Today

Review for Final..

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Rao's Cheat Sheet.

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A million slides.

Notes: Logic/Proofs.

Statement?

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Statement? “3 = 4+1”?

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Statement: $\forall n, P(n)$. Base: $P(0)$. Step: $P(k) \implies P(k+1)$.
Simple: $\sum_i i = n(n+1)/2$.

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Strong induction: primes have a factorization.

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Stable Marriage

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Improvement Lemma. Optimality/Pessimality.

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Multiplicative inverses!

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A statement is a true or false.

Statements?

$3 = 4 - 1$? Statement!

$3 = 5$? Statement!

3 ? Not a statement!

$n = 3$? Not a statement...but a predicate.

Predicate: Statement with free variable(s).

Example: $x = 3$ Given a value for x, becomes a statement.

Predicate?

$n > 3$? Predicate: $P(n)$!

$x = y$? Predicate: $P(x, y)$!

$x + y$? No. An expression, not a statement.

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Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

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You got this!

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Propositional Expressions and Logical Equivalence

Connecting Statements

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You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$

Connecting Statements

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Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

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Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
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Propositional Expressions and Logical Equivalence

$$(A \implies B) \equiv (\neg A \vee B)$$
$$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$$

Proofs: truth table or manipulation of known formulas.

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$$(A \implies B) \equiv (\neg A \vee B)$$
$$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$$

Proofs: truth table or manipulation of known formulas.

$$(\forall x)(P(x) \wedge Q(x)) \equiv (\forall x)P(x) \wedge (\forall x)Q(x)$$

..and then proofs...

Direct: $P \implies Q$

..and then proofs...

Direct: $P \implies Q$

Example: a is even $\implies a^2$ is even.

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Approach: What is even?

..and then proofs...

Direct: $P \implies Q$

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Approach: What is even? $a = 2k$

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Integers closed under multiplication!

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$\neg P \implies$ **false**

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Example: a^2 is odd $\implies a$ is odd.

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$\neg P \implies R \wedge \neg R$

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Useful for prove something does not exist:

..and then proofs...

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$

..and then proofs...

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$\neg P \implies$ **false**

$\neg P \implies R \wedge \neg R$

Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

Example: finite set of primes

..and then proofs...

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

Example: finite set of primes does not exist.

Example: rogue couple does not exist.

...jumping forward..

Contradiction in induction:

...jumping forward..

Contradiction in induction:
    contradict place where induction step doesn't hold.

...jumping forward..

Contradiction in induction:
 contradict place where induction step doesn’t hold.

Well Ordering Principle.

...jumping forward..

Contradiction in induction:
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Well Ordering Principle.
 Stable Marriage:

...jumping forward..

Contradiction in induction:
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Well Ordering Principle.
Stable Marriage:
first day where women does not improve.

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...and then induction...

$$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$$

...and then induction...

$$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$$

Thm: For all $n \geq 1$, $8|3^{2n}-1$.

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

...and then induction...

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Base: $8 | 3^2 - 1$.

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Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.

...and then induction...

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Induction Step: Prove $P(n+1)$

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Induction on n.

Base: $8 | 3^2 - 1$.

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Induction Step: Prove $P(n+1)$

$3^{2n+2} - 1 =$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

Thm: For all $n \geq 1$, $8|3^{2n} - 1$.

Induction on n.

Base: $8|3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.

Induction Step: Prove $P(n+1)$

$$3^{2n+2} - 1 = 9(3^{2n}) - 1$$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$3^{2n+2} - 1 = 9(3^{2n}) - 1$ (by induction hypothesis)

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

Thm: For all $n \geq 1$, $8|3^{2n}-1$.

Induction on n.

Base: $8|3^2-1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n}-1 = 8d)$

Induction Step: Prove $P(n+1)$

$$\begin{aligned} 3^{2n+2}-1 &= 9(3^{2n})-1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1)-1 \end{aligned}$$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n)$.

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

Induction on n.

Base: $8 | 3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$$\begin{aligned} 3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\ &= 9(8d + 1) - 1 \\ &= 72d + 8 \end{aligned}$$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

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$$
\begin{aligned}
3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\
&= 9(8d+1) - 1 \\
&= 72d + 8 \\
&= 8(9d+1)
\end{aligned}
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...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\ P(n).$

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Divisible by 8.

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Divisible by 8.

□

Stable Marriage: a study in definitions and WOP.

n-men, *n*-women.

Stable Marriage: a study in definitions and WOP.

n-men, *n*-women.

Each person has completely ordered preference list

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No, for roommates problem.

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Man optimal $\implies$ Woman pessimal.

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Woman optimal $\implies$ Man pessimal.

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Connected Graph: one connected component.

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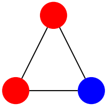

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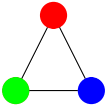

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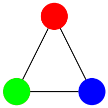

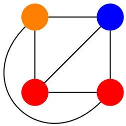

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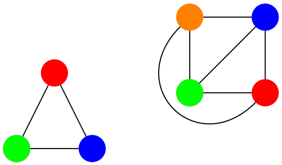

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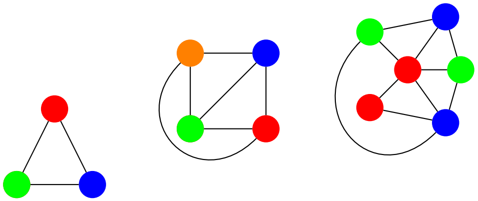

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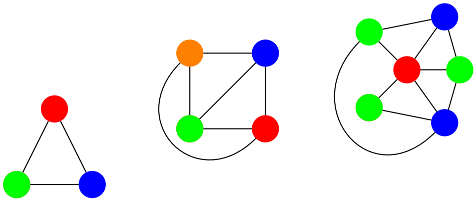

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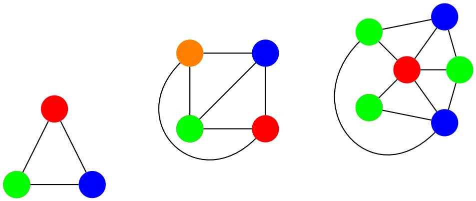

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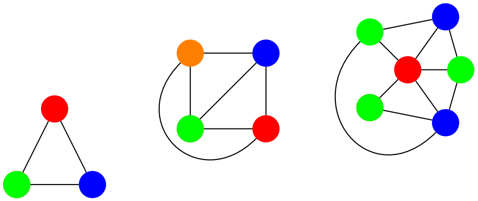

Notice that the last one, has one three colors.

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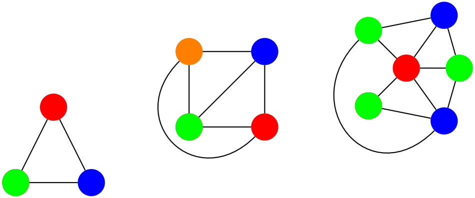

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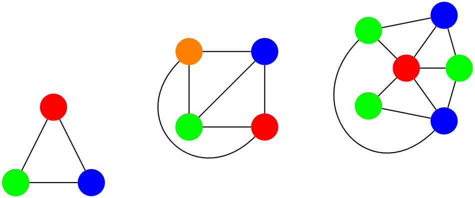

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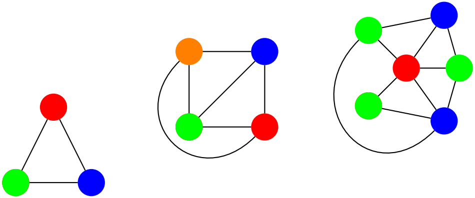

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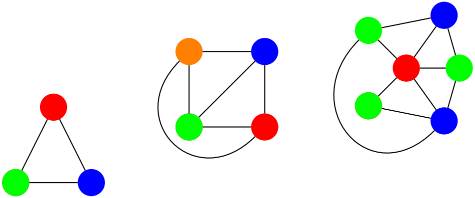

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Interesting things to do.

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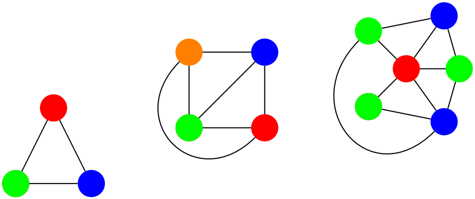

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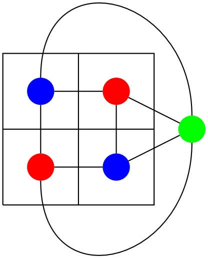
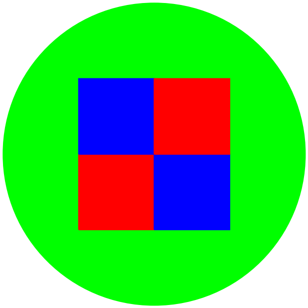
Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





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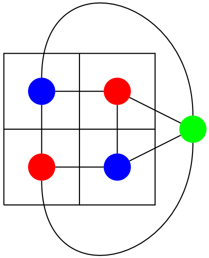

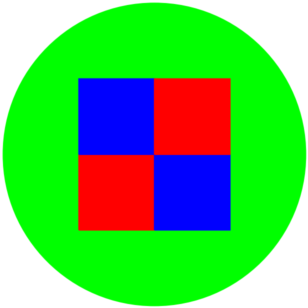

Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

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Recall: $e \leq 3v - 6$ for any planar graph.

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Total degree: $2e$

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Inductively color remaining graph.

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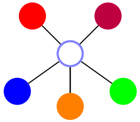

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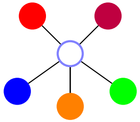

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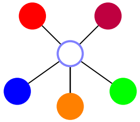

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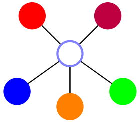

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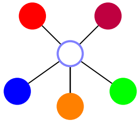

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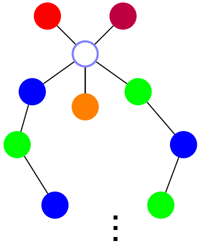

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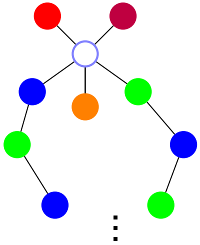

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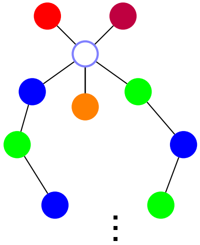

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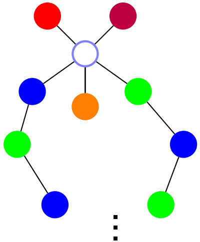

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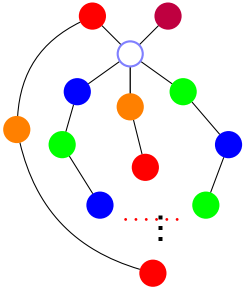

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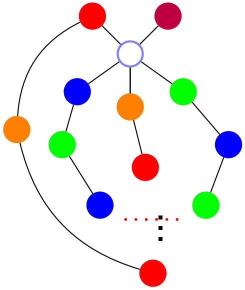

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Planar.

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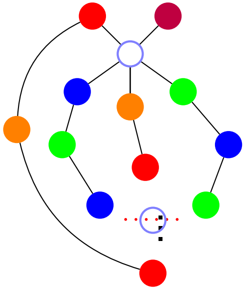

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Planar. $\Longrightarrow$ paths intersect at a vertex!

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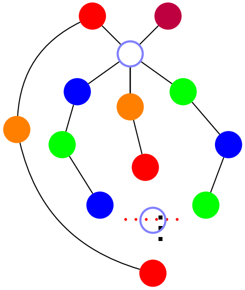

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What color is it?

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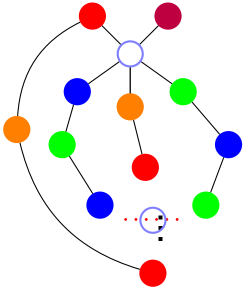

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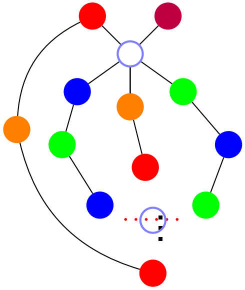

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Planar. $\implies$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.

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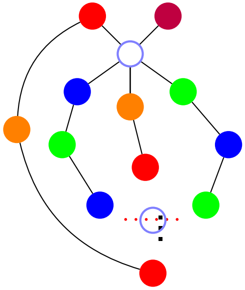

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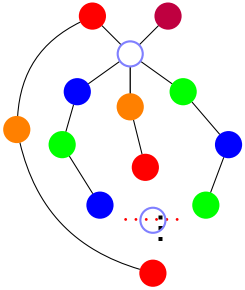

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What color is it?
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Contradiction.

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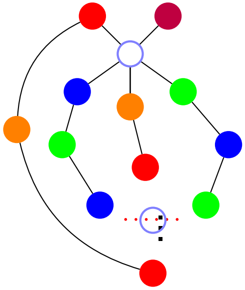

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Contradiction. Can recolor one of the neighbors.
And recolor “center” vertex.

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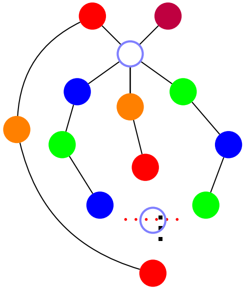
Proof:
Again with the degree 5 vertex. Again recurse.



Assume neighbors are colored all differently.
  Otherwise done.
Switch green to blue in component.
  Done. Unless blue-green path to blue.
Switch red to orange in its component.
  Done. Unless red-orange path to red.

Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
  Must be blue or green to be on that path.
  Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.
And recolor "center" vertex. □

Four Color Theorem

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Theorem: Any planar graph can be colored with four colors.

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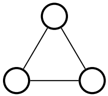

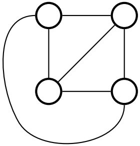

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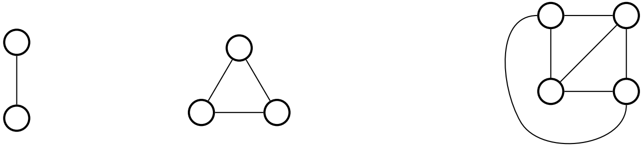

K_n, $|V| = n$

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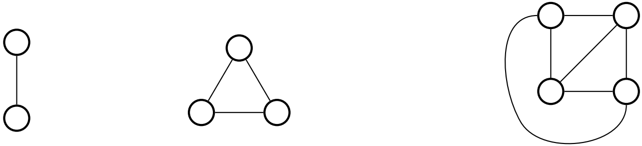

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every edge present.

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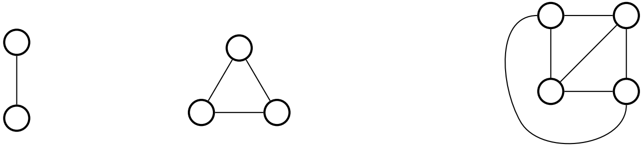

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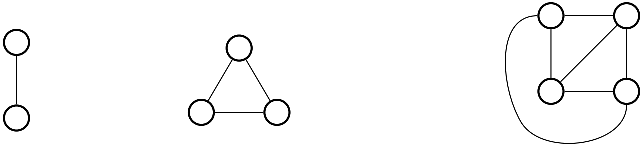

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degree of vertex? $|V| - 1$.

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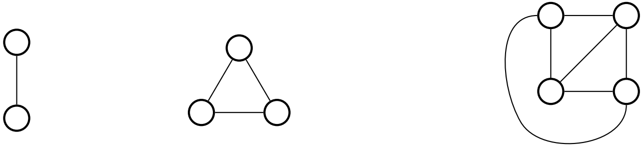

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Very connected.

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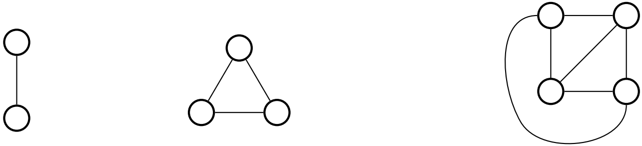

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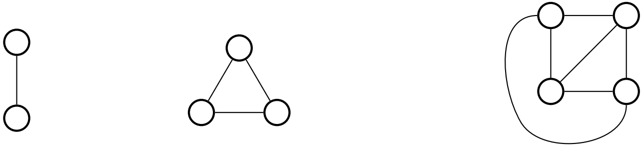

K_n, $|V| = n$

every edge present.
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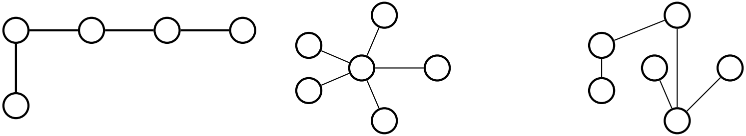
Very connected.
Lots of edges: $n(n-1)/2$.

Trees.



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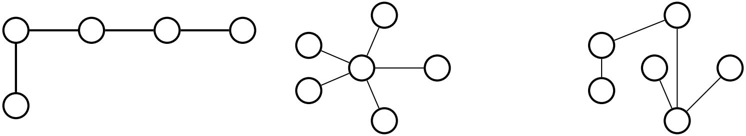

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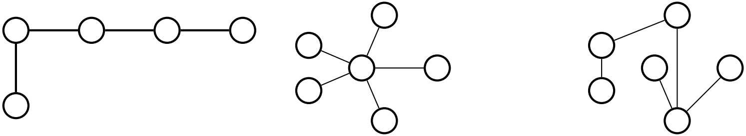

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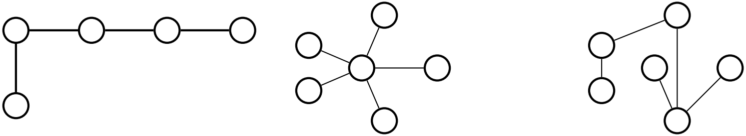

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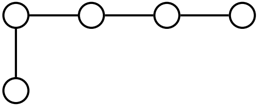

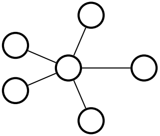

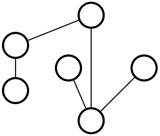

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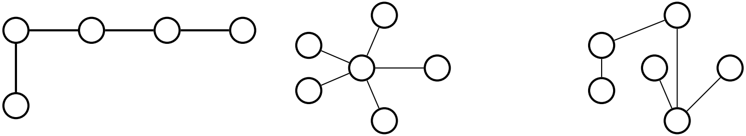

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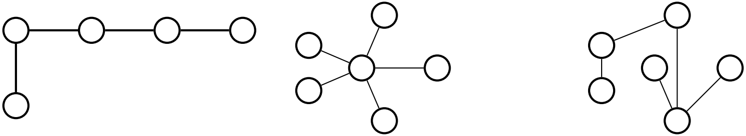

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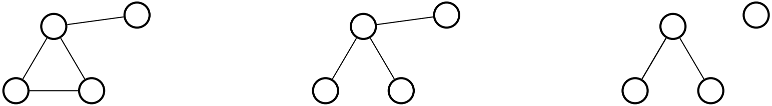
To tree or not to tree!



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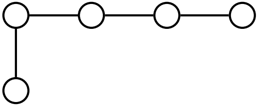
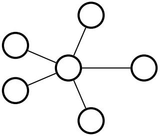
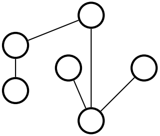

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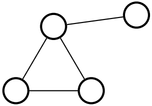
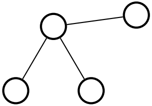
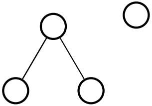

Minimally connected, minimum number of edges to connect.

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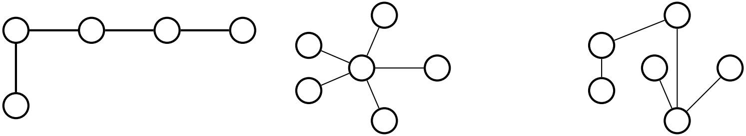

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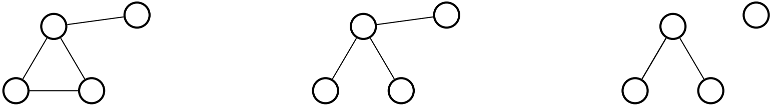

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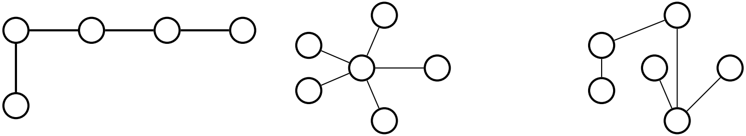

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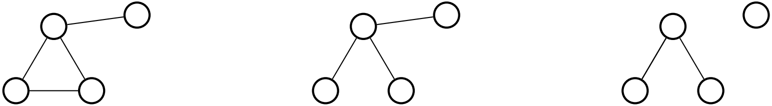

Minimally connected, minimum number of edges to connect.

Property:
Can remove a single node and break into components of size at most $|V|/2$.

Hypercube

Hypercubes.

Hypercube

Hypercubes. Really connected.

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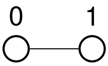

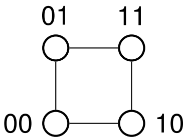

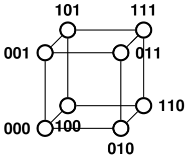

Recursive Definition.

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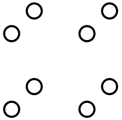

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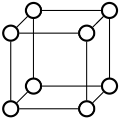

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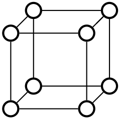

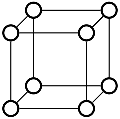

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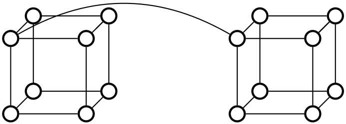

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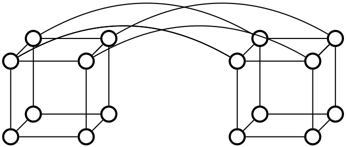

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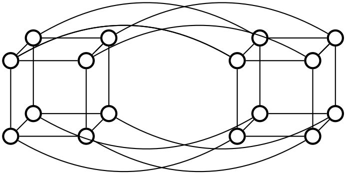

Hypercube:properties

Rudrata Cycle: cycle that visits every node.

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Good communication network!

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$d = e^{-1} = -17 = 43 = \pmod{60}$

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x has inverse modulo m if and only if $gcd(x,m) = 1$.

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Modular Arithmetic Inverses and GCD

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$D(E(m,K),k) = (m^e)^d \mod N = m.$

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$S(C) = D(C)$.
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Welsh-Berlekamp

Idea: Error locator polynomial of degree k with zeros at errors.

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For all points $i = 1, \ldots, i, n+2k$, $P(i)E(i) = R(i)E(i) \pmod{p}$
since $E(i) = 0$ at points where there are errors.
Let $Q(x) = P(x)E(x)$.

$Q(x) = a_{n+k-1}x^{n+k-1} + \cdots a_0$.
$E(x) = x^k + b_{k-1}x^{k-1} + \cdots b_0$.
Gives system of $n+2k$ linear equations.

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Counting

First Rule

Counting

First Rule
Second Rule

Counting

First Rule
Second Rule
Stars/Bars

Counting

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Second Rule
Stars/Bars
Common Scenarios: Sampling, Balls in Bins.

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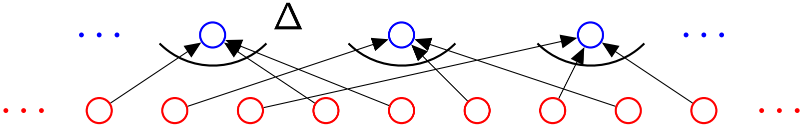
Example: visualize.

First rule: $n_1 \times n_2 \cdots \times n_3$**. Product Rule.**
Second rule: when order doesn't matter divide..when possible.



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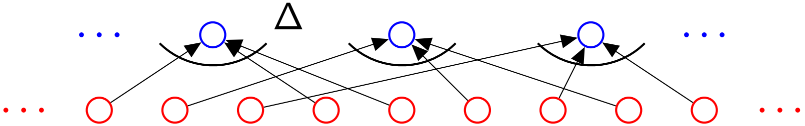

3 card Poker deals: 52

Example: visualize.

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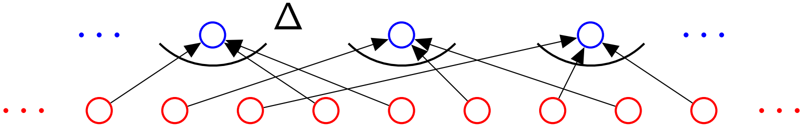

3 card Poker deals: 52×51

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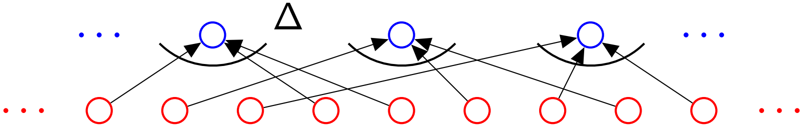

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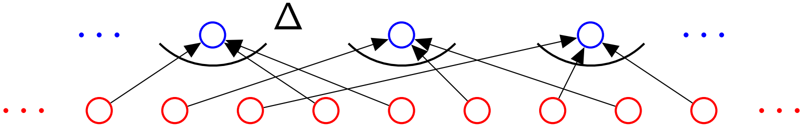

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$.

Example: visualize.

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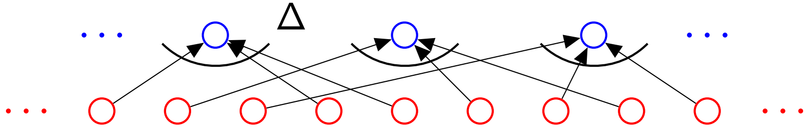

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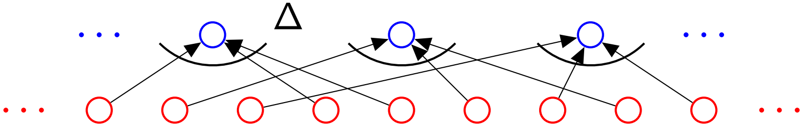

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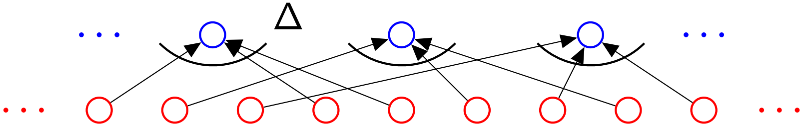

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Hand: Q, K, A.

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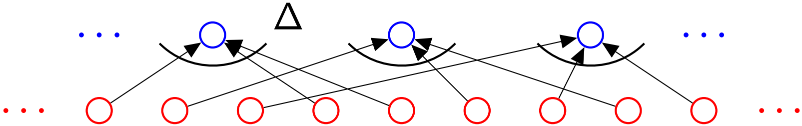

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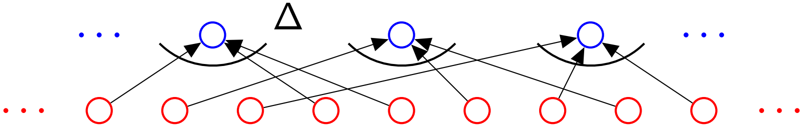

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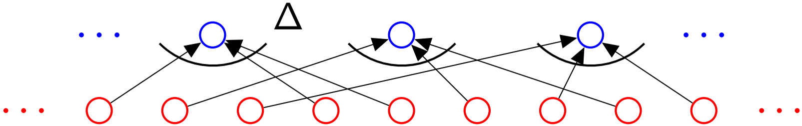

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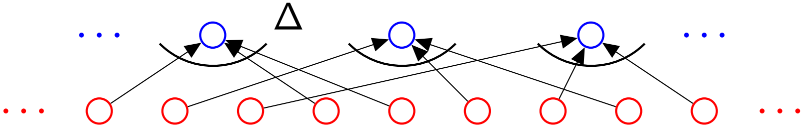

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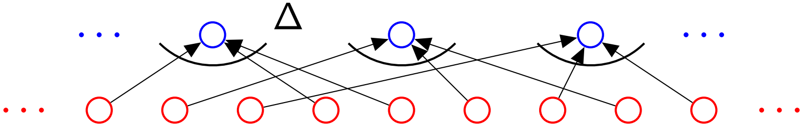

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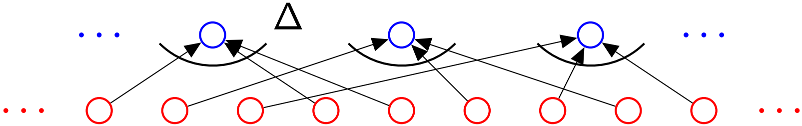

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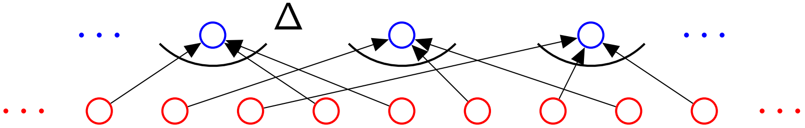

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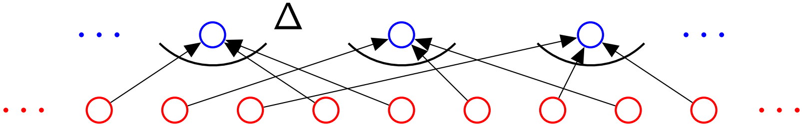

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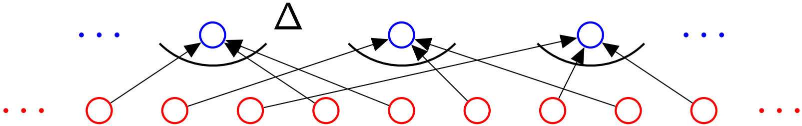

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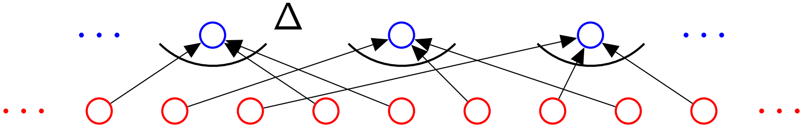

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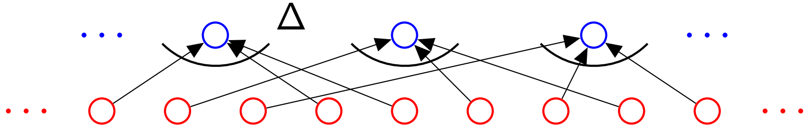

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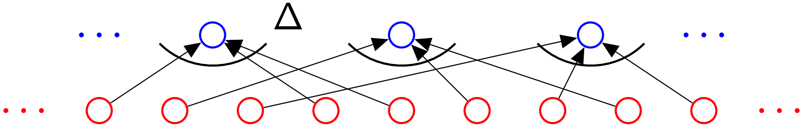

3 card Poker deals: $52 \times 51 \times 50 = \frac{52!}{49!}$. First rule.
Poker hands: Δ?

Hand: Q, K, A.

Deals: Q, K, A, Q, A, K, K, A, Q, K, A, Q, A, K, Q, A, Q, K.

$\Delta = 3 \times 2 \times 1$ First rule again.
Total: $\frac{52!}{49!3!}$ Second Rule!

Choose k out of n.

Ordered set: $\frac{n!}{(n-k)!}$

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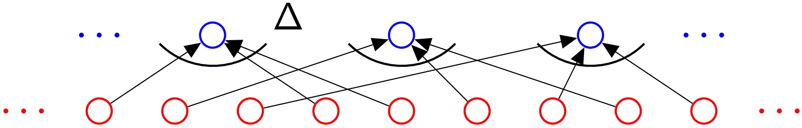

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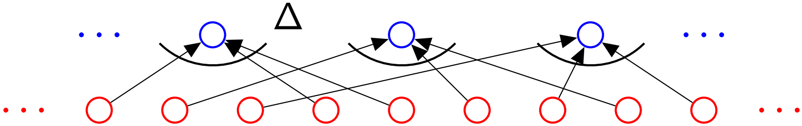

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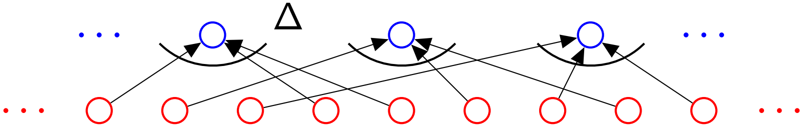

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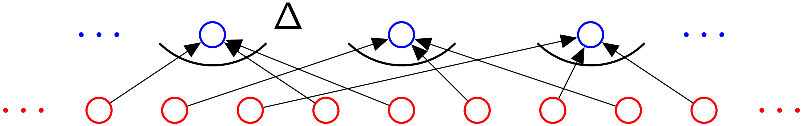

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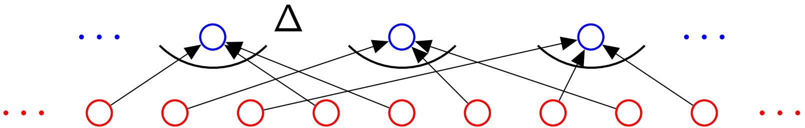

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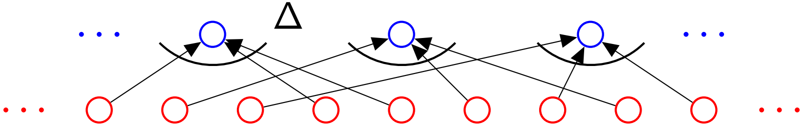

Orderings of ANAGRAM?

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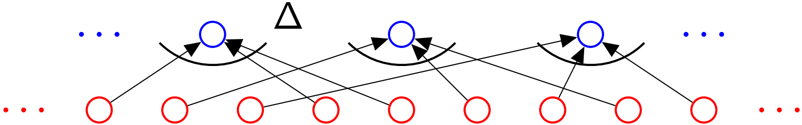

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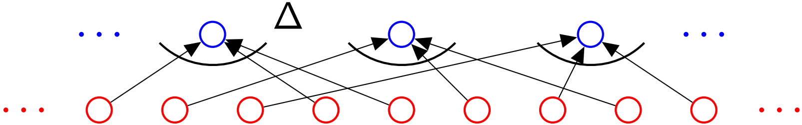

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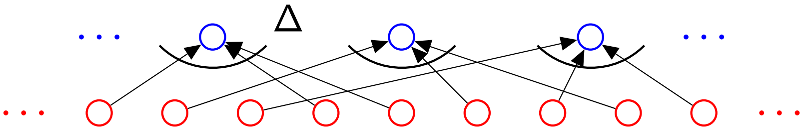

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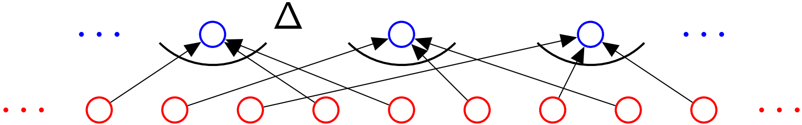

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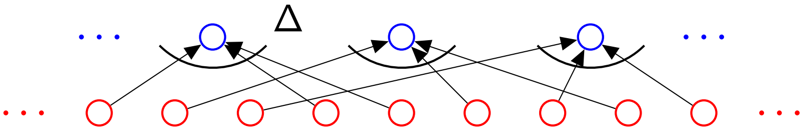

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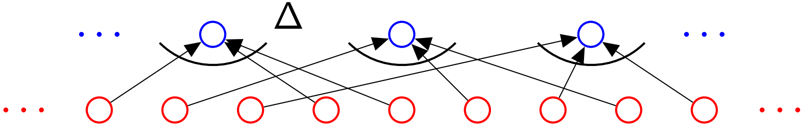

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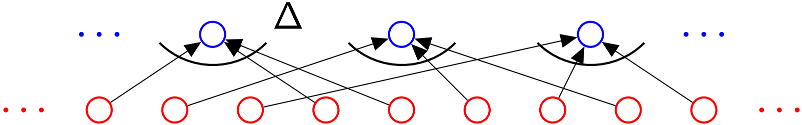

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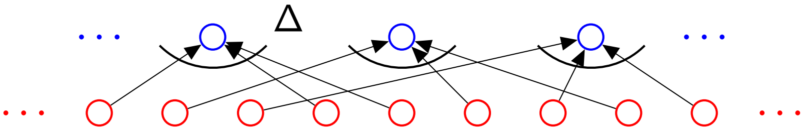

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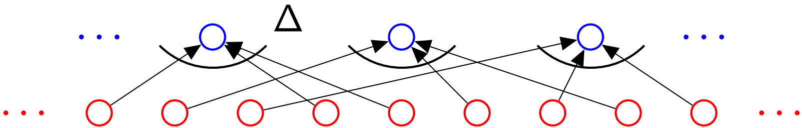

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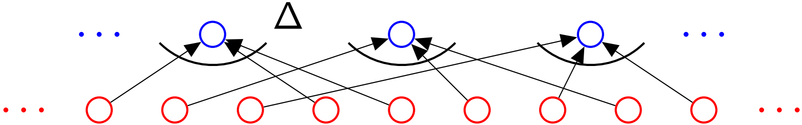

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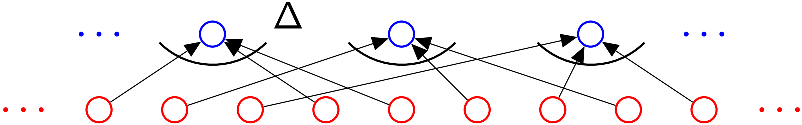

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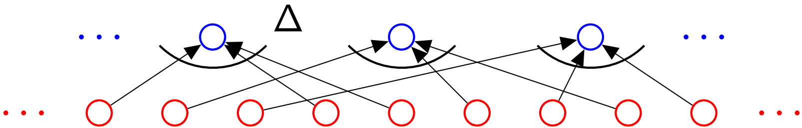

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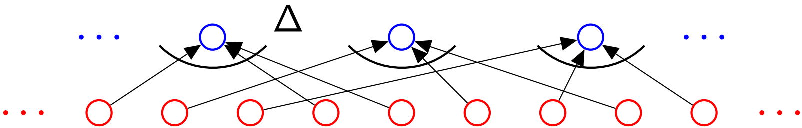

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5 indistinguishable balls into 52 bins only one ball in each bin
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Example: Poker hands.

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Example: Poker hands.

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Dividing 5 dollars among Alice, Bob and Eve.

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$S \cap T$ = phone numbers with 7 as first and second digit.

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$|S \cup T| = |S| + |T| - |S \cap T|$.

Example: How many 10-digit phone numbers have 7 as their first or second digit?

S = phone numbers with 7 as first digit.$|S| = 10^9$

T = phone numbers with 7 as second digit. $|T| = 10^9$.

$S \cap T$ = phone numbers with 7 as first and second digit. $|S \cap T| = 10^8$.

Simple Inclusion/Exclusion

Sum Rule: For disjoint sets S and T, $|S \cup T| = |S| + |T|$

Example: How many permutations of n items start with 1 or 2?
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Answer: $|S| + |T| - |S \cap T| = 10^9 + 10^9 - 10^8$.

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So, ${n \choose k-1} + {n \choose k} = {n+1 \choose k}$. □

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Isomporphism principle.

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Example.

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Cardinality of $[0, 1]$ smaller than all the reals?

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$[0,1]$ is same cardinality as nonnegative reals!

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All countably infinite sets are the same cardinality as each other.

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Countably infinite (same cardinality as naturals)

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Diagonalization: power set of Integers.

The set of all subsets of N.

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Any program is a fixed length string.

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P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

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Another view: diagonalization.

Any program is a fixed length string.
Fixed length strings are enumerable.
Program halts or not any input, which is a string.

	P_1	P_2	P_3	$\cdots$
P_1	H	H	L	$\cdots$
P_2	L	L	H	$\cdots$
P_3	L	H	H	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Halt - diagonal.
Turing - is not Halt.
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Halt does not exist!

Undecidable problems.

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Find exit points and add statement: **Print** “Hello World.”

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$\implies$ no program can take any set of integer equations

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Undecidability for Diophantine set of equations
$\Longrightarrow$ no program can take any set of integer equations
and always output correct answer.

Simulate

1) Any program with finite time/space.

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2) Can output all programs that halt on themselves?

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1) Run it and check!

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2) Like enumerating pairs of natural numbers.
– (program, time). and run program for that time.
Each program that halts, halts at some time.

Final format

Time: approximately 180 minutes

Final format

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Many short answers.

Final format

Time: approximately 180 minutes

Many short answers.
Get at ideas that we study.

Final format

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Many short answers.
Get at ideas that we study.
Know material well:

Final format

Time: approximately 180 minutes

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Get at ideas that we study.
Know material well: fast,

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Get at ideas that we study.
Know material well: fast, correct.

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Know material medium:

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Many short answers.
Get at ideas that we study.
Know material well: fast, correct.
Know material medium: slower,

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Wrapup.

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Watch Piazza for Logistics!

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If you sent me email about Final conflicts

Wrapup.

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If you sent me email about Final conflicts
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Wrapup.

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