CS70 - Lecture 6

Graphs: Coloring; Special Graphs

1. Review of L5
2. Planar Five Color Theorem
3. Special Graphs:
   - Trees: Three characterizations
   - Hypercubes: Strongly connected!

Administration

- You need to submit you grading option (HW or Test Only) by Thursday night at 10pm!

- Instruction is at Piazza @241

- If you don't submit your response on time, the default will be the homework option.

Review of L5

- Definitions: graph, walk, tour, path, cycle, Eulerian tour
- There exists a Eulerian Tour iff connected and even degrees.

  Only if: If v has odd degree, you get stuck there.

  This solves the Konigsberg problem.

  If: Induction on e (= number of edges).

- Euler Formula: Planar + Connected $\Rightarrow v + f = e + 2$.

  Proof: Induction on e.

- Planar $\Rightarrow 2e \geq 3f \Rightarrow 3v \geq e + 6 \Rightarrow K_5$ is non-planar
- Planar + Bipartite $\Rightarrow 2e \geq 4f \Rightarrow 2v \geq e + 4 \Rightarrow K_{3,3}$ is non-planar

Review of L5: Preliminaries

- ► Was Euler alive before or after Euclid?
- ► After! Euler = 1707- 1783. Euclid $\approx$ 300BC
- ► Was Euler alive before of after Newton?
- ► After! Newton = 1642 - 1726.
- ► What was Euler's first name?
- ► Leonhard.
- ► Was Euler a freak of nature?
- ► Definitely! $e^{i\pi} = -1$, γ, graphs, number theory, physics, astronomy, more than 800 papers, ...
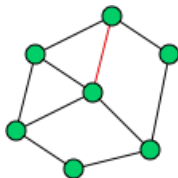
Review of L5: Q1

Question: What is this argument?

A   B   C
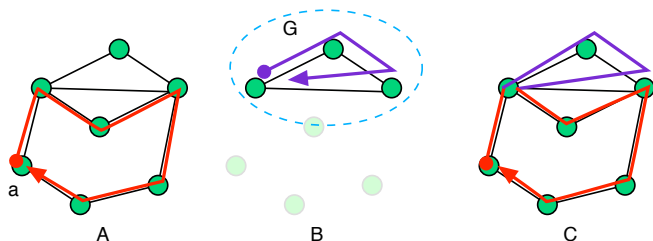
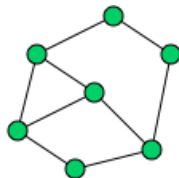Proof by induction on *e* of the existence of a Eulerian tour in a connected even-degrees graph.

There is one in *G*, so that there is one in the original graph.

Review of L5: Q2

What is this argument?

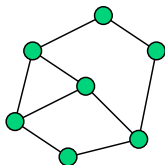| e edges | e' = e − 1 |
| f faces | f' = f − 1 |
| v vertices | v' = v |

A proof by induction on *e* of Euler's formula: $v + f = e + 2$.
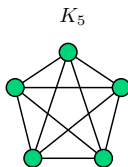
Review of L5: Q3

What is this argument?

K_5

$2e = |E/F \text{ adjacencies }| \geq 3f$ (*)

$3v + 3f = 3e + 6$ (**)

$3v + 2e \geq 3e + 6$

$3v \geq e + 6$
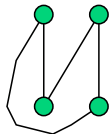
$v = 5, e = 10$

A proof that K_5 is non-planar.

Where does (*) come from? Every cycle has at least 3 edges.

Where does (**) come from? Euler's formula.

Let's remember: Planar $\Rightarrow e \leq 3v - 6$

Review of L5: Q3

What is this argument?

$K_{3,3}$
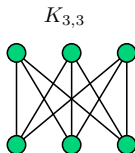
Bipartite
$2e = |E/F \text{ adjacencies}| \geq 4f$ (*)
$e \geq 2f$
$2v + 2f = 2e + 4$ (**)
$2v + e \geq 2e + 4$
$2v \geq e + 4$
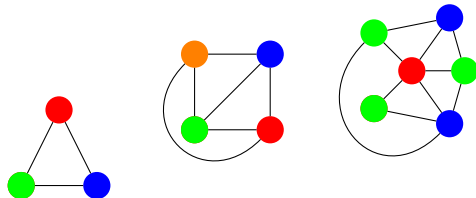
$e = 9, v = 6$

A proof that $K_{3,3}$ is non-planar.

Where does (*) come from? Every cycle has at least 4 edges.

Where does (**) come from? Euler's formula.

Graph Coloring.

Given $G = (V, E)$, a coloring of a G assigns colors to vertices V where for each edge the endpoints have different colors.

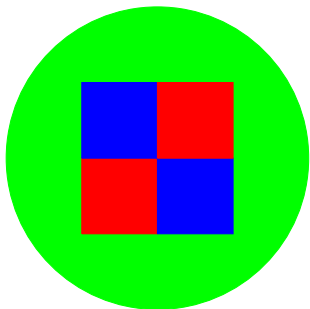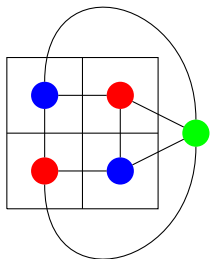Notice that the last graph has one three-color coloring.
$\rightarrow$ Fewer colors than number of vertices.
$\rightarrow$ Fewer colors than the maximum degree of the nodes.

Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.

Four color theorem is about planar graphs!

It says that every planar graph (or map) can be colored with four colors!

Stated in 1852. Proved in 1976 ... by reducing the problem to 1936 cases (400 pages of analysis) and checking these cases by computer!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

Proof:
Induction on v.

Recall: $e \leq 3v - 6$ for any planar graph.

Total degree (sum of the degrees): $2e$
Average degree: $\frac{2e}{v} \leq \frac{2(3v-6)}{v} \leq 6 - \frac{12}{v} < 6$.

$\Rightarrow$ There exists a vertex x with degree < 6 or at most 5.

Remove vertex x of degree at most 5.
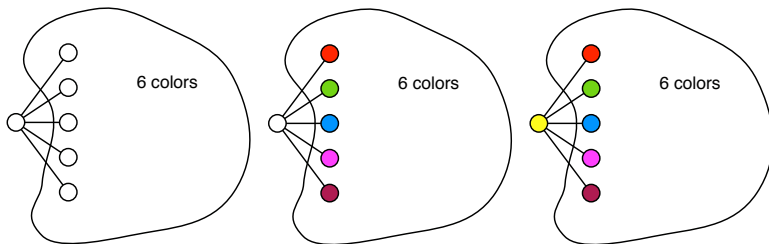Inductively color remaining graph with the six colors.
One of the six colors is available for x since only five neighbors...

$\square$

Six color theorem.

Theorem: Every planar graph can be colored with six colors.
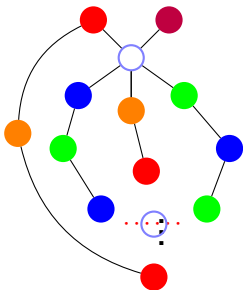
A picture of the proof by induction:

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Proof:
Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.

Assume neighbors are colored all differently. Otherwise done.
Switch green to blue in component.
    Done. Unless blue-green path to blue.
Switch red to orange in its component.
    Done. Unless red-orange path to red.

Planar. $\implies$ paths intersect at a vertex!

What color is it?
    Must be blue or green to be on that path.
    Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.
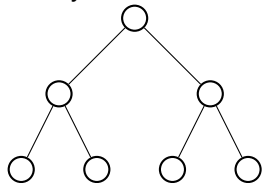And recolor "center" vertex.

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof: Not Today!

A Tree, a tree.

Graph $G = (V, E)$.
Binary Tree!
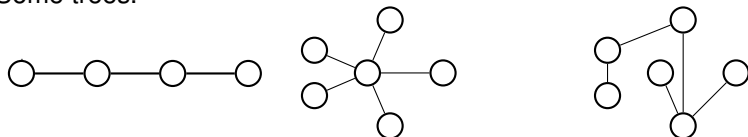

More generally.

Trees.

Definitions: (Equivalent, as we prove later)

A connected graph without a cycle.
A connected graph with $|V| - 1$ edges.
A connected graph where any edge removal disconnects it.
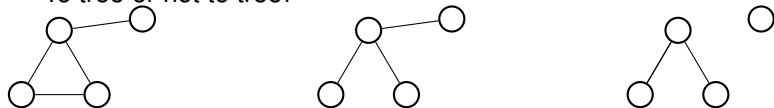
Some trees.

No cycle and connected? Yes.
$|V| - 1$ edges and connected? Yes.
Removing any edge disconnects it. Harder to check, but yes.
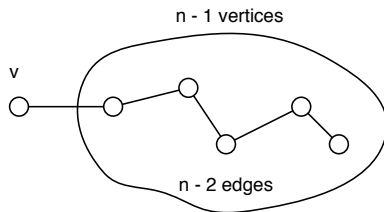
To tree or not to tree!

Equivalence of Definitions.

Theorem These properties of a graph are equivalent: (i.e., each implies any other)

(1) A connected graph without a cycle.

(2) A connected graph with $|V| - 1$ edges.

(3) A connected graph where any edge removal disconnects it.

Proof: $(1) \Rightarrow (2)$

Assume true for $|V| \leq n - 1$. Consider G with $|V| = n$.

Why is there a unpopular v with degree only 1? Otherwise: cycle (enter-leave-enter-leave-enter). $\Rightarrow G$ has $n - 1$ edges. $\qquad \square$
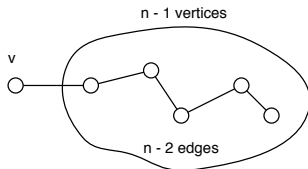
Equivalence of Definitions.

Theorem These properties of a graph are equivalent: (i.e., each implies any other)

(1) A connected graph without a cycle.

(2) A connected graph with $|V| - 1$ edges.

(3) A connected graph where any edge removal disconnects it.

Proof: $(2) \Rightarrow (1)$

Assume true for $|V| \leq n - 1$. Consider G connected with $|V| = n$ vertices and $n - 1$ edges.

$\Rightarrow$ There must be some v with degree 1. Otherwise, sum of degrees $\geq 2n$. But sum of degrees $= 2|E| = 2(n - 1)$.

Remove v. Get connected graph G' without a cycle. Same for G. $\quad \square$
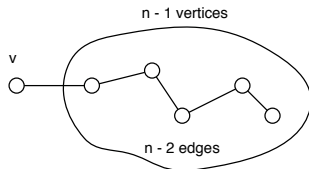
Equivalence of Definitions.

Theorem These properties of a graph are equivalent: (i.e., each implies any other)

  (1) A connected graph without a cycle.
  (2) A connected graph with $|V| - 1$ edges.
  (3) A connected graph where any edge removal disconnects it.

Proof: $(1) \Rightarrow (3)$

Assume true for $|V| \leq n - 1$. Consider G with $|V| = n$ and (1).

There is some v with degree 1. (Otherwise, there is a cycle.)

If you remove the edge of v, you disconnect G. If you remove any other edge, you disconnect G', by induction hypothesis. $\square$
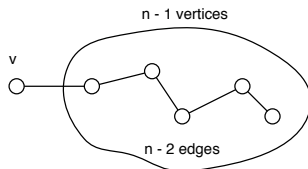
Equivalence of Definitions.

Theorem These properties of a graph are equivalent: (i.e., each implies any other)
  (1) A connected graph without a cycle.
  (2) A connected graph with $|V| - 1$ edges.
  (3) A connected graph where any edge removal disconnects it.

Proof: $(3) \Rightarrow (1)$

Assume true for $|V| \le n - 1$. Consider G with $|V| = n$ and (3).

There is some v with degree 1. Otherwise, removing an edge would not disconnect G.

G' has no cycle, by induction hypothesis. So G has no cycle. $\qquad\square$

Hypercubes.

Complete graphs, really connected! But lots of edges.
$|V|(|V|-1)/2$

Trees, Cool, but very few edges. $(|V|-1)$
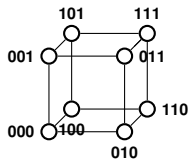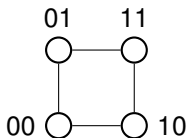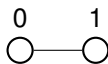They just falls apart! Watch for Monterey Pines!

Hypercubes. Really connected. $|V|\log|V|$ edges!
Also represents bit-strings nicely.

$G = (V, E)$
$|V| = \{0,1\}^n,$
$|E| = \{(x,y)|x \text{ and } y \text{ differ in one bit position.}\}$

2^n vertices. number of n-bit strings!
$n2^{n-1}$ edges.
  2^n vertices each of degree n
    total degree is $n2^n$ and half as many edges!
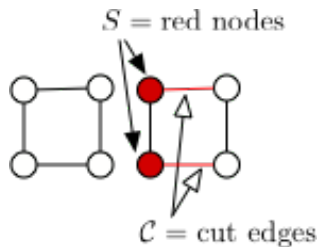
Recursive Definition.

A 0-dimensional hypercube is a node labelled with the empty string of bits.

An *n*-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$.
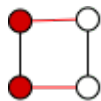
Cuts

$S = $ red nodes
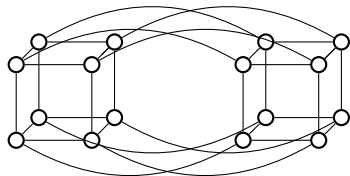
$\mathcal{C} = $ cut edges

Take a connected graph $G = (V, E)$ and some set $S \subset V$.

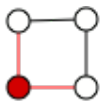The cut $\mathcal{C}$ is the set of edges that attach S to $V - S$.

Hypercube: Can't cut me!

Thm: In a hypercube, $|\mathscr{C}| \geq \min\{|S|, |V - S|\}$.

Examples:
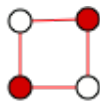

$|S| = 2$
$|\mathcal{C}| = 2$

$|S| = 1$
$|\mathcal{C}| = 2$

$|S| = 3$
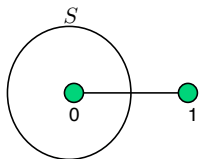$|\mathcal{C}| = 2$

$|S| = 2$
$|\mathcal{C}| = 4$

Proof of Large Cuts.

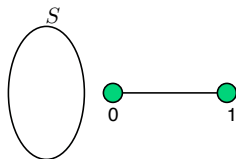**Thm:** In a hypercube, $|\mathscr{C}| \geq \min\{|S|, |V - S|\}$.

Proof:

Induction on n. Recall, $V = \{0, 1\}^n, \ldots$.

$\rightarrow$ Base Case: $n = 1$, $V = \{0, 1\}$.

$S = \{0\}, |\text{ cut edges }| = 1$ $S = \emptyset, |\text{ cut edges }| = 0$

Induction Step Idea
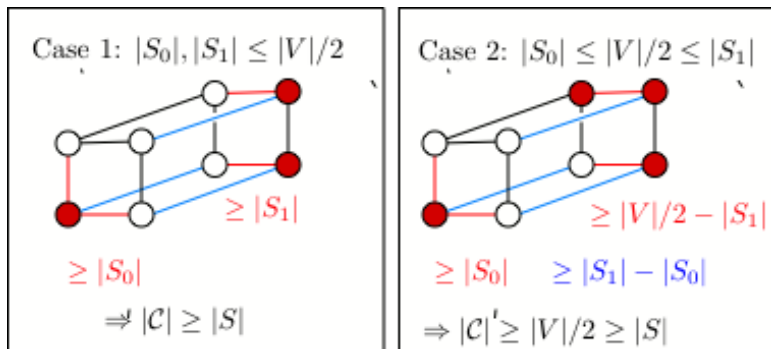
Thm: In a hypercube, $|\mathscr{C}| \geq \min\{|S|, |V - S|\}$.

Proof:

Induction on n. Recall, $V = \{0,1\}^n, \ldots$.

$\rightarrow$ Base Case: $n = 1$, $V = \{0,1\}$.

$\rightarrow$ Induction step: Assume (without loss of generality) that $|S| \leq |V|/2$.

Case 1: $|S_0|, |S_1| \leq |V|/2$

$\geq |S_1|$

$\geq |S_0|$

$\Rightarrow |\mathcal{C}| \geq |S|$

Case 2: $|S_0| \leq |V|/2 \leq |S_1|$

$\geq |V|/2 - |S_1|$

$\geq |S_0|$ $\geq |S_1| - |S_0|$

$\Rightarrow |\mathcal{C}| \geq |V|/2 \geq |S|$

Here S_0 be the part of S in left cube, S_1 in right-cube. Red edges are cut in each half-cube and blue edges across.

Hypercubes and Boolean Functions.

The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0,1\}^n$.

Central area of study in computer science!

Yes/No Computer Programs $\equiv$ Boolean function on $\{0,1\}^n$

Central object of study.

Summary of L6

1. Review of L5

    Eulerian Tour iff connected and even degrees

    Euler Formula, K_5 and $K_{3,3}$ are non-planar

2. Planar Five Color Theorem

3. Special Graphs:

    ▸ Trees: Three characterizations
    ▸ Hypercubes: Strongly connected!
        Any cut $(S, V - S)$ has at least $\min\{|S|, |V - S|\}$ edges

4. The power of induction!

Have a nice weekend!