CS70 - Lecture 6

Graphs: Coloring; Special Graphs

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1. Review of L5

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1. Review of L5
2. Planar Five Color Theorem

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Graphs: Coloring; Special Graphs

1. Review of L5
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3. Special Graphs:

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1. Review of L5
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1. Review of L5
2. Planar Five Color Theorem
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   - Trees: Three characterizations
   - Hypercubes:

CS70 - Lecture 6

Graphs: Coloring; Special Graphs

1. Review of L5
2. Planar Five Color Theorem
3. Special Graphs:
    - Trees: Three characterizations
    - Hypercubes: Strongly connected!

Administration

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- If you don't submit your response on time, the default will be the homework option.

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- Planar + Bipartite $\Rightarrow 2e \geq 4f \Rightarrow 2v \geq e + 4 \Rightarrow K_{3,3}$ is non-planar

Review of L5: Preliminaries

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- Definitely! $e^{i\pi} = -1$, γ, graphs, number theory, physics, astronomy, more than 800 papers, ...

Review of L5: Q1

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Question: What is this argument?

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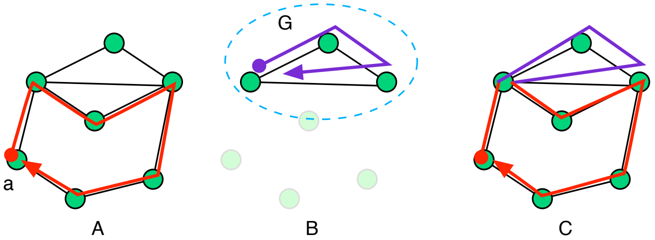

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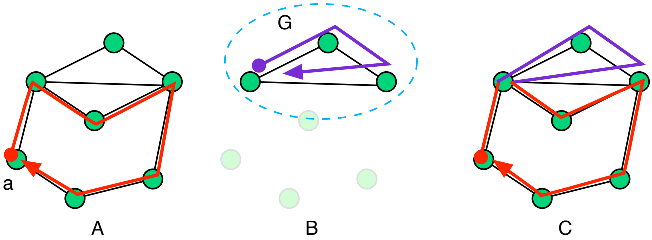

Proof by induction on *e* of the existence of a Eulerian tour in a connected even-degrees graph.

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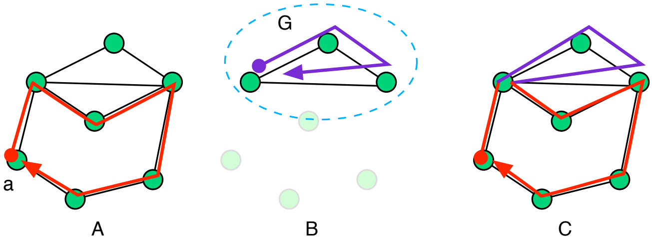

Proof by induction on *e* of the existence of a Eulerian tour in a connected even-degrees graph.

There is one in *G*, so that there is one in the original graph.

Review of L5: Q2

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What is this argument?

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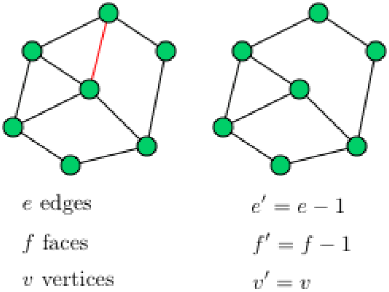

Review of L5: Q2

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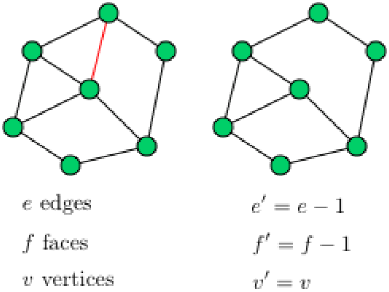

A proof by induction on e of Euler's formula: $v + f = e + 2$.

Review of L5: Q3

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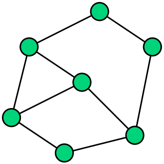

$2e = |E/F \text{ adjacencies }| \geq 3f$ (*)

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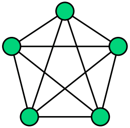
K_5



$v = 5, e = 10$

Review of L5: Q3

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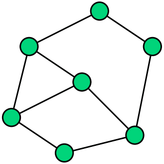

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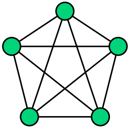

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A proof that K_5 is non-planar.

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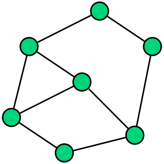

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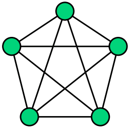

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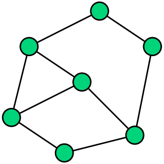

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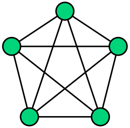

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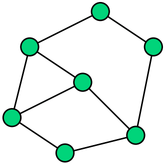

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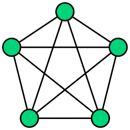

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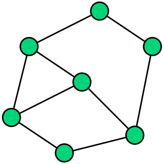

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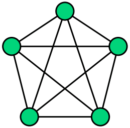

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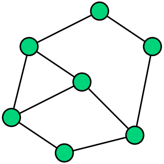

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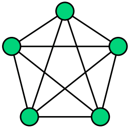

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Let's remember: Planar $\Rightarrow e \leq 3v - 6$

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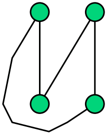

Bipartite

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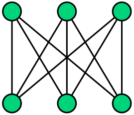

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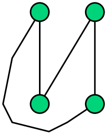

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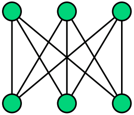

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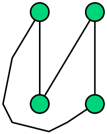

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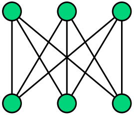

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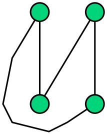

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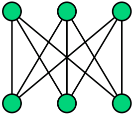

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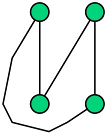

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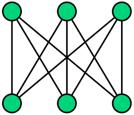

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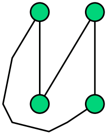

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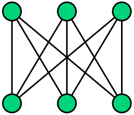

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Where does (*) come from? Every cycle has at least 4 edges.

Where does (**) come from? Euler's formula.

Graph Coloring.

Given $G = (V, E)$, a coloring of a G assigns colors to vertices V where for each edge the endpoints have different colors.

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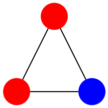

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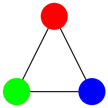

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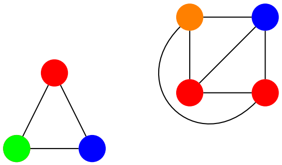

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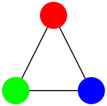

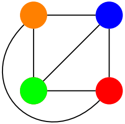

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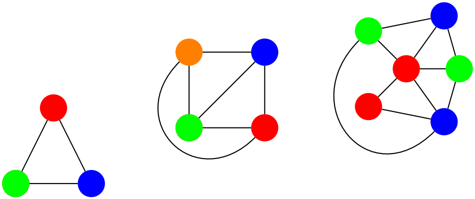

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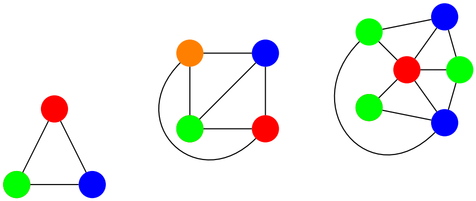

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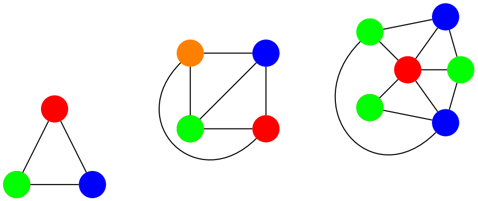

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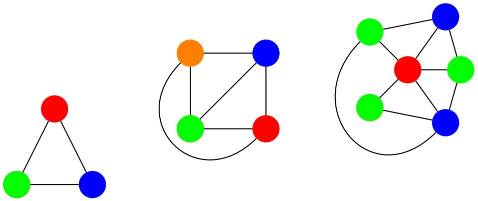

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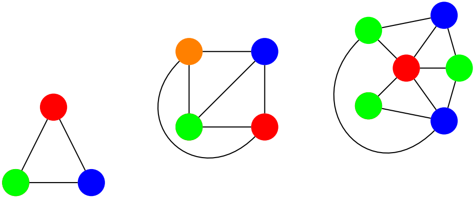

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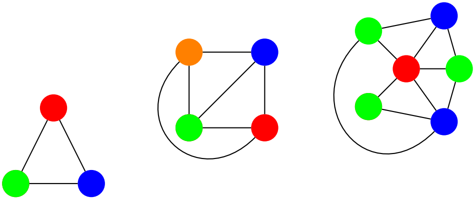

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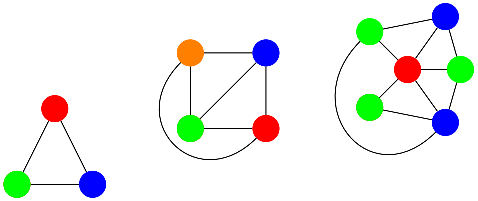

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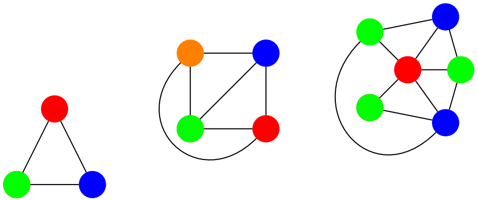

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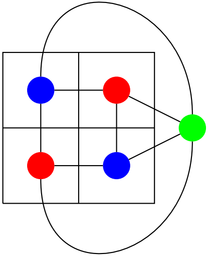

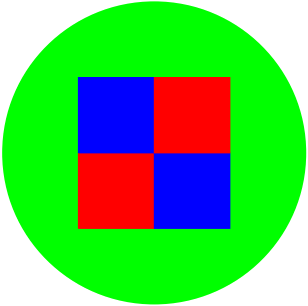

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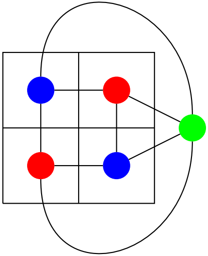

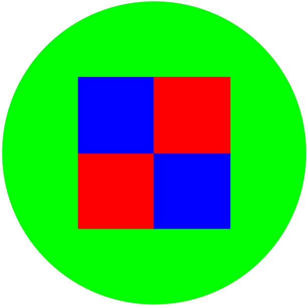

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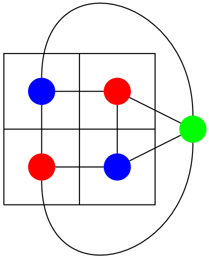

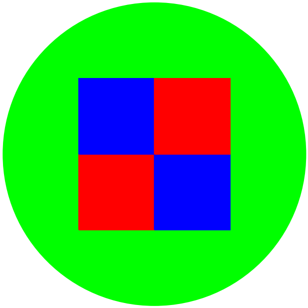

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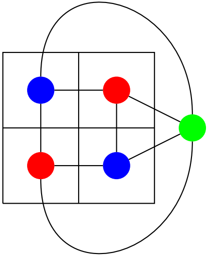

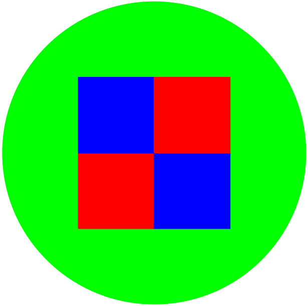

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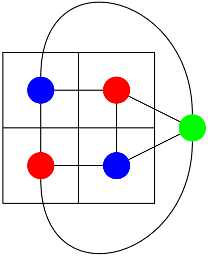

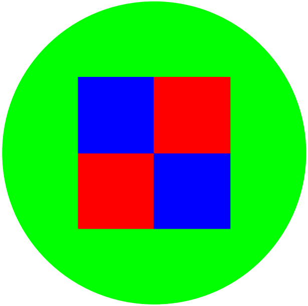

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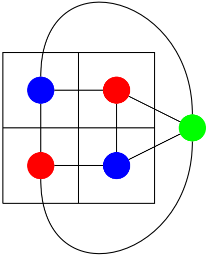

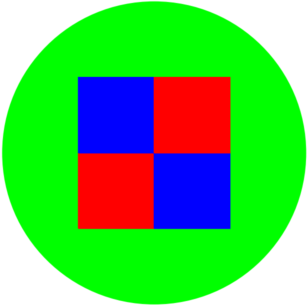

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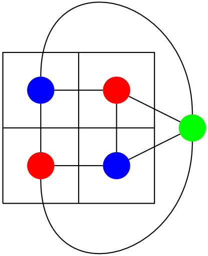

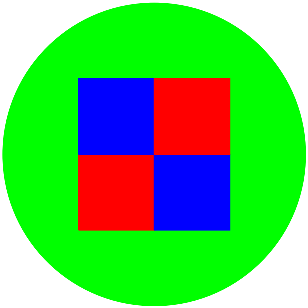

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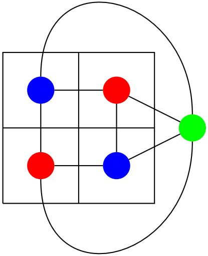

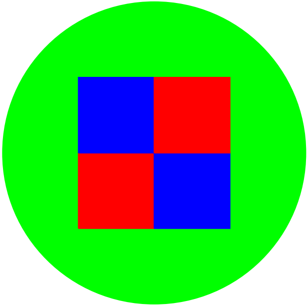

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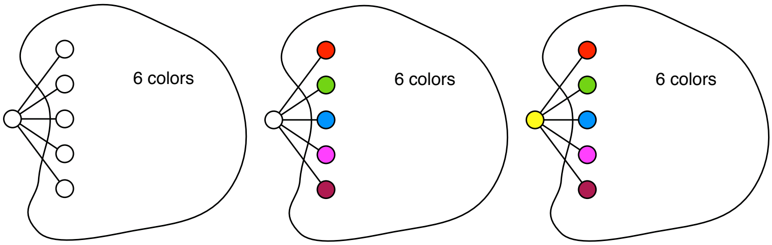

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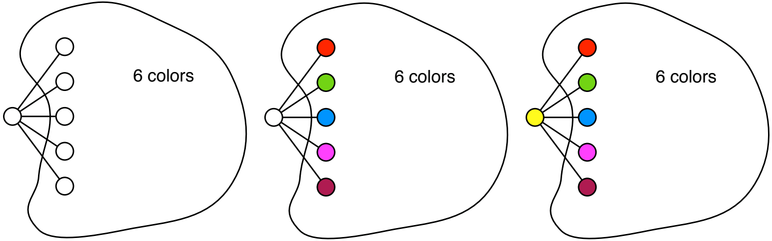

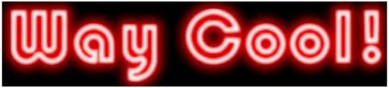

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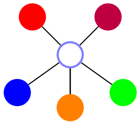

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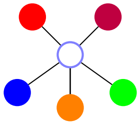

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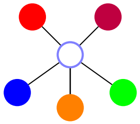

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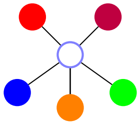

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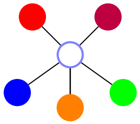

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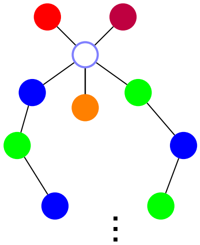

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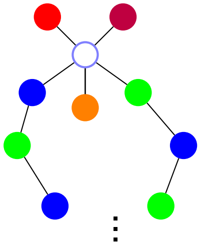

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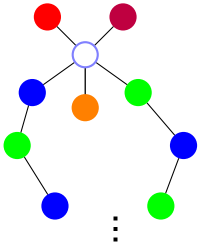

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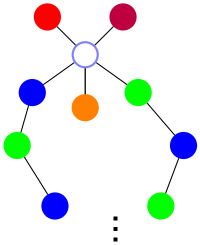

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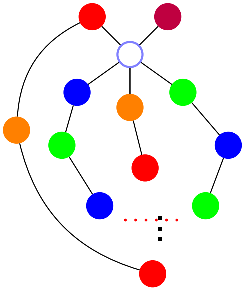

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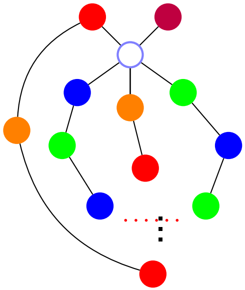
Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.



Assume neighbors are colored all differently.
Otherwise done.
Switch green to blue in component.
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Planar.

Five color theorem

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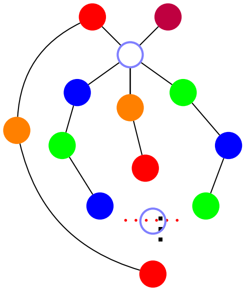

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Planar. $\implies$ paths intersect at a vertex!

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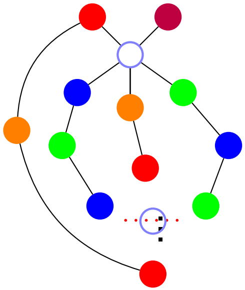

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What color is it?

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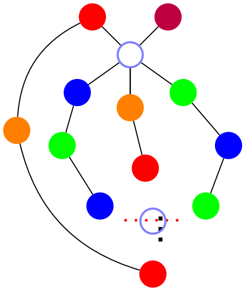

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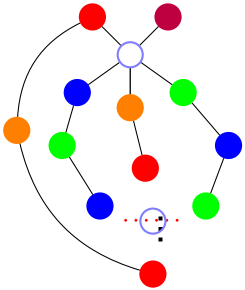

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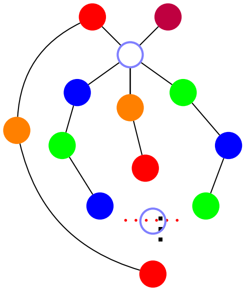

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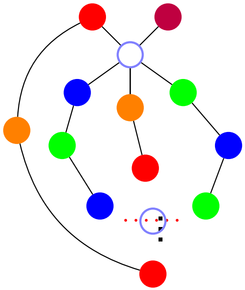

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What color is it?
Must be blue or green to be on that path.
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Contradiction.

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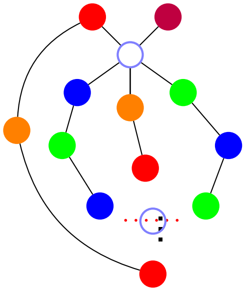

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 Must be blue or green to be on that path.
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Contradiction. Can recolor one of the neighbors.
And recolor "center" vertex.

Five color theorem

Theorem: Every planar graph can be colored with five colors.

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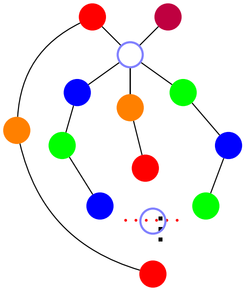
Preliminary Observation: You can switch two colors in a legal coloring. Obvious!

Consider again the degree 5 vertex. Again recurse: Assume five colors.



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What color is it?
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Contradiction. Can recolor one of the neighbors.
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Four Color Theorem

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof:

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof: Not Today!

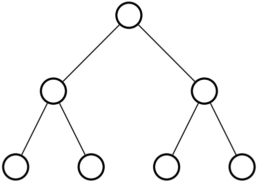
Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof: Not Today!

A Tree, a tree.

Graph $G = (V, E)$.
Binary Tree!



More generally.

Trees.

Definitions:

Trees.

Definitions: (Equivalent, as we prove later)

Trees.

Definitions: (Equivalent, as we prove later)

A connected graph without a cycle.

Trees.

Definitions: (Equivalent, as we prove later)

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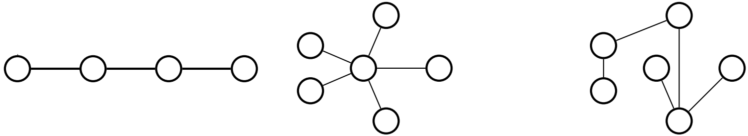
Some trees.



No cycle and connected?

Trees.

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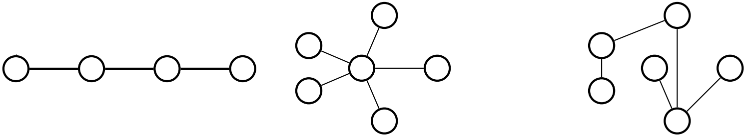

No cycle and connected? Yes.

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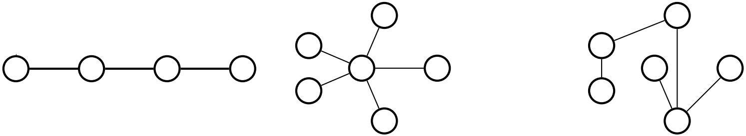

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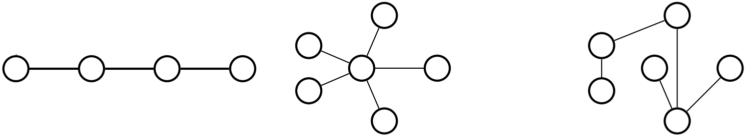

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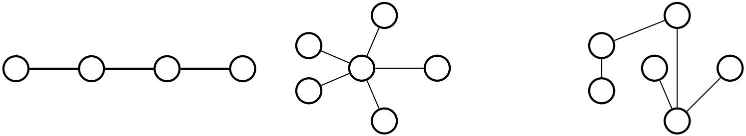

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Removing any edge disconnects it.

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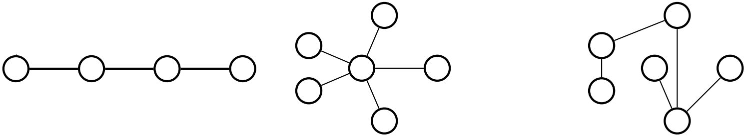

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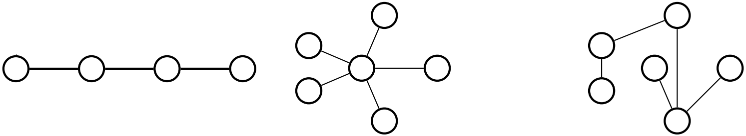

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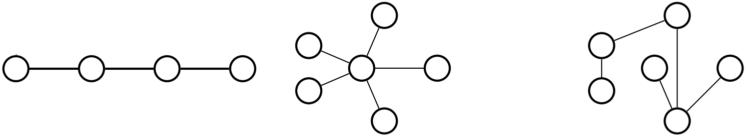

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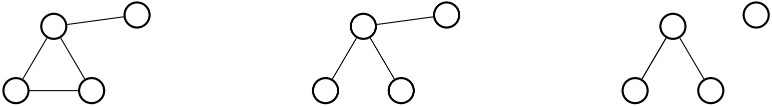
To tree or not to tree!



Equivalence of Definitions.

Theorem These properties of a graph are equivalent:

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Theorem These properties of a graph are equivalent: (i.e., each implies any other)

(1) A connected graph without a cycle.
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Proof: (1) $\Rightarrow$ (2)

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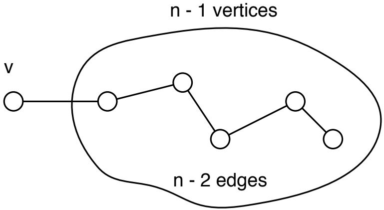

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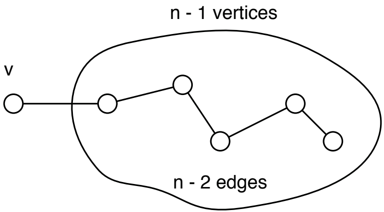

Why is there a unpopular v with degree only 1?

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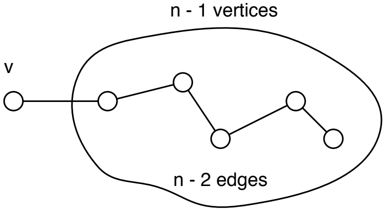

Why is there a unpopular v with degree only 1? Otherwise: cycle

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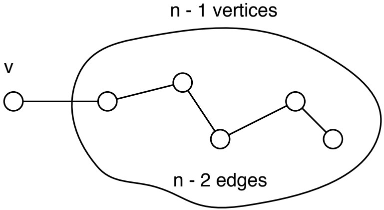

Why is there a unpopular v with degree only 1? Otherwise: cycle (enter-leave-enter-leave-enter).

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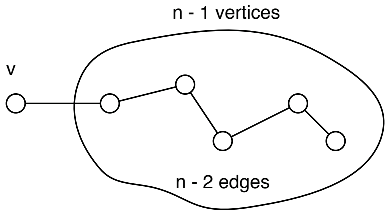

Why is there a unpopular v with degree only 1? Otherwise: cycle (enter-leave-enter-leave-enter). $\Rightarrow$ G has $n-1$ edges. $\square$

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Assume true for $|V| \leq n-1$. Consider G connected with $|V| = n$ vertices and $n-1$ edges.

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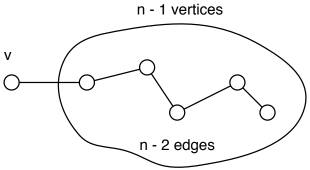

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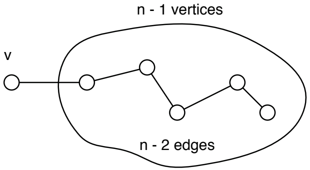

$\Rightarrow$ There must be some v with degree 1.

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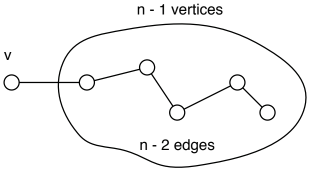

$\Rightarrow$ There must be some v with degree 1. Otherwise, sum of degrees $\geq 2n$.

Equivalence of Definitions.

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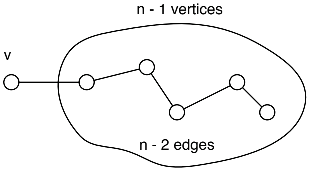

$\Rightarrow$ There must be some v with degree 1. Otherwise, sum of degrees $\geq 2n$. But sum of degrees $= 2|E| = 2(n-1)$.

Equivalence of Definitions.

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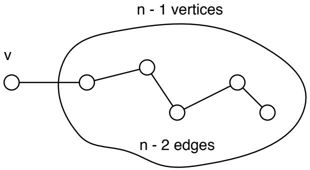

$\Rightarrow$ There must be some v with degree 1. Otherwise, sum of degrees $\geq 2n$. But sum of degrees $= 2|E| = 2(n-1)$.

Remove v.

Equivalence of Definitions.

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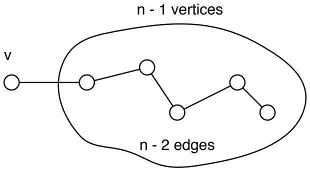

$\Rightarrow$ There must be some v with degree 1. Otherwise, sum of degrees $\geq 2n$. But sum of degrees $= 2|E| = 2(n-1)$.

Remove v. Get connected graph G' without a cycle.

Equivalence of Definitions.

Theorem These properties of a graph are equivalent: (i.e., each implies any other)

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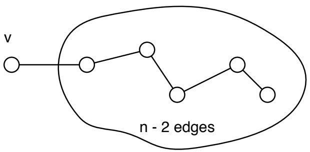
(2) A connected graph with $|V| - 1$ edges.

(3) A connected graph where any edge removal disconnects it.

Proof: (2) $\Rightarrow$ (1)

Assume true for $|V| \leq n-1$. Consider G connected with $|V| = n$ vertices and $n-1$ edges.



$\Rightarrow$ There must be some v with degree 1. Otherwise, sum of degrees $\geq 2n$. But sum of degrees $= 2|E| = 2(n-1)$.

Remove v. Get connected graph G' without a cycle. Same for G. $\square$

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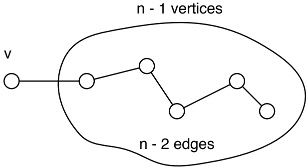

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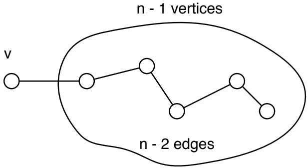

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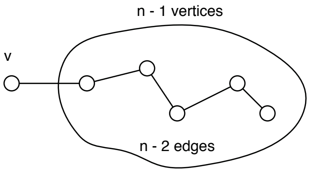

If you remove the edge of v, you disconnect G. If you remove any other edge, you disconnect G', by induction hypothesis. $\square$

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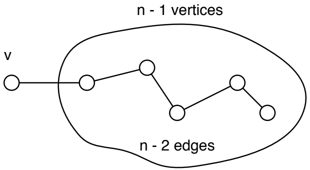

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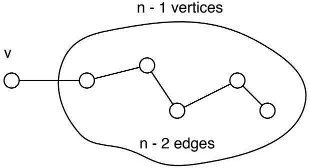

G' has no cycle, by induction hypothesis. So G has no cycle. $\square$

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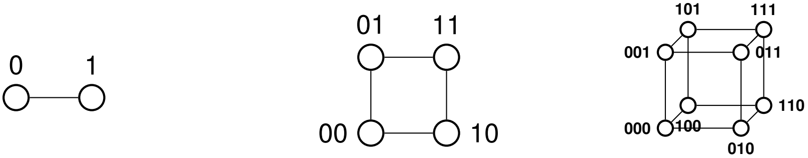

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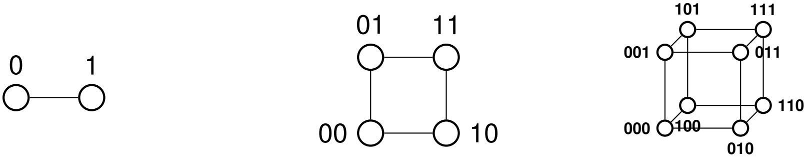

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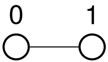

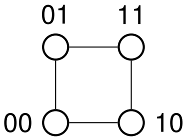

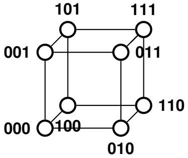

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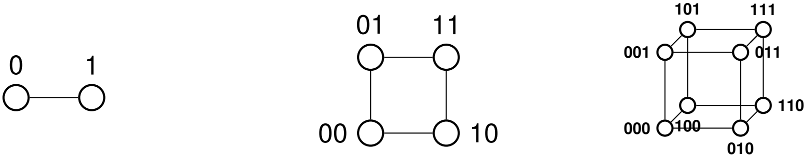

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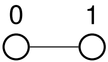

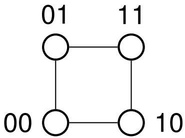

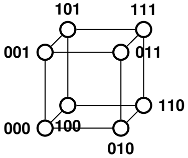

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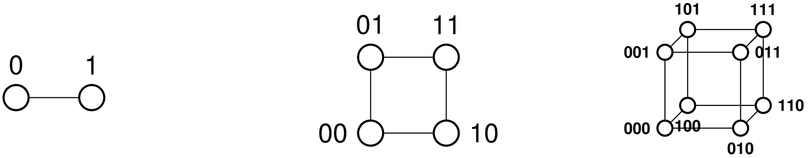

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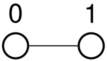

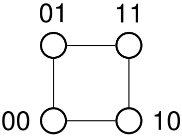

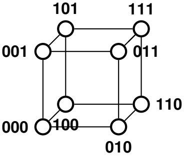

2^n vertices. number of n-bit strings!

$n2^{n-1}$ edges.

  2^n vertices each of degree n

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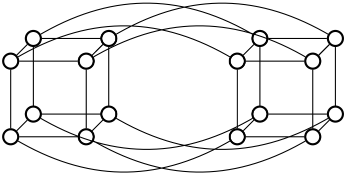

Cuts

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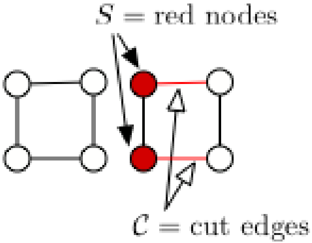

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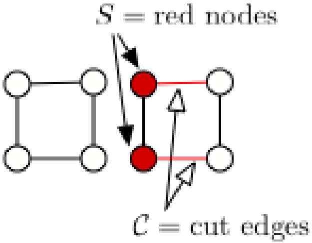

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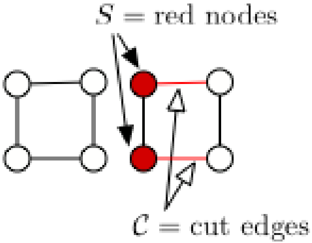

Take a connected graph $G = (V, E)$ and some set $S \subset V$.

The cut $\mathscr{C}$ is the set of edges that attach S to $V - S$.

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Thm: In a hypercube, $|\mathscr{C}| \geq \min\{|S|, |V - S|\}$.

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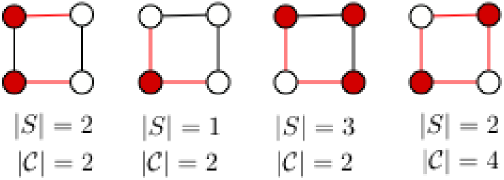
Examples:



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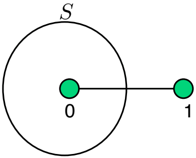
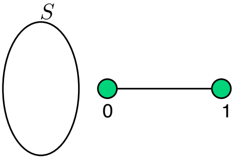


$S = \{0\}, |\text{ cut edges }| = 1$



$S = \emptyset, |\text{ cut edges }| = 0$

Induction Step Idea

Thm: In a hypercube, $|\mathscr{C}| \geq \min\{|S|, |V-S|\}$.

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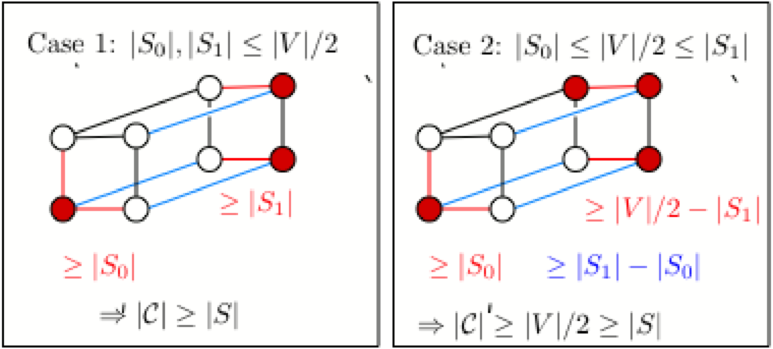

Here S_0 be the part of S in left cube, S_1 in right-cube. Red edges are cut in each half-cube and blue edges across.

Hypercubes and Boolean Functions.

The cuts in the hypercubes are exactly the transitions from 0 sets to 1 set on boolean functions on $\{0,1\}^n$.

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4. The power of induction!

Have a nice weekend!