CS70: Jean Walrand: Lecture 26.

Continuous Probability

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Continuous Probability

1. Examples
2. Events
3. Continuous Random Variables
4. Expectation
5. Bayes' Rule
6. Multiple Random Variables

Continuous Probability - Pick a real number.

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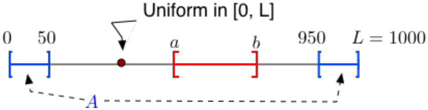

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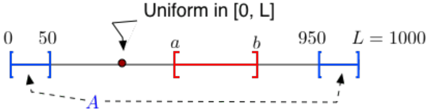

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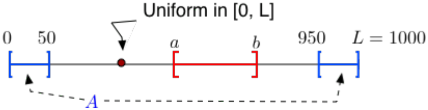

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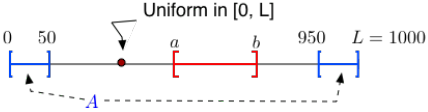

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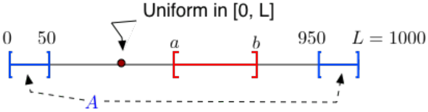

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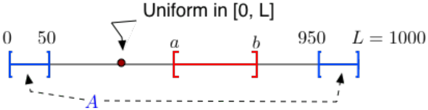

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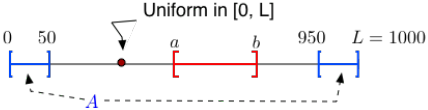

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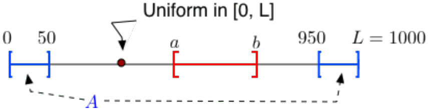

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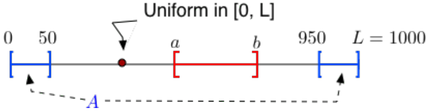

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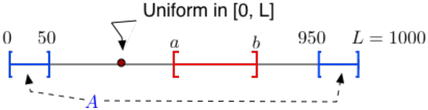

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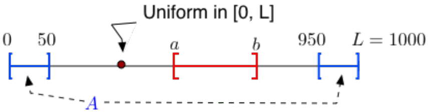

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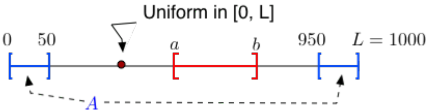

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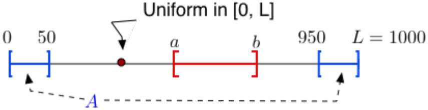

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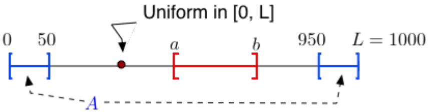

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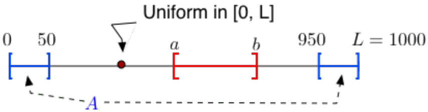

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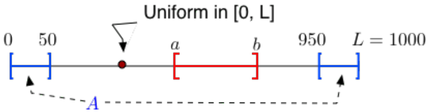

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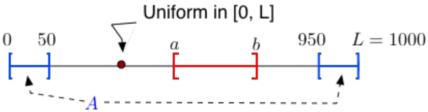

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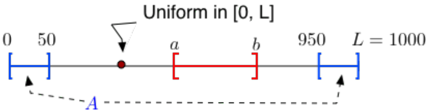

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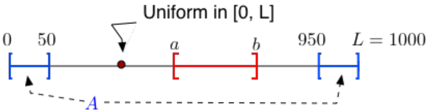

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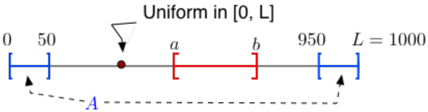

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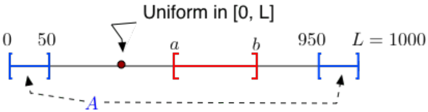

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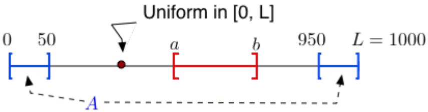

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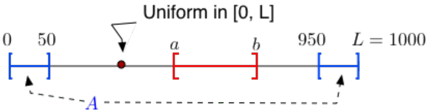

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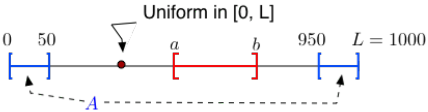

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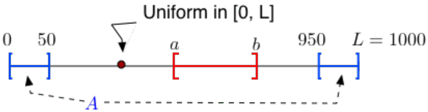

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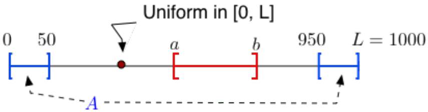

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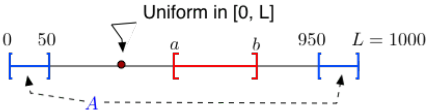

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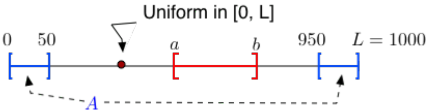

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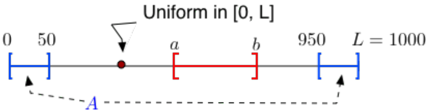

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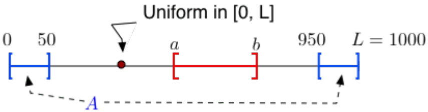

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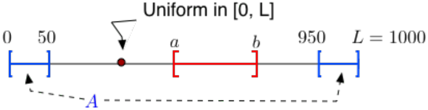

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Thus, the probability is a function from events to $[0, 1]$. Can any function make sense? No! At least, it should be additive!. In our example, $Pr[[0, 50] \cup [950, 1000]] = Pr[[0, 50]] + Pr[[950, 1000]]$.

Shooting..

A James Bond example. In Spectre, Mr. Hinx

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Assume the gas tank is a one foot circle and the DB10 is an expensive 4×5 rectangle.

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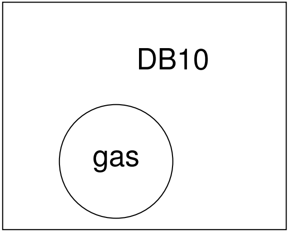

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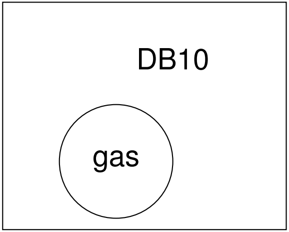

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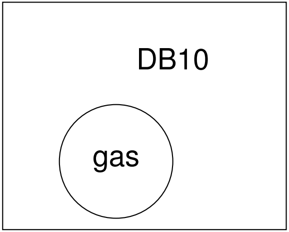

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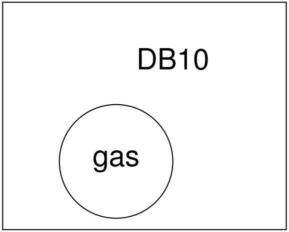

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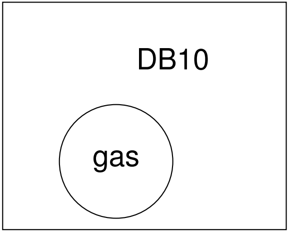

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Since uniform, probability of event is $\frac{\pi}{20}$.

Continuous Random Variables: CDF

$Pr[a < X \leq b]$ instead of $Pr[X = a]$.

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Example: hitting random location on gas tank.

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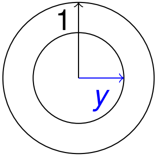

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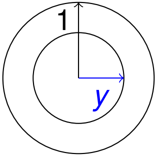

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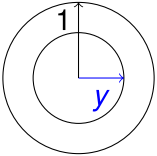

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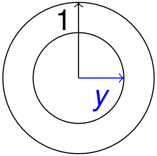

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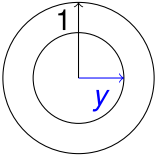

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Probability between .5 and .6 of center?

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&= \lim_{\delta \to 0} \frac{F_X(x + \delta) - F_X(x)}{\delta} \\
&= \frac{d(F_X(x))}{dx}.
\end{aligned}$$

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Definition: (Density) A **probability density function** for a random variable X with cdf $F_X(x) = Pr[X \leq x]$ is the function $f_X(x)$ where

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Use whichever is convenient.

Target

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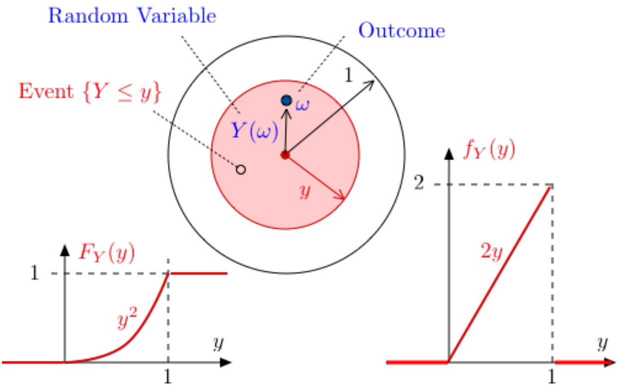

Random Variable
Outcome
1
Event $\{Y \leq y\}$
ω
$Y(\omega)$
y
$f_Y(y)$
2
$2y$
$F_Y(y)$
1
y^2
y
1
y
1

$U[a, b]$

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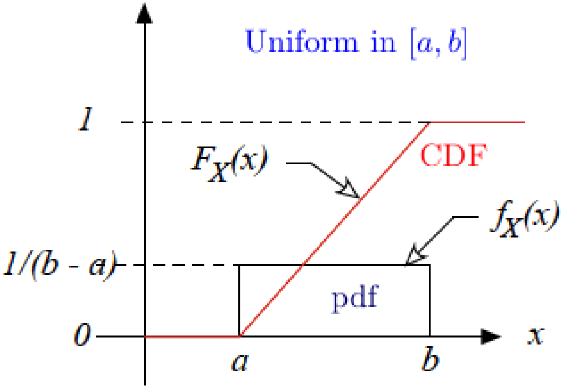

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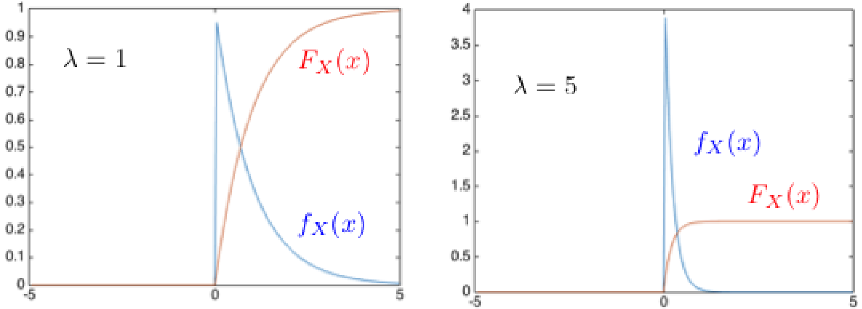

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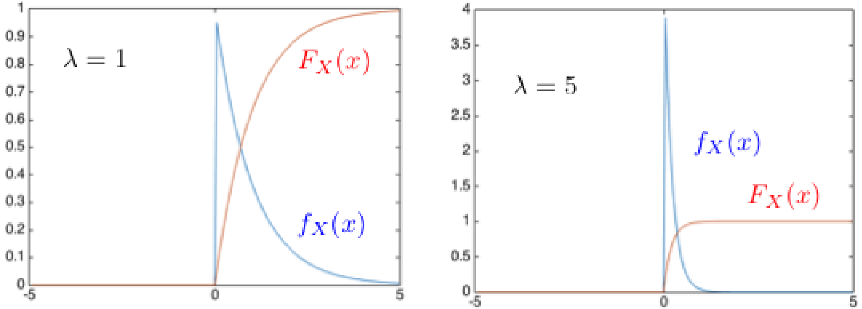

Note that $Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

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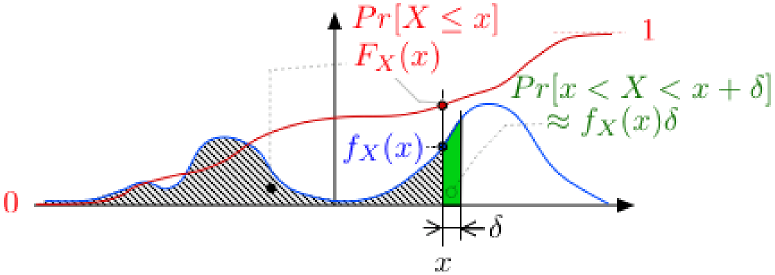
Recall that $Pr[X \in (x, x+\delta)] \approx f_X(x)\delta$. Think of X taking discrete values $n\delta$ for $n = \ldots, -2, -1, 0, 1, 2, \ldots$ with $Pr[X = n\delta] = f_X(n\delta)\delta$.

A Picture



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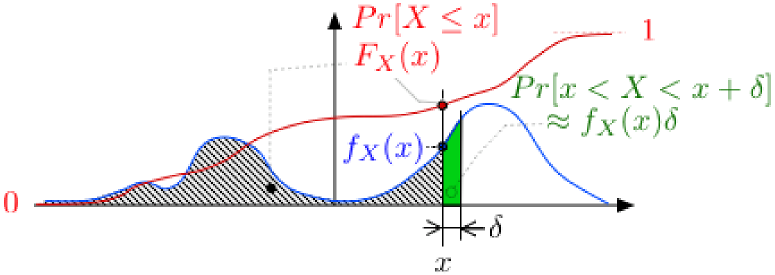

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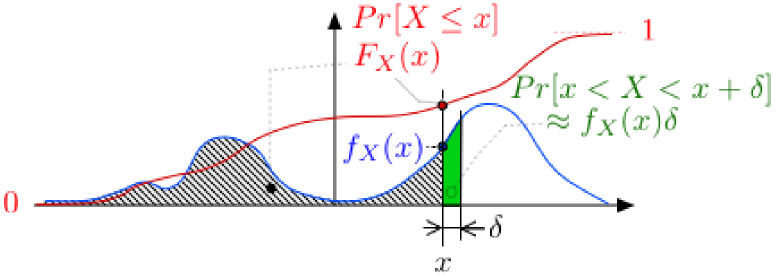

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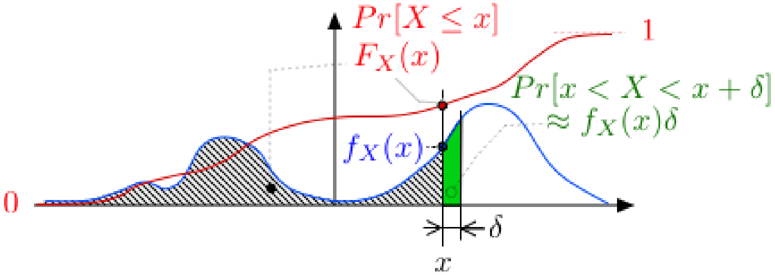

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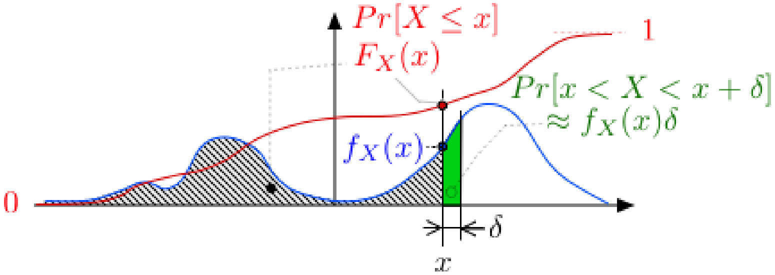

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Thus, $a \times Expo(\lambda) = Expo(\lambda/a)$.

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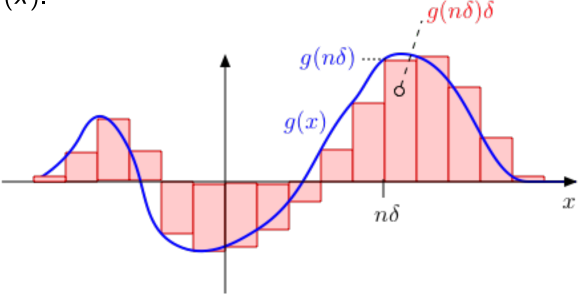

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Fact Expectation is linear. **Proof:** As in the discrete case.

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