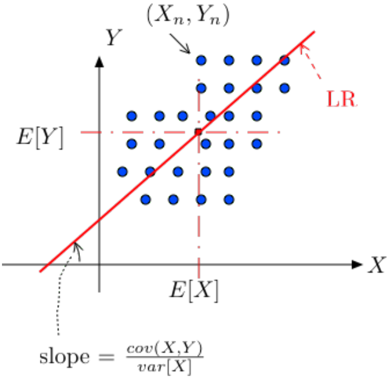
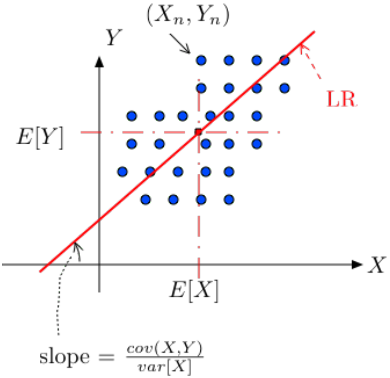


Note that

- the LR line goes through $(E[X], E[Y])$

LR: Another Figure



Note that

- the LR line goes through $(E[X], E[Y])$
- its slope is $\frac{cov(X,Y)}{var(X)}$.

Conditional Expectation: Motivation

There are many situations where a good guess about Y given X is not linear.

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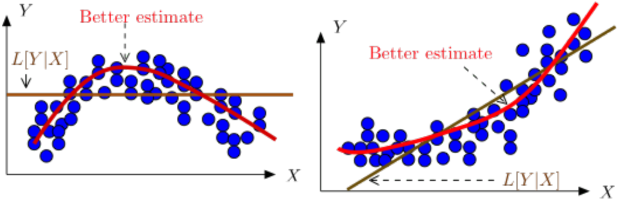

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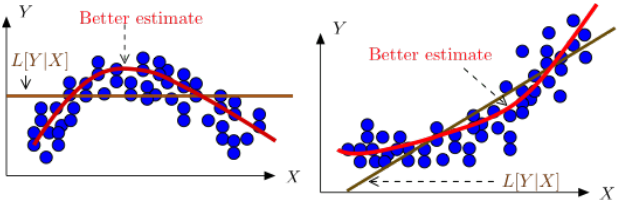

Our goal:

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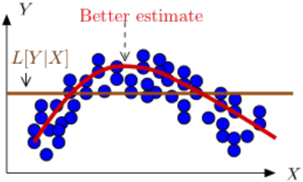

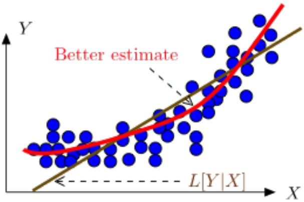

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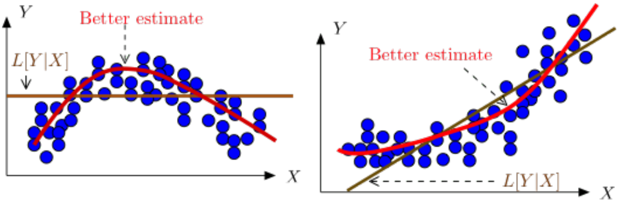

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That is, find the function $g(\cdot)$ so that $g(X)$ is the best guess about Y given X.

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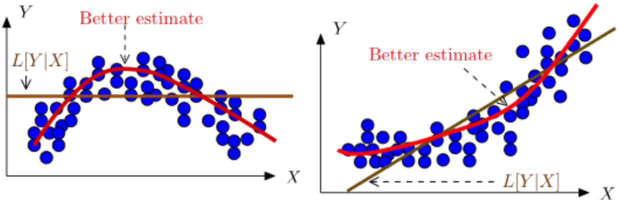

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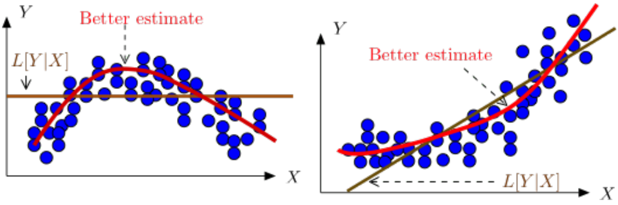

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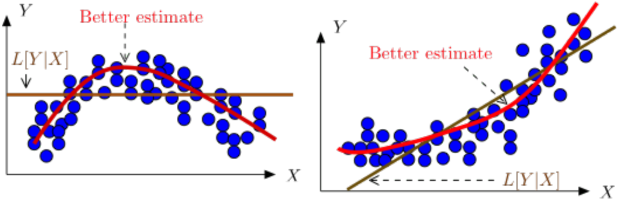

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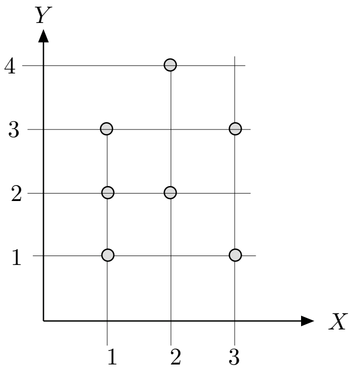

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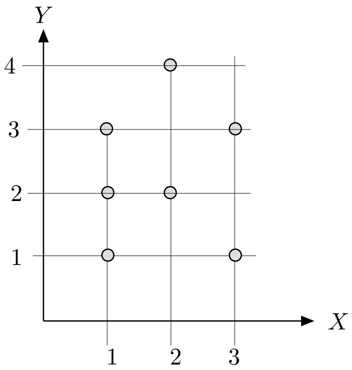

Without any observation, our guess for Y is

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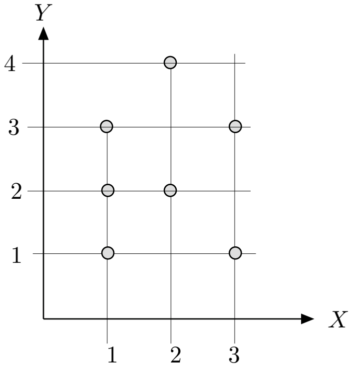

Without any observation, our guess for Y is $E[Y] =$

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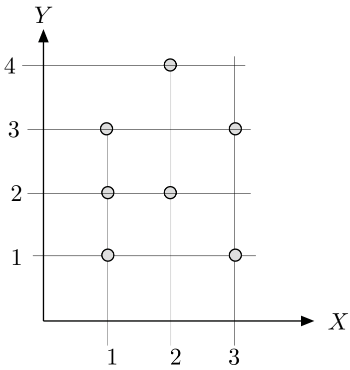

Without any observation, our guess for Y is $E[Y] = 2.3$.

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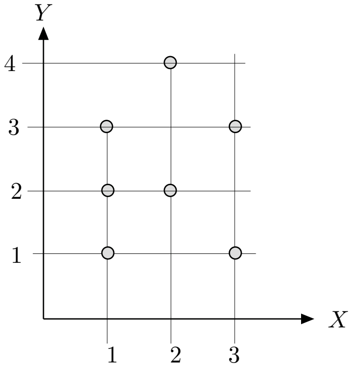
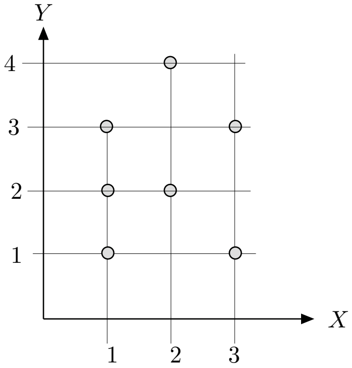


Without any observation, our guess for Y is $E[Y] = 2.3$.

Assume now we observe X.

Conditional Expectation: Intuition



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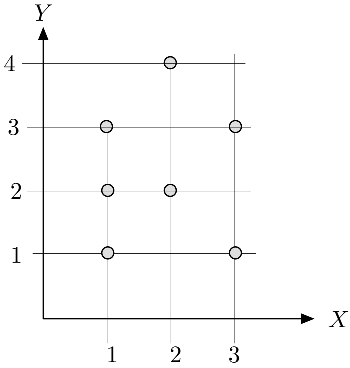
Assume now we observe X. We can calculate
$L[Y|X] = a + bX$

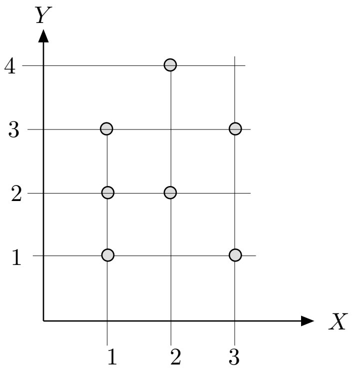
Conditional Expectation: Intuition



Without any observation, our guess for Y is $E[Y] = 2.3$.

Assume now we observe X. We can calculate
$L[Y|X] = a + bX \approx 2.1 + 0.1x$.

Conditional Expectation: Intuition



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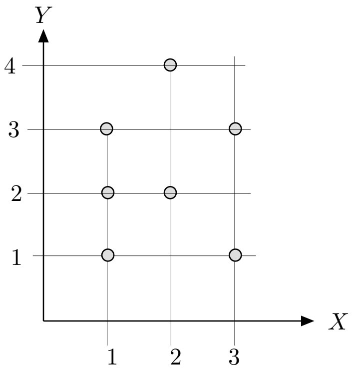

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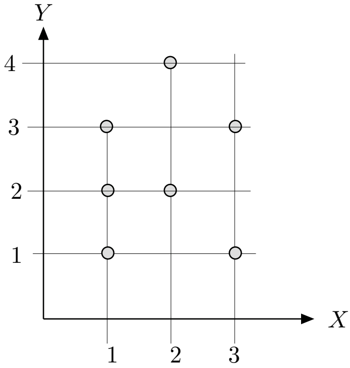
A better guess when $X = 1$ is

Conditional Expectation: Intuition



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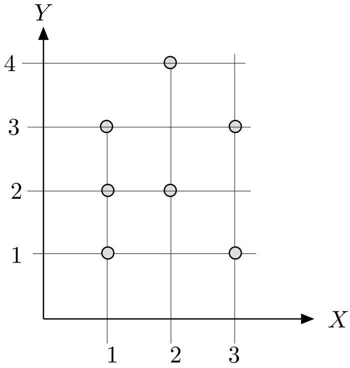
A better guess when $X = 1$ is 2;

Conditional Expectation: Intuition



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A better guess when $X = 1$ is 2; when $X = 2$:

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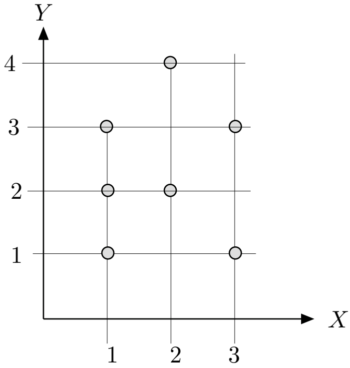

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A better guess when $X = 1$ is 2; when $X = 2$: 3;

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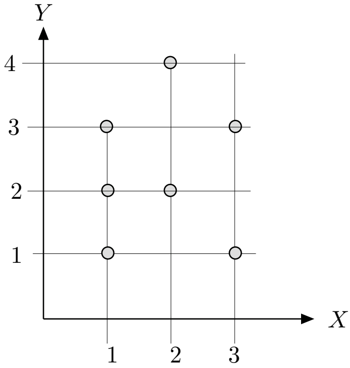

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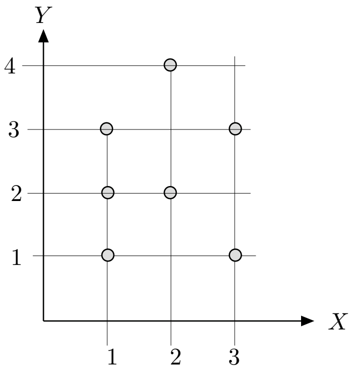

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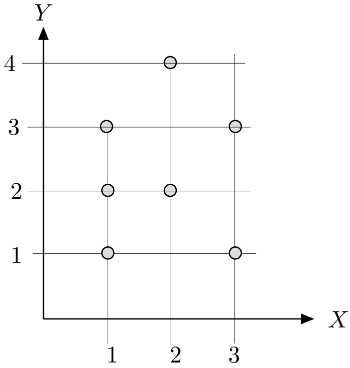

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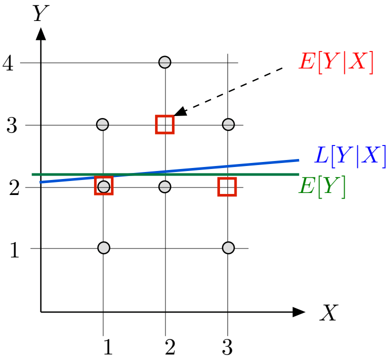

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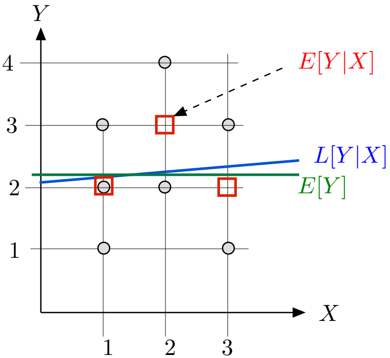
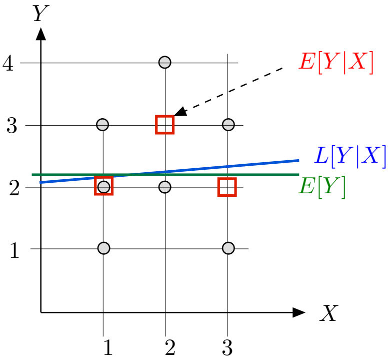


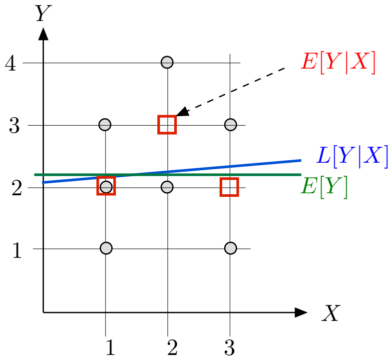
Here, $E[Y|X = 1]$ is the mean value of Y given that $X = 1$.

Conditional Expectation: Intuition



Here, $E[Y|X=1]$ is the mean value of Y given that $X=1$. Also, $E[Y|X=2]$ is the mean value of Y given that $X=2$

Conditional Expectation: Intuition



Here, $E[Y|X=1]$ is the mean value of Y given that $X=1$. Also, $E[Y|X=2]$ is the mean value of Y given that $X=2$ and $E[Y|X=3]$ is the mean value of Y given that $X=3$.

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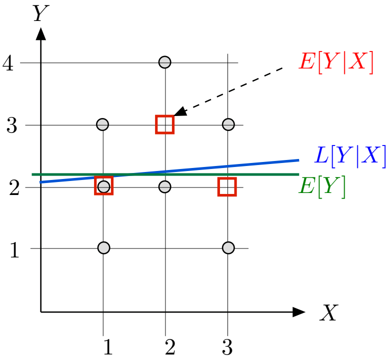

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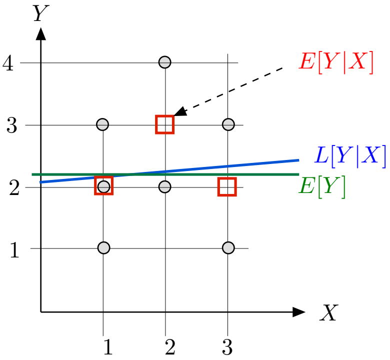
When we know that $X=1$, Y has a new distribution:

Conditional Expectation: Intuition



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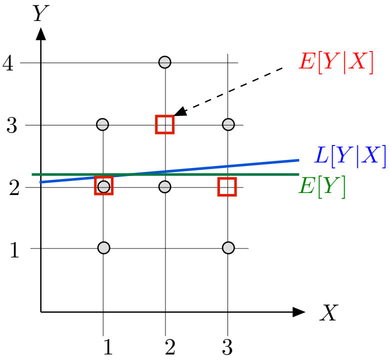
When we know that $X=1$, Y has a new distribution: Y is uniform in $\{1,2,3\}$.

Conditional Expectation: Intuition



Here, $E[Y|X=1]$ is the mean value of Y given that $X=1$. Also, $E[Y|X=2]$ is the mean value of Y given that $X=2$ and $E[Y|X=3]$ is the mean value of Y given that $X=3$.

When we know that $X=1$, Y has a new distribution: Y is uniform in $\{1,2,3\}$.
Thus, our guess is $E[Y|X=1]=1(1/3)+2(1/3)+3(1/3)=2$.

Conditional Expectation

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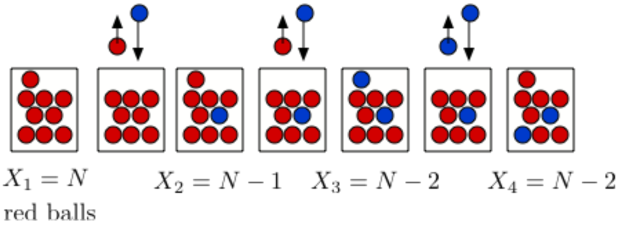
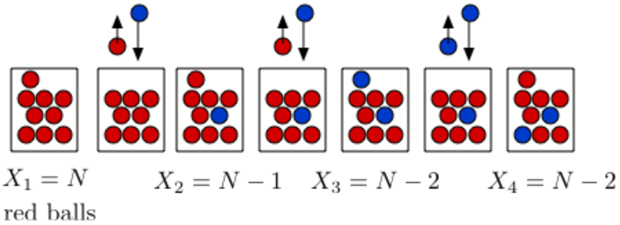
□

Application: Diluting

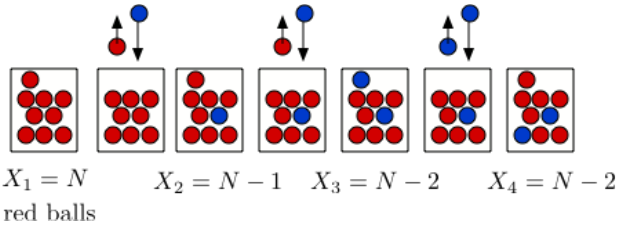


Application: Diluting



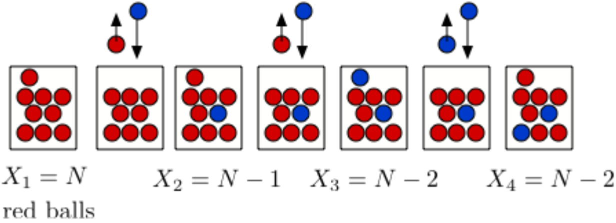
At each step, pick a ball from a well-mixed urn.

Application: Diluting



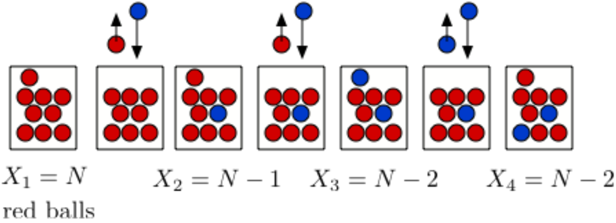
At each step, pick a ball from a well-mixed urn. Replace it with a blue ball.

Application: Diluting



At each step, pick a ball from a well-mixed urn. Replace it with a blue ball. Let X_n be the number of red balls in the urn at step n.

Application: Diluting



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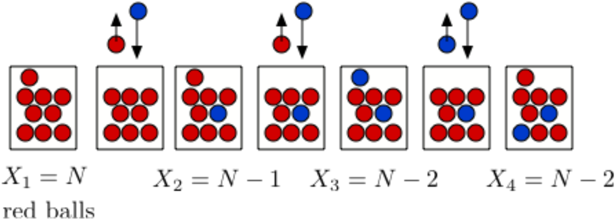

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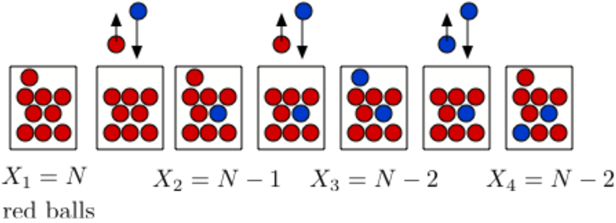
Application: Diluting



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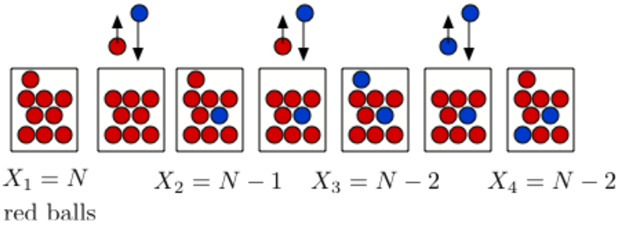

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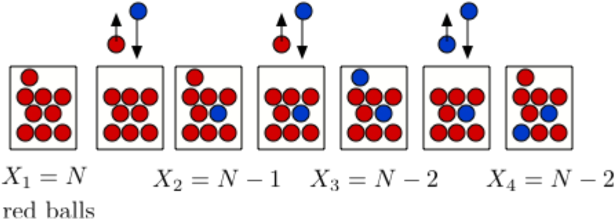

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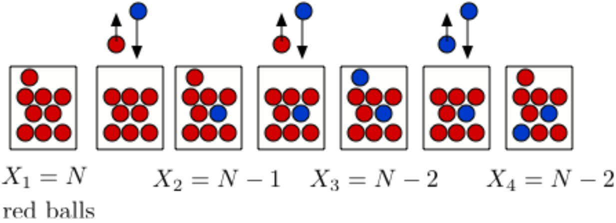

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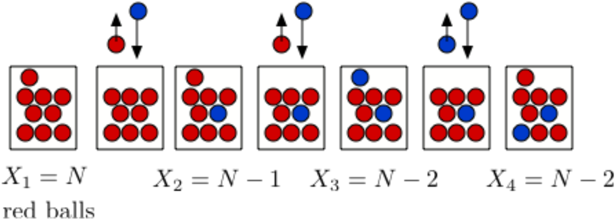
with $\rho := (N-1)/N$.

Application: Diluting



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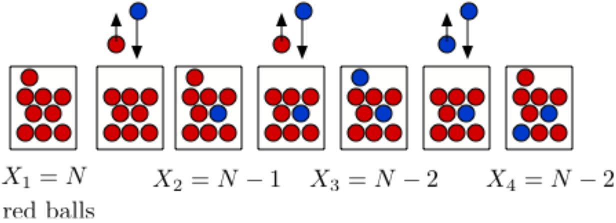

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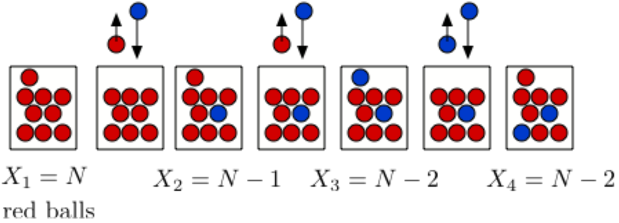

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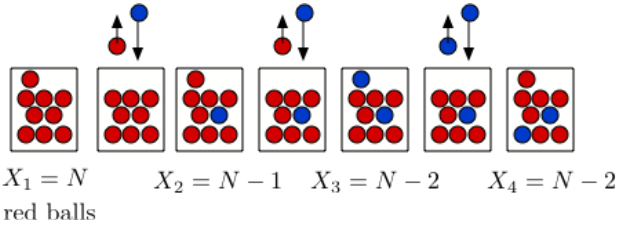

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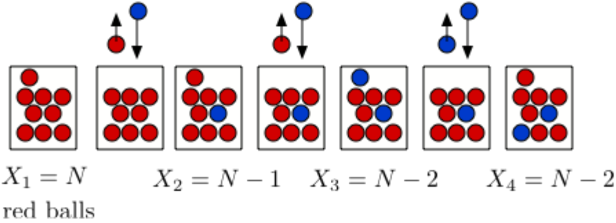
$$\implies E[X_n] = \rho^{n-1} E[X_1]$$

Application: Diluting



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$$E[X_{n+1}] = E[E[X_{n+1}|X_n]] = \rho E[X_n], n \geq 1.$$

$$\Longrightarrow E[X_n] = \rho^{n-1} E[X_1] = N(\frac{N-1}{N})^{n-1}, n \geq 1.$$

Diluting

Here is a plot:

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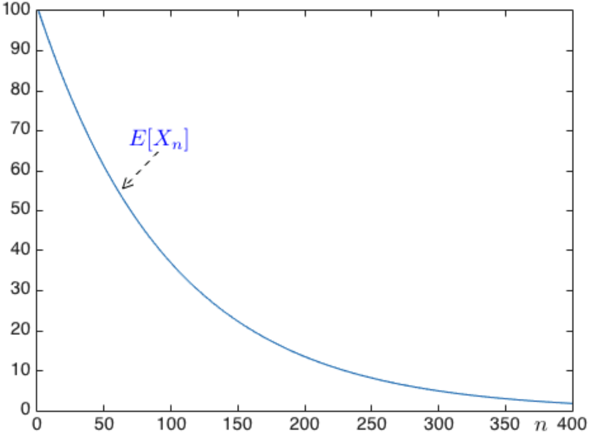
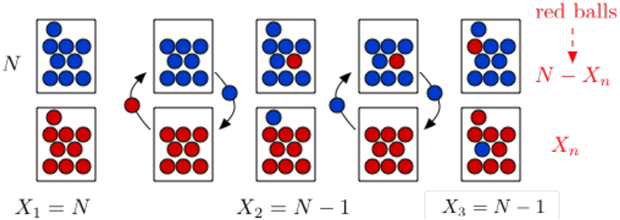


Application: Mixing



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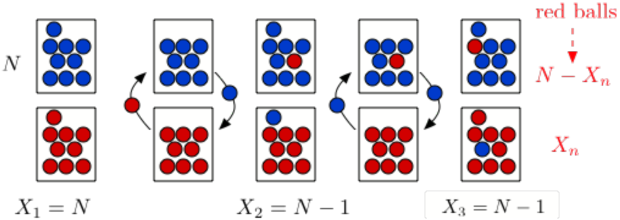
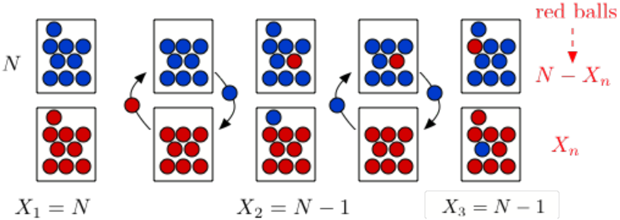


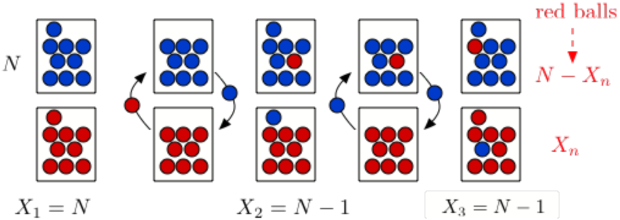
At each step, pick a ball from each well-mixed urn.

Application: Mixing



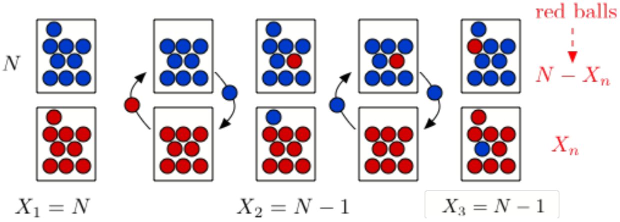
At each step, pick a ball from each well-mixed urn. We transfer them to the other urn.

Application: Mixing



At each step, pick a ball from each well-mixed urn. We transfer them to the other urn. Let X_n be the number of red balls in the bottom urn at step n.

Application: Mixing



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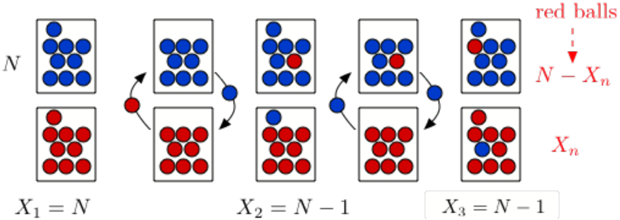

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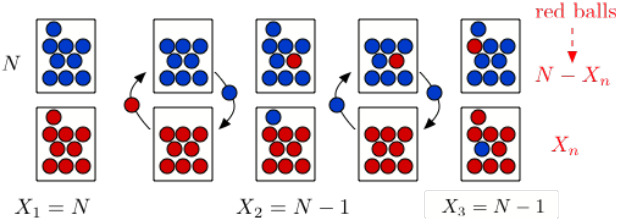
Application: Mixing



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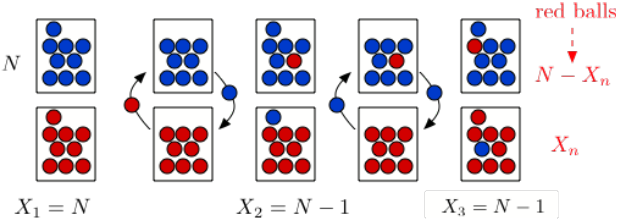
where $p = (1 - m/N)^2$ (B goes up, R down)

Application: Mixing



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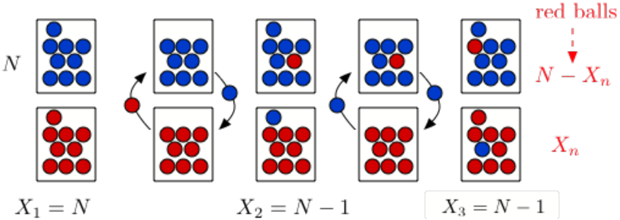
Application: Mixing



At each step, pick a ball from each well-mixed urn. We transfer them to the other urn. Let X_n be the number of red balls in the bottom urn at step n. What is $E[X_n]$?

Given $X_n = m$, $X_{n+1} = m + 1$ w.p. p and $X_{n+1} = m - 1$ w.p. q

where $p = (1 - m/N)^2$ (B goes up, R down) and $q = (m/N)^2$ (R goes up, B down).

Thus,
$E[X_{n+1}|X_n] = X_n + p - q$

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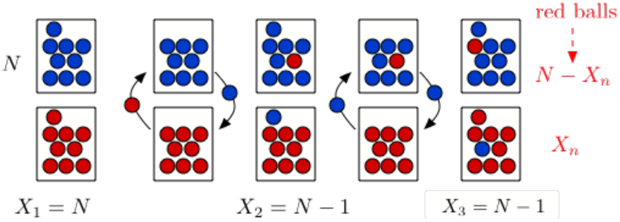

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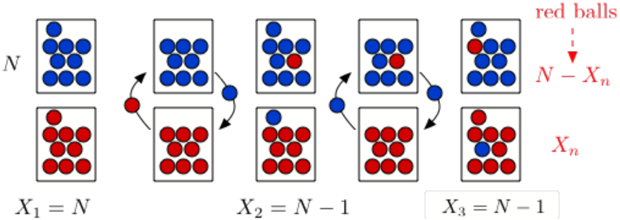

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Thus,
$E[X_{n+1}|X_n] = X_n + p - q = X_n + 1 - 2X_n/N = 1 + \rho X_n, \ \rho := (1 - 2/N).$

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Hence,

$$E[X_n] = \frac{1 - \rho^{n-1}}{1 - \rho} + \rho^{n-1} N, n \geq 1.$$

Application: Mixing

Here is the plot.

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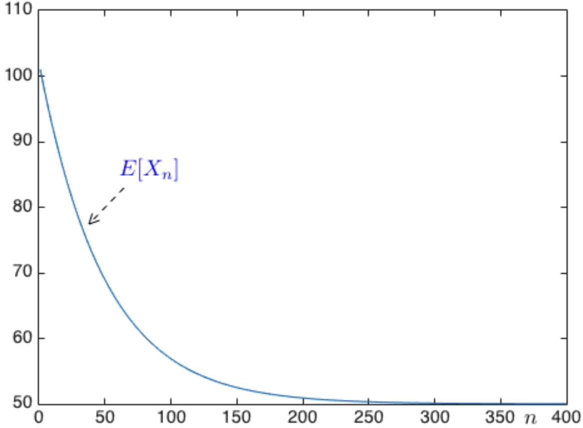

Application: Going Viral

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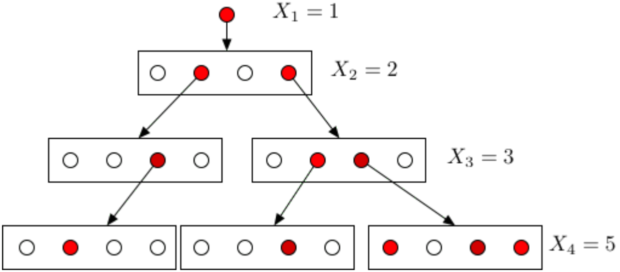

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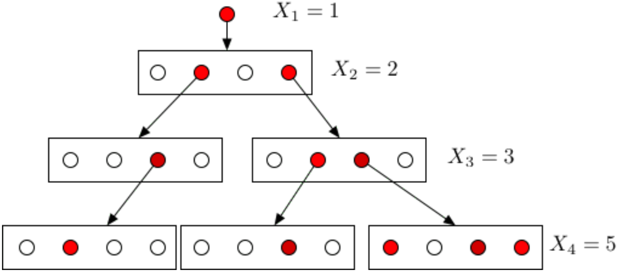
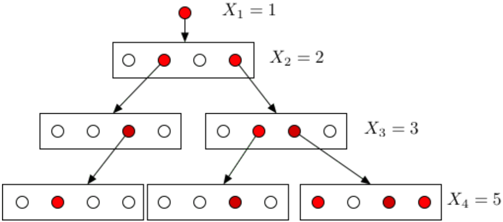


In this example, $d = 4$.

Application: Going Viral



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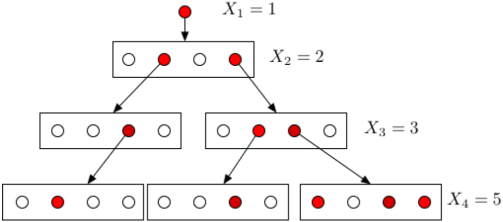
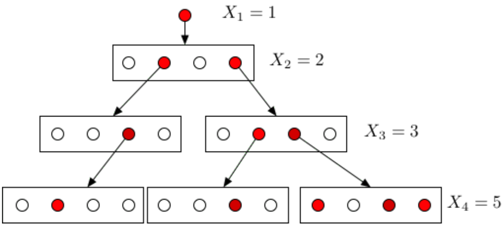


Fact:

Application: Going Viral



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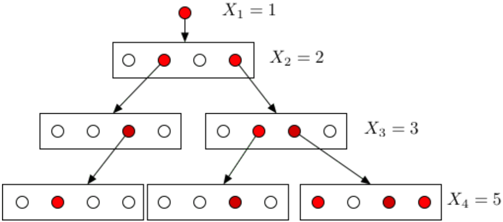
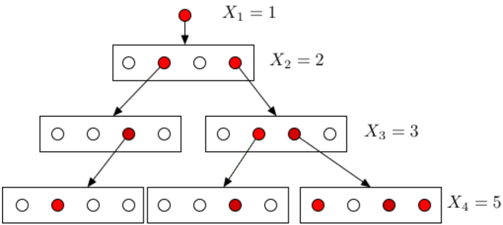


Fact: Let $X = \sum_{n=1}^{\infty} X_n$. Then, $E[X] < \infty$ iff $pd < 1$.

Application: Going Viral



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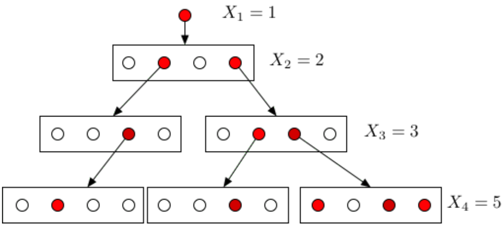
Application: Going Viral



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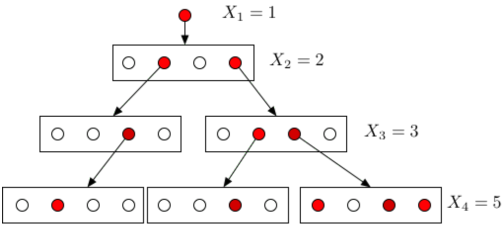

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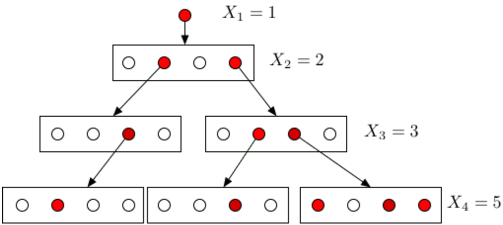

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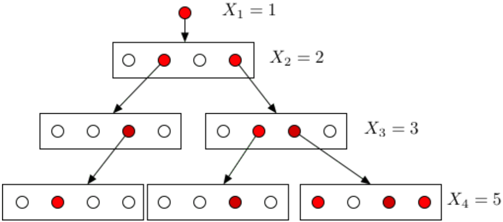

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If $pd < 1$, then $E[X_1 + \cdots + X_n] \leq (1 - pd)^{-1}$

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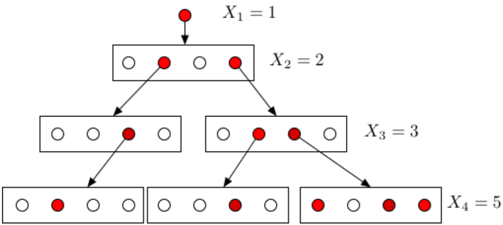

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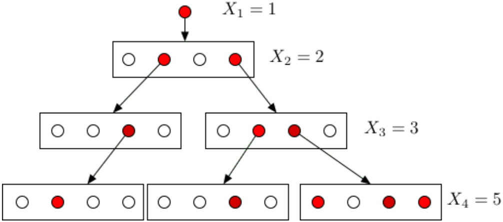

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If $pd \geq 1$, then for all C one can find n s.t.

$$E[X] \geq E[X_1 + \cdots + X_n] \geq C. \qquad \square$$

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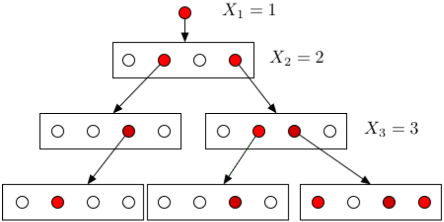

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In fact, one can show that $pd \geq 1 \implies Pr[X = \infty] > 0$.

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Hence, $E[X_1 + \cdots + X_Z] = E[\mu Z] = \mu E[Z].$ $\square$

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Theorem
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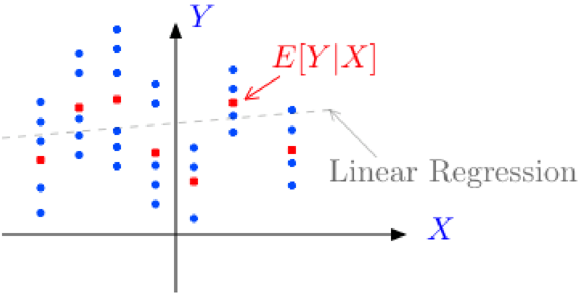

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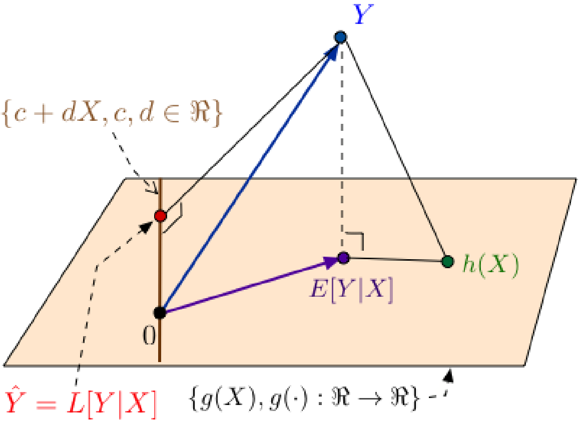
Thus, $E[(Y-h(X))^2] \geq E[(Y-g(X))^2]$. □

$E[Y|X]$ and $L[Y|X]$ as projections



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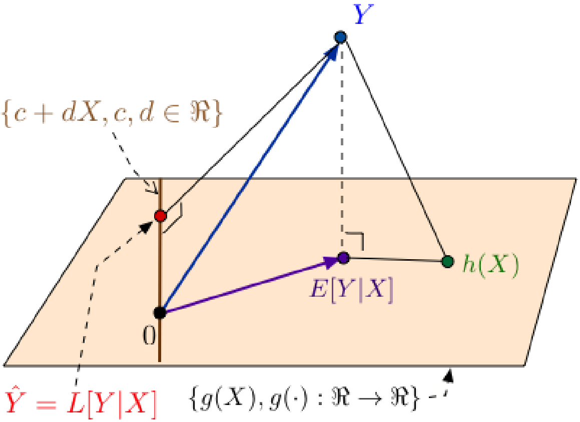

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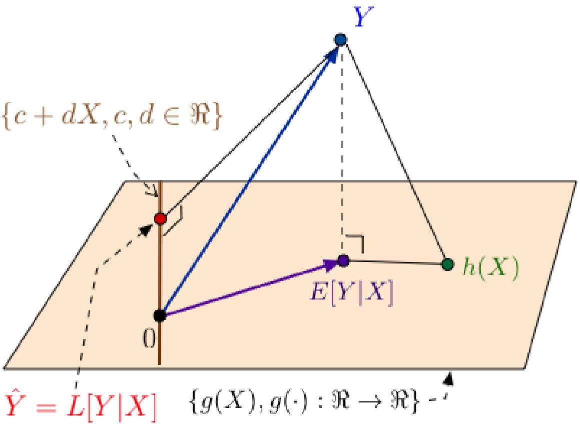

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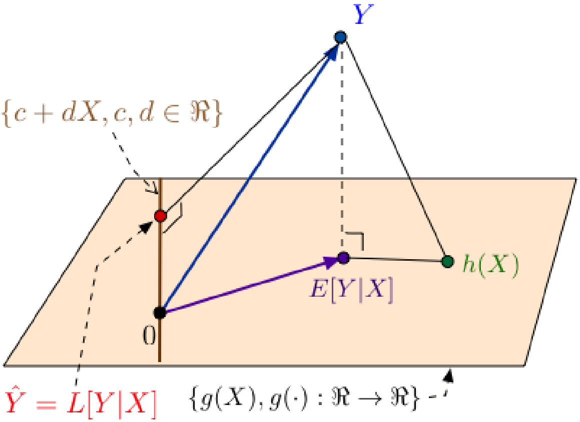
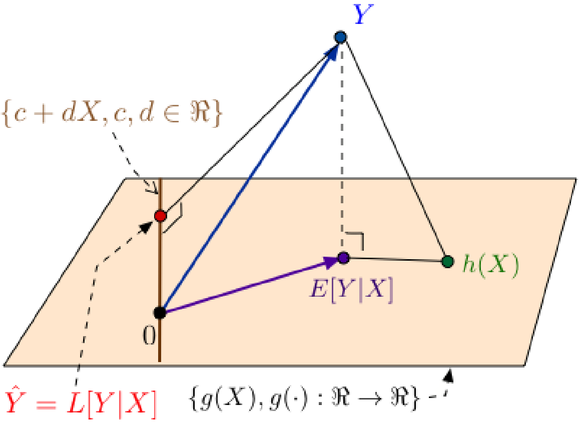


$L[Y|X]$ is the projection of Y on $\{a + bX, a, b \in \Re\}$: LLSE

$E[Y|X]$ is the projection of Y on $\{g(X), g(\cdot) : \Re \to \Re\}$:

$E[Y|X]$ and $L[Y|X]$ as projections



$L[Y|X]$ is the projection of Y on $\{a + bX, a, b \in \Re\}$: LLSE

$E[Y|X]$ is the projection of Y on $\{g(X), g(\cdot) : \Re \to \Re\}$: MMSE.

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- MMSE: $E[Y|X]$ minimizes $E[(Y - g(X))^2]$ over all $g(\cdot)$