CS70: Jean Walrand: Lecture 25.

Markov Chains: Distributions

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Markov Chains: Distributions

1. Review
2. Distribution
3. Irreducibility
4. Convergence

Review

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- Markov Chain:

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Distribution of X_n

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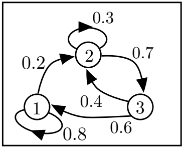

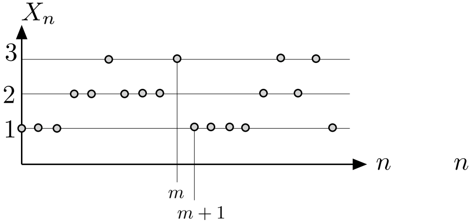

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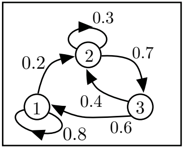

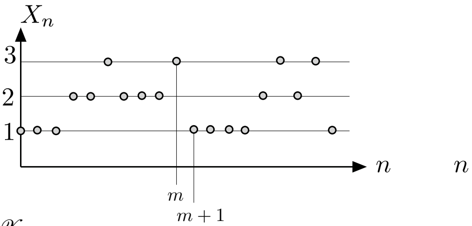

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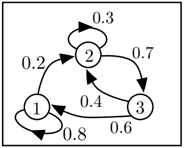

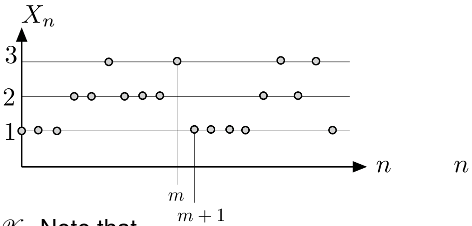

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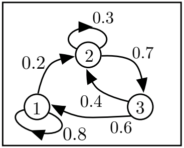

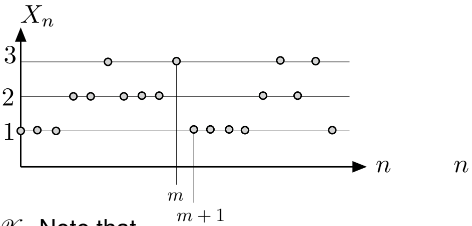

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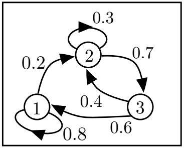

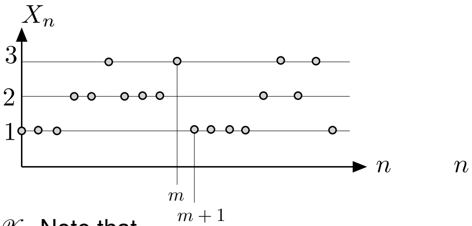

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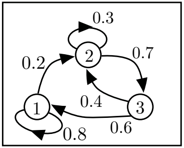

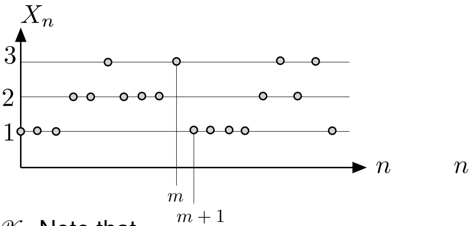

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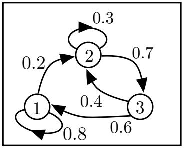

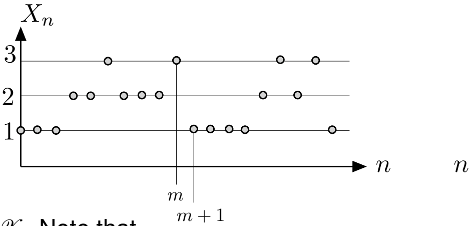

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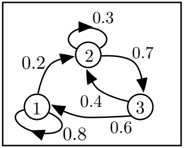

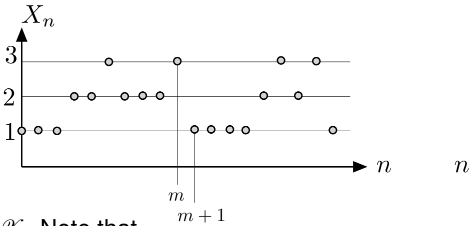

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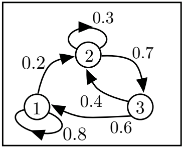

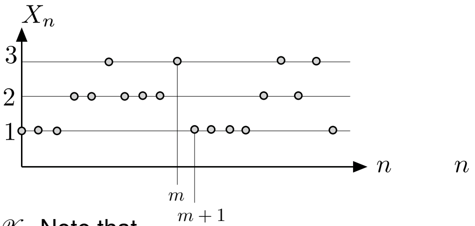

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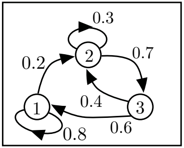

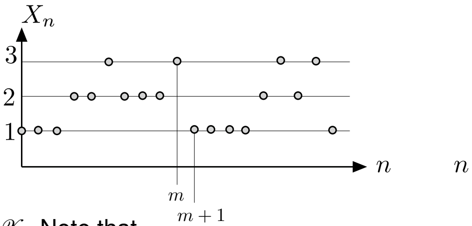

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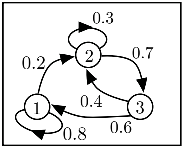

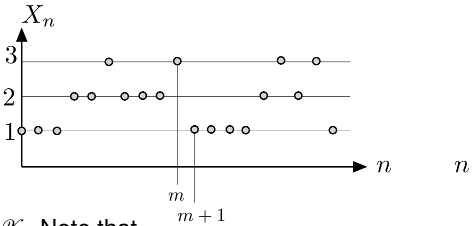

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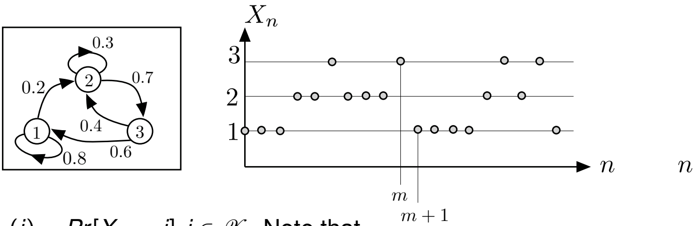

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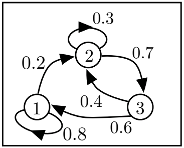

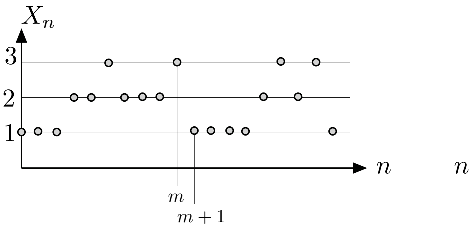

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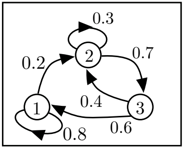

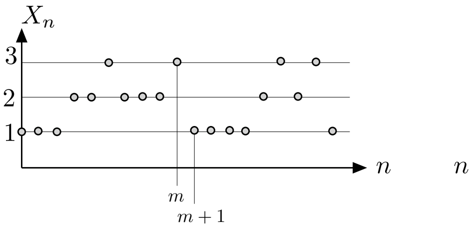

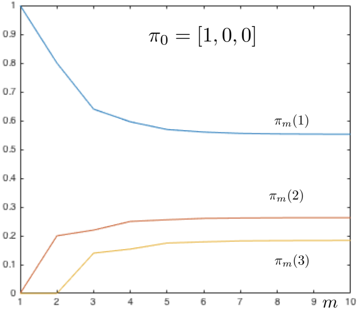

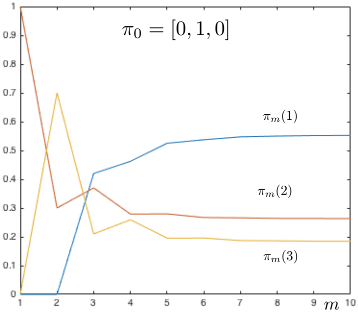

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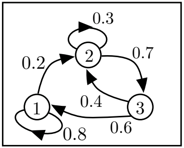
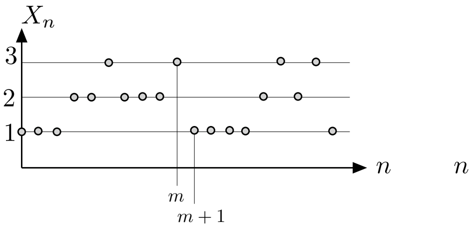
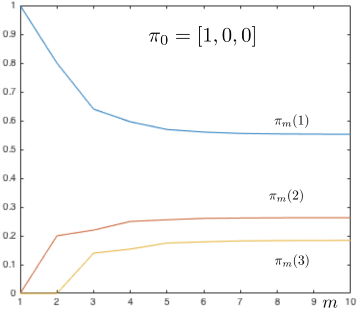
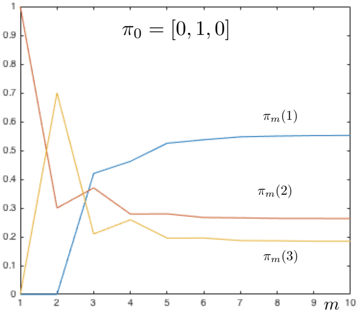


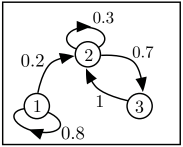


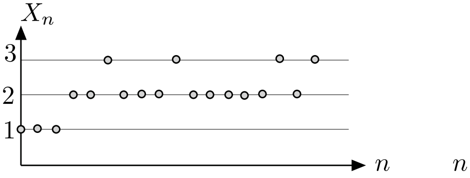




As m increases, π_m converges to a vector that does not depend on π_0.

Distribution of X_n





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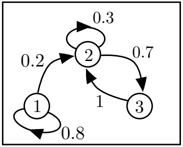
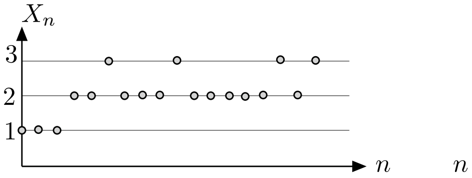
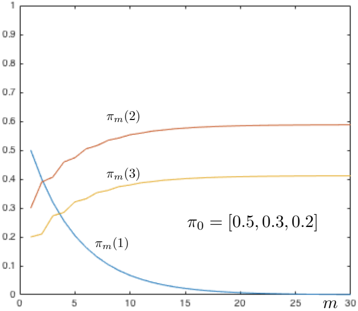
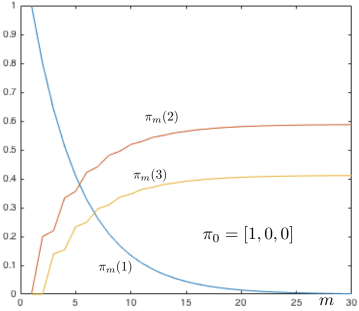
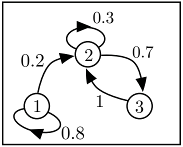


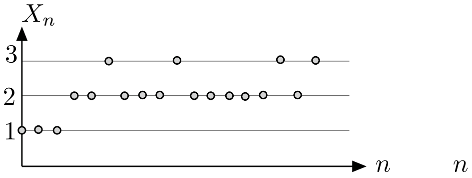


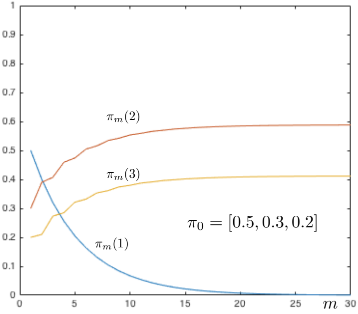


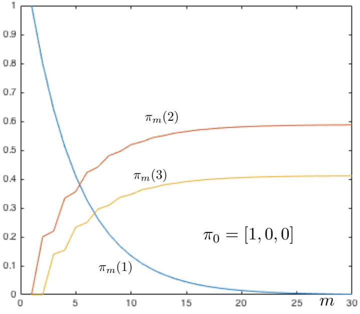


Distribution of X_n









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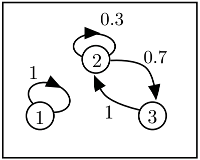

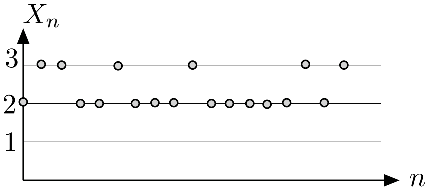

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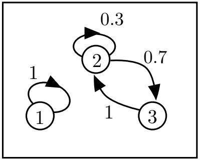

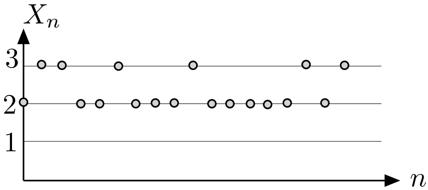

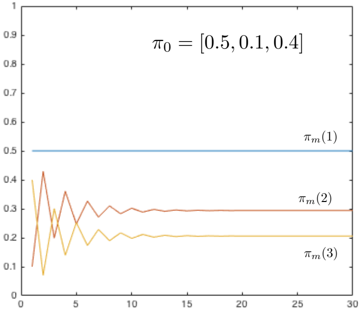

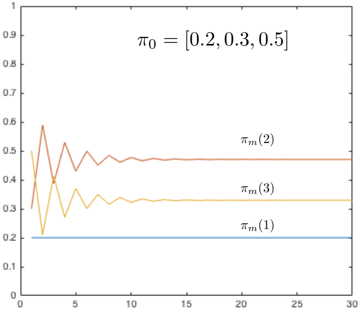

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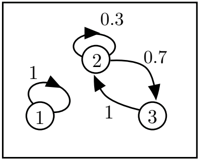

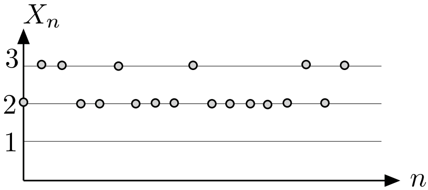

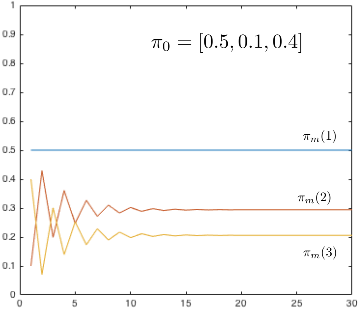

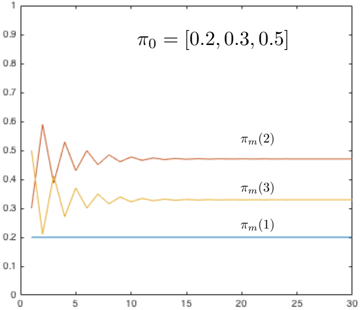

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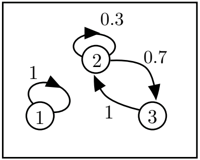

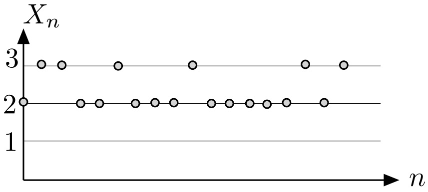

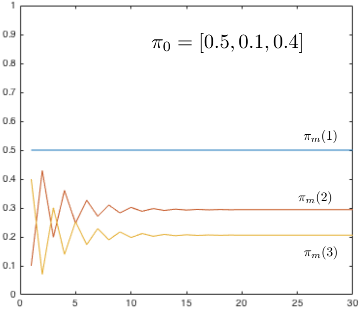

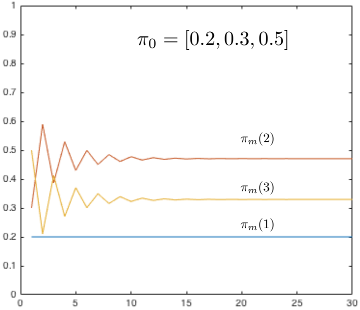

As m increases, π_m converges to a vector that depends on π_0 (obviously, since $\pi_m(1) = \pi_0(1), \forall m$).

Balance Equations

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Question: Is there some π_0 such that $\pi_m = \pi_0, \forall m$?

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Thus, $Pr[\text{enter } i] = Pr[\text{leave } i]$.

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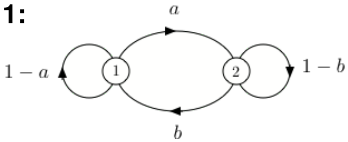

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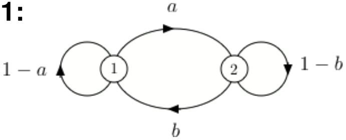

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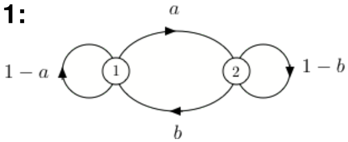

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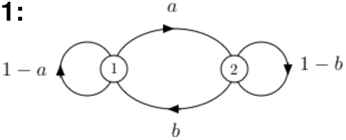

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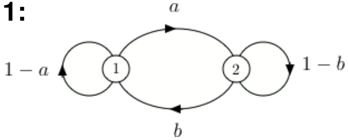

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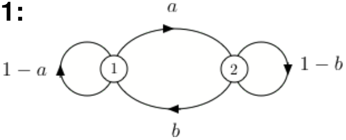

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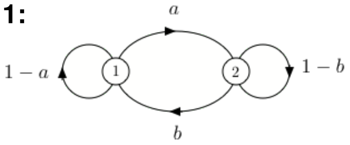

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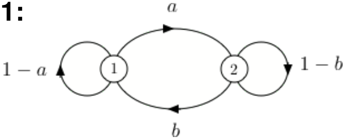

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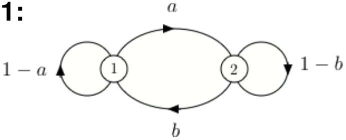

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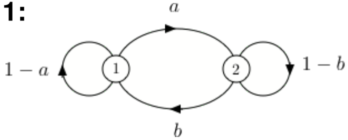

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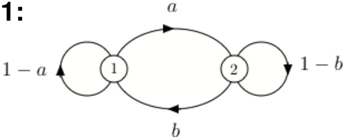

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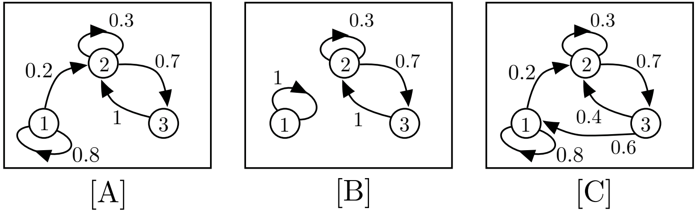

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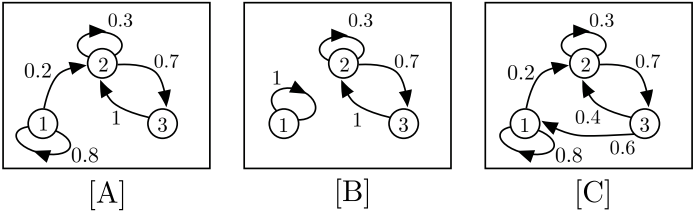

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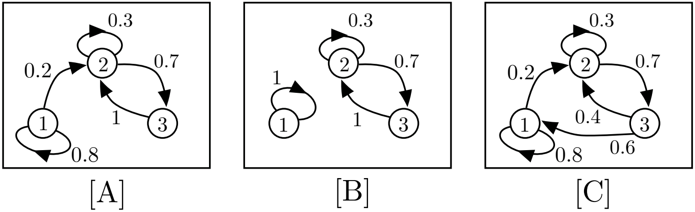

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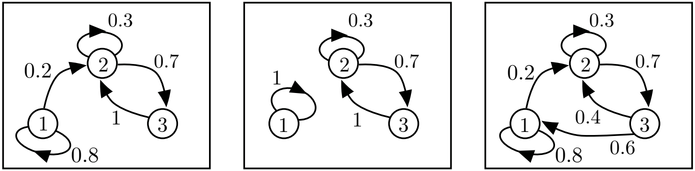

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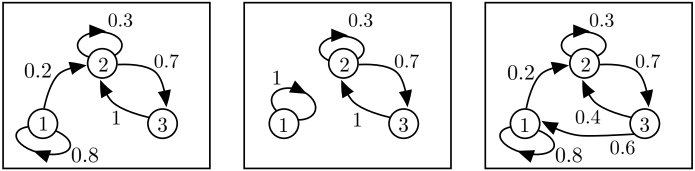

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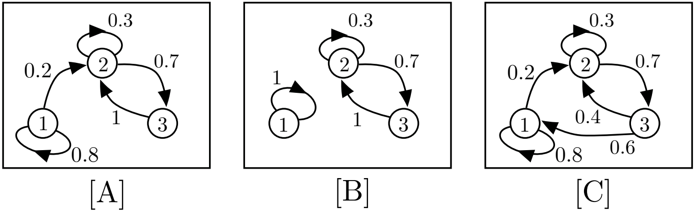

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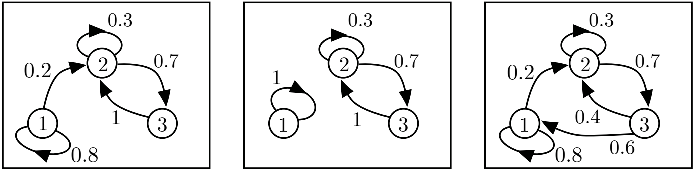

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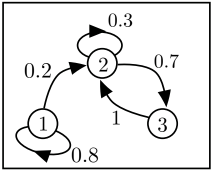

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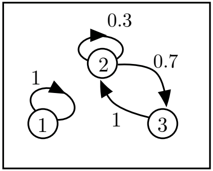

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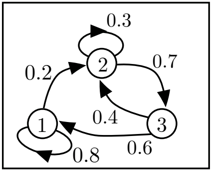

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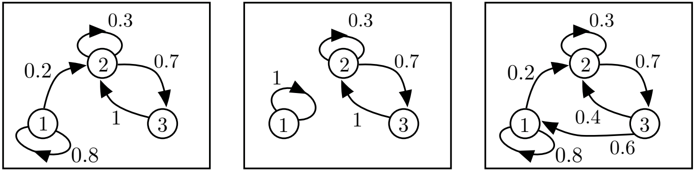

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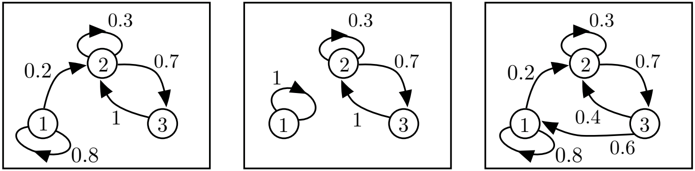

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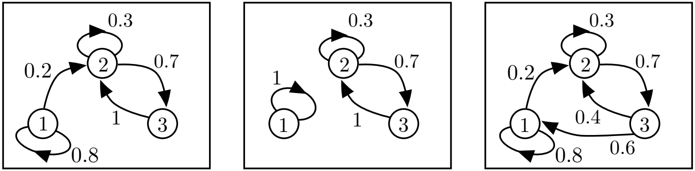

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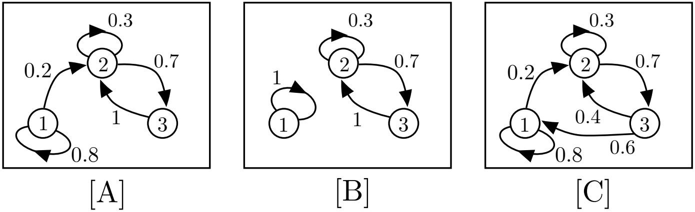

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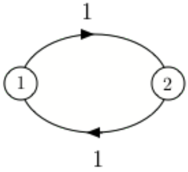

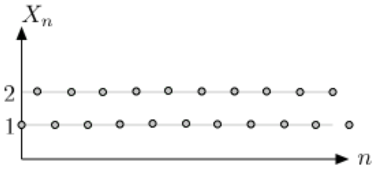

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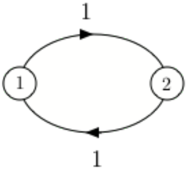

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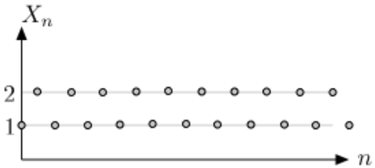

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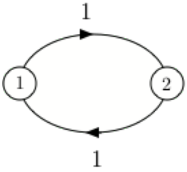

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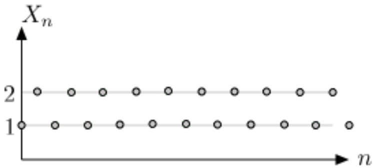

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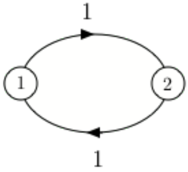

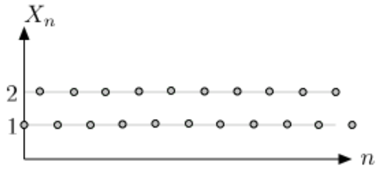

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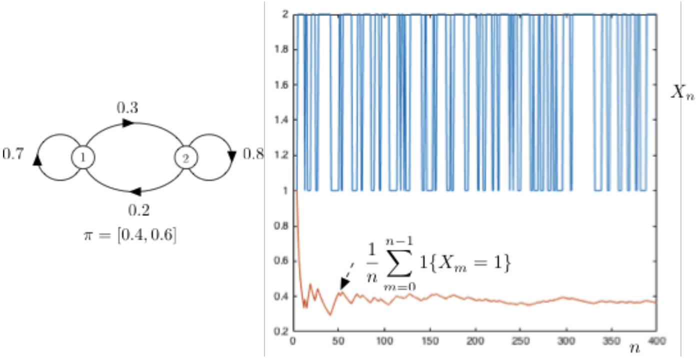

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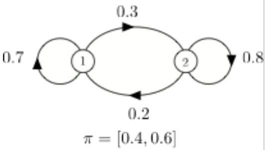

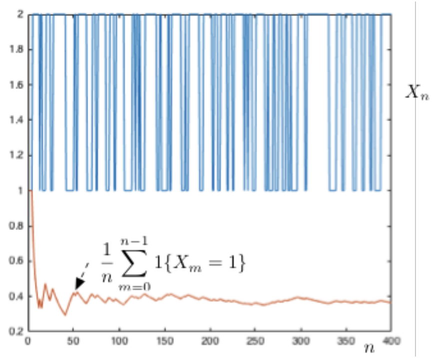

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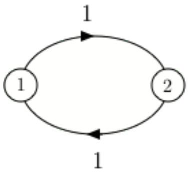

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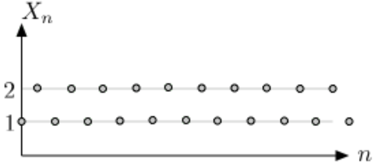

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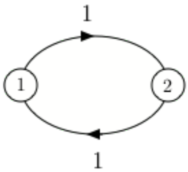
Question: Assume that the MC is irreducible. Does π_n approach the unique invariant distribution π?

Answer: Not necessarily. Here is an example:



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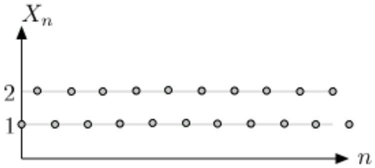

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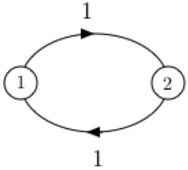

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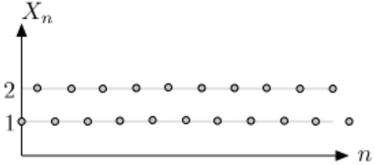

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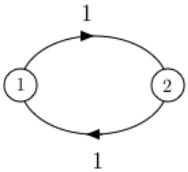

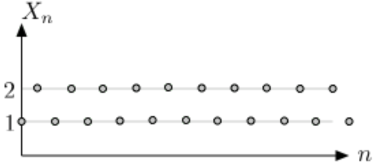

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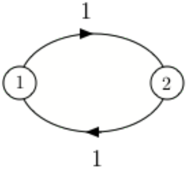

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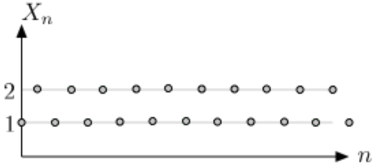

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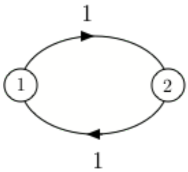

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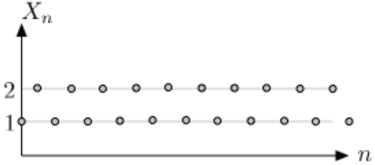

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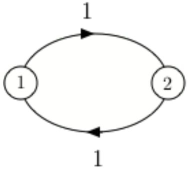

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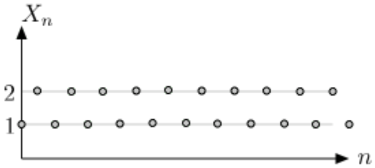

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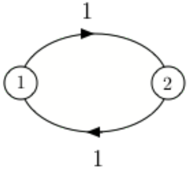

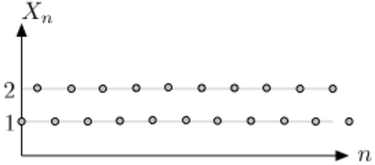

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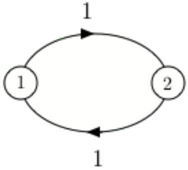

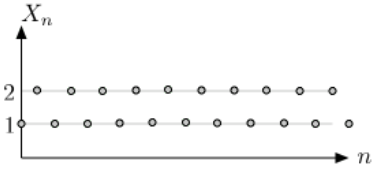

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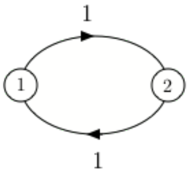

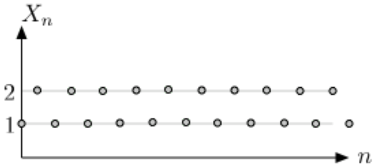

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Thus, if $\pi_0 = [1, 0]$, $\pi_1 = [0, 1], \pi_2 = [1, 0], \pi_3 = [0, 1]$, etc.

Hence, π_n does not converge to $\pi = [1/2, 1/2]$.

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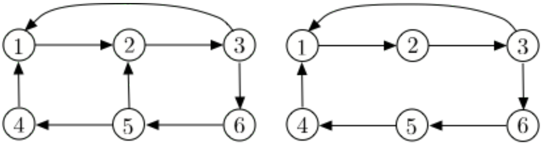

[A] [B]

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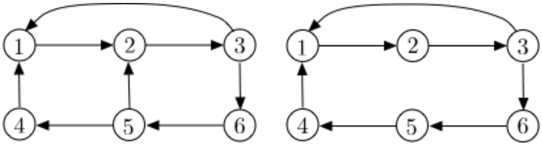

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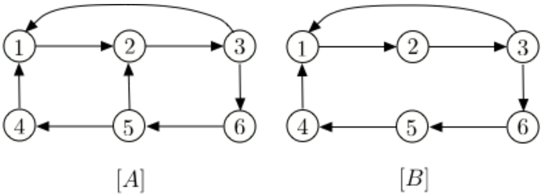

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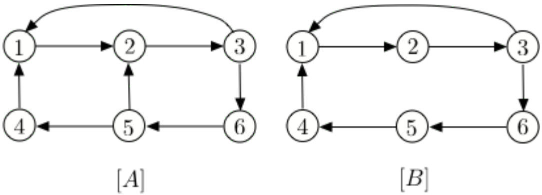

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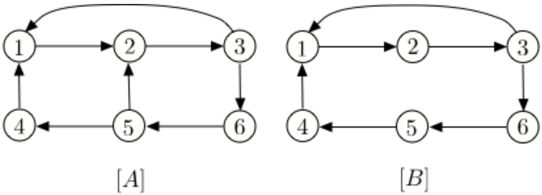

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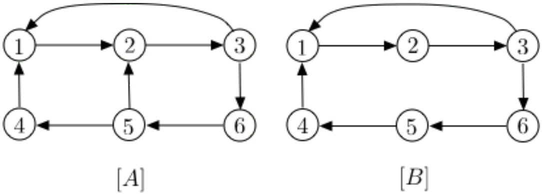

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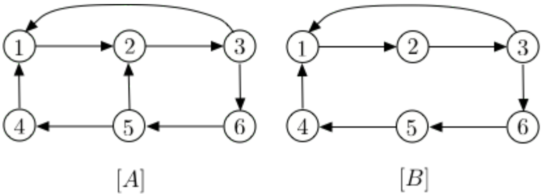

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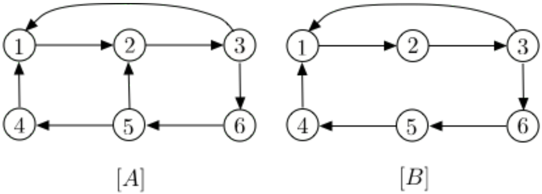

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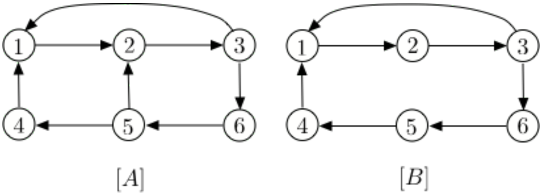

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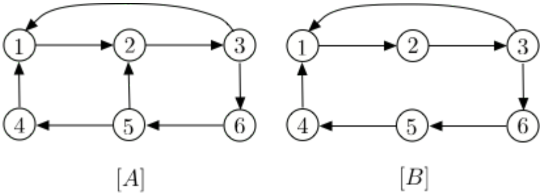

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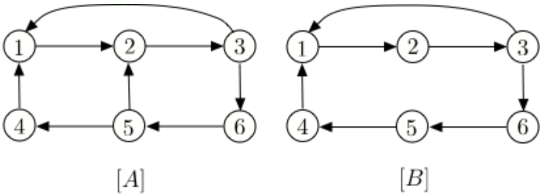

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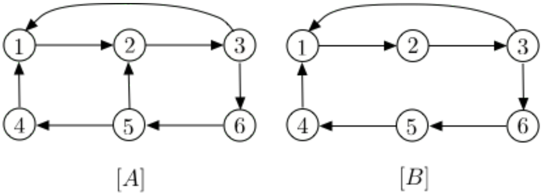

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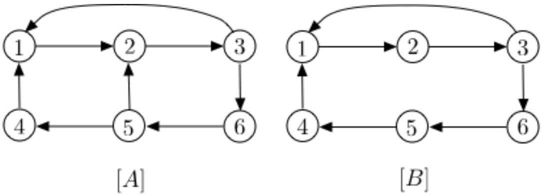

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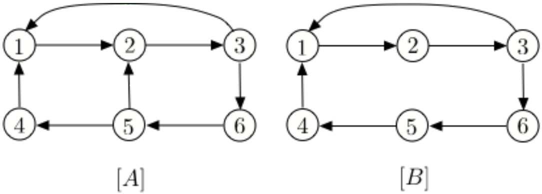

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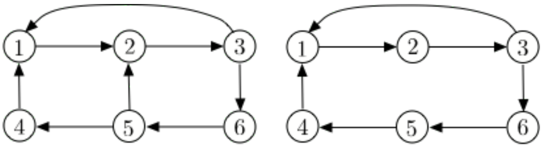
$$d(i) := \text{g.c.d.}\{n > 0 \mid Pr[X_n = i \mid X_0 = i] > 0\}$$

has the same value for all states i.

Proof: See Lecture notes 24. □

Definition If $d(i) = 1$, the Markov chain is said to be aperiodic. Otherwise, it is periodic with period $d(i)$.

Example



[A] [B]

[A]: $\{n > 0 \mid Pr[X_n = 1|X_0 = 1] > 0\} = \{3, 6, 7, 9, 11, \ldots\} \Rightarrow d(1) = 1.$

$\{n > 0 \mid Pr[X_n = 2|X_0 = 2] > 0\} = \{3, 4, \ldots\} \Rightarrow d(2) = 1.$

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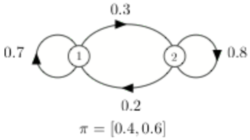

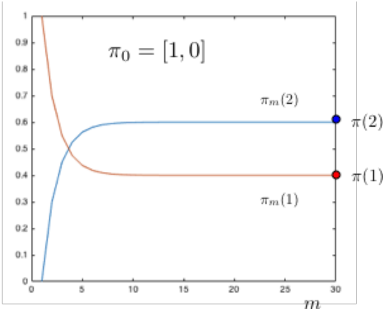

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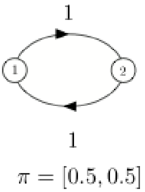

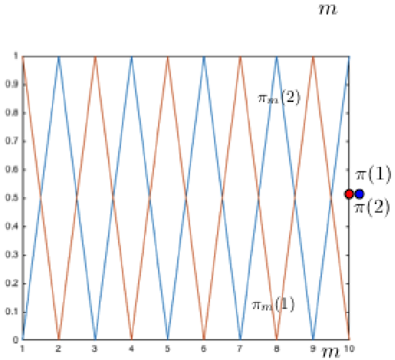

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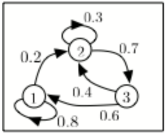

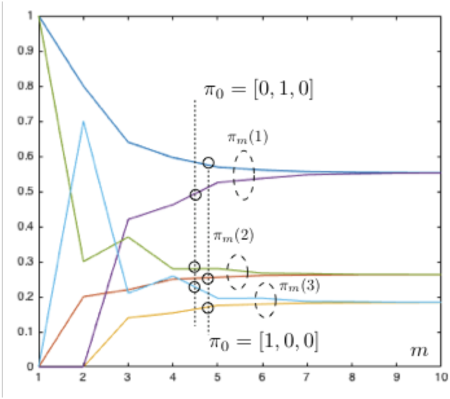

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There are relatively few problems for which one can prove such a clean result. However, there is a systematic approach to calculate the optimal strategy for many problems. We explain that approach next on this problem.

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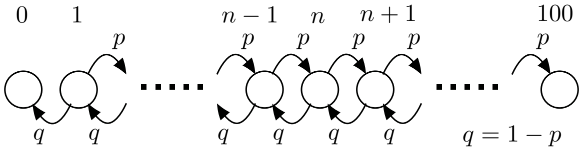

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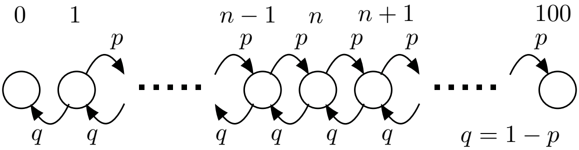

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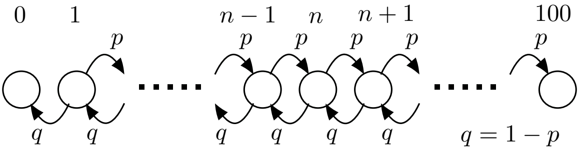

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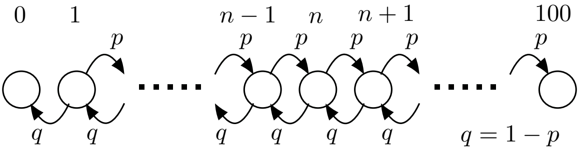

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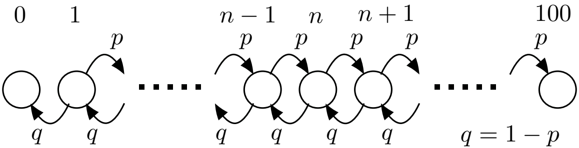

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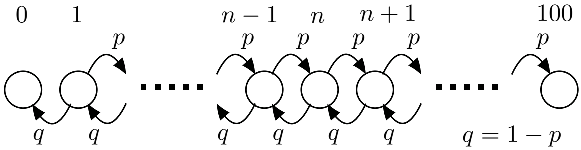

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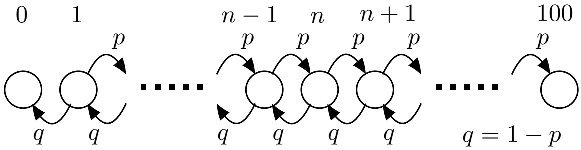

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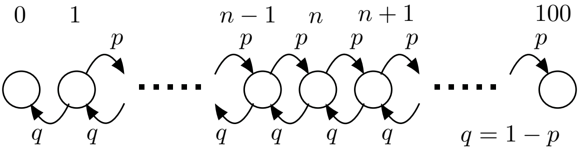

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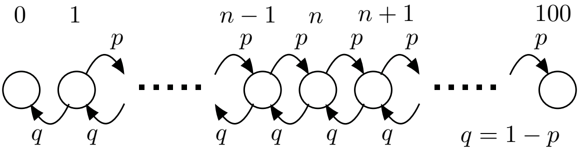

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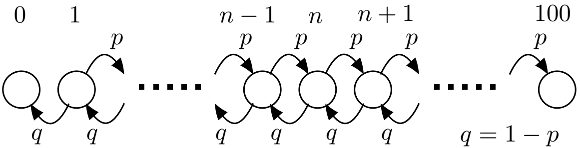

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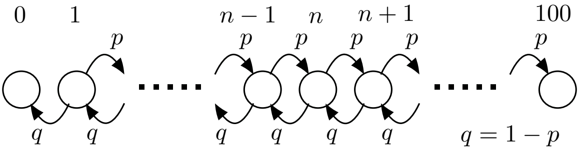

Let $\alpha(n)$ be the probability of reaching 100 before 0, starting from n, for $n = 0, 1, \ldots, 100$.

$$\alpha(0) = 0; \alpha(100) = 1.$$
$$\alpha(n) = p\alpha(n+1) + q\alpha(n-1), 0 < n < 100.$$

Solving, we find

$$\alpha(n) = \frac{1-\rho^n}{1-\rho^{100}} \text{ with } \rho = qp^{-1}.$$

For $p = 0.46$, we get $\alpha(10) \approx 3.5 \times 10^{-6}$.

Bold: Estimate

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We can do better.

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We can do better. Let us bet all we have.

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With $p = 0.46$, we see that we get to 100, at least with probability $(0.46)^4 = 0.0448$.

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With $p = 0.46$, we see that we get to 100, at least with probability $(0.46)^4 = 0.0448$. This is much better than 3.5×10^{-6}.

Thus, the probability of winning the game (i.e., getting to 100 before 0) is at least 0.0448 when playing bold.

Bold: Analysis

What is the exact probability of winning when playing bold?

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What is the exact probability of winning when playing bold? Here is the corresponding MC:

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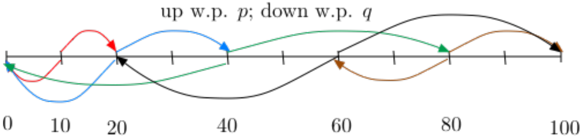

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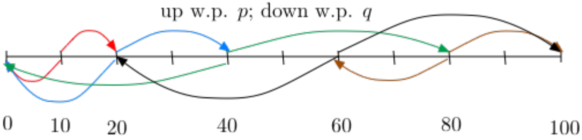

The FSE for $\alpha(n) = Pr[T_{100} < T_0 \mid X_0 = n]$ are

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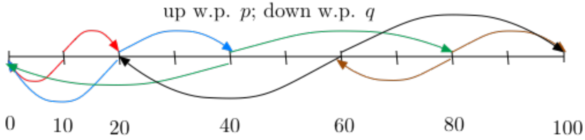

The FSE for $\alpha(n) = Pr[T_{100} < T_0 \mid X_0 = n]$ are

$$\alpha(10) = p\alpha(20) + q0;$$

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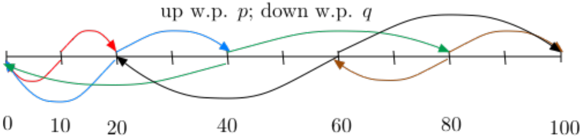

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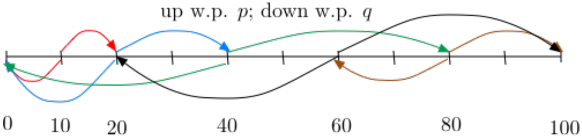

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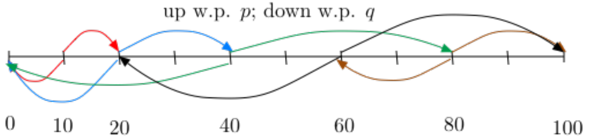

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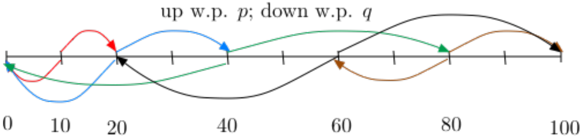

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To solve, let $\alpha(10) = x$.

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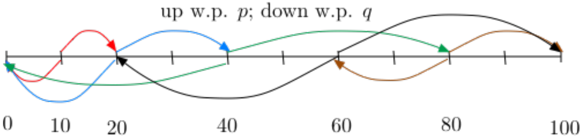

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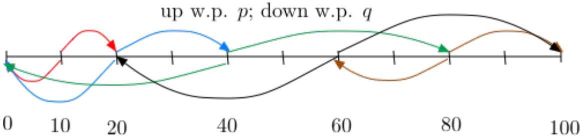

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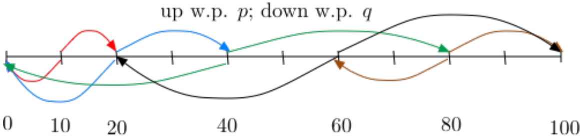

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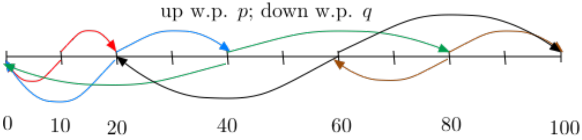

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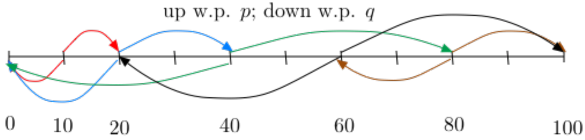

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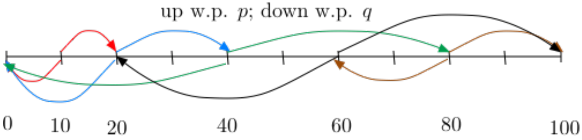

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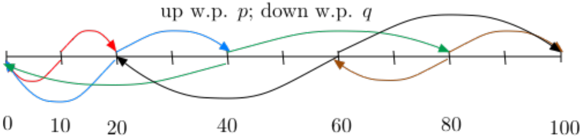

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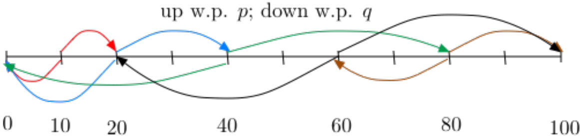

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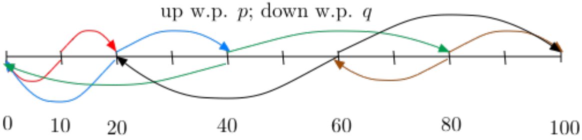

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We solve the last two equations for x.

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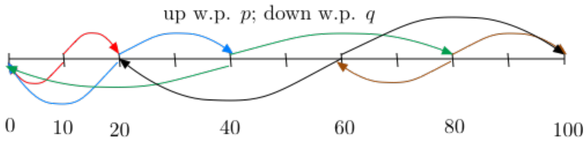

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We solve the last two equations for x.
We find $x = p^2(1+q)/(p^{-2} - q^2)$

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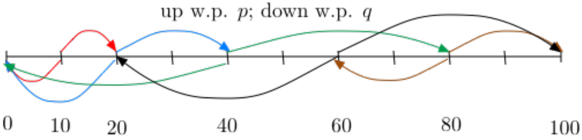

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We solve the last two equations for x.
We find $x = p^2(1+q)/(p^{-2} - q^2) \approx 0.0735$.

Optimal Strategy

Optimal Strategy

Note: The material on the remaining slides of this lecture will not be on the final.

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How do we calculate the optimal strategy?

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How do we calculate the optimal strategy?

Here is a systematic approach. Assume you can only play 0 time.

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Here is a systematic approach. Assume you can only play 0 time. Let $V(0, n)$ be you maximum probability of winning the game

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How do we calculate the optimal strategy?

Here is a systematic approach. Assume you can only play 0 time. Let $V(0, n)$ be you maximum probability of winning the game if you start with n.

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Here is a systematic approach. Assume you can only play 0 time. Let $V(0, n)$ be you maximum probability of winning the game if you start with n. Clearly,

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Here is a systematic approach. Assume you can only play 0 time. Let $V(0, n)$ be you maximum probability of winning the game if you start with n. Clearly, $V(0, 100) =$

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Here is a systematic approach. Assume you can only play 0 time. Let $V(0,n)$ be you maximum probability of winning the game if you start with n. Clearly, $V(0,100) = 1$. Also, for $n = 0,\ldots,99$, one has $V(0,n) =$

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Let $V(k,n)$ be the maximum probability of winning the game

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Let $V(k,n)$ be the maximum probability of winning the game if we can play k times

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Let $V(k, n)$ be the maximum probability of winning the game if we can play k times and we start with n.

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Let $V(k,n)$ be the maximum probability of winning the game if we can play k times and we start with n.

Then,

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Conclusion: You might as well stop at the first card!.

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See EE126, CS188, EE223.