CS70: Lecture 8. Outline.

1. Finish Up Extended Euclid.
2. Cryptography
3. Public Key Cryptography
4. RSA system
   4.1 Efficiency: Repeated Squaring.
   4.2 Correctness: Fermat's Theorem.
   4.3 Construction.
5. Warnings.

Extended GCD Algorithm.

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ext-gcd(x,y)
  if y = 0 then return(x, 1, 0)
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Theorem: Returns (d, a, b), where $d = gcd(a, b)$ and

$$d = ax + by.$$

Correctness.

Proof: Strong Induction.[1]

[1] Assume d is $gcd(x,y)$ by previous proof.

Correctness.

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Ind hyp: **ext-gcd**$(y, \mod(x, y))$ returns (d, a, b) with
$d = ay + b(\mod(x, y))$

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Iterative Algorithm? A bit easier. Later.

Wrap-up

Conclusion: Can find multiplicative inverses in $O(n)$ time!

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≤ 80 divisions.

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Property: $A \oplus B \oplus B = A$.

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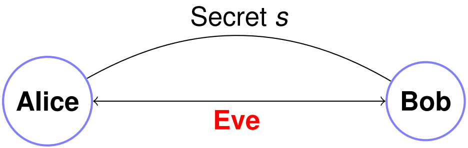

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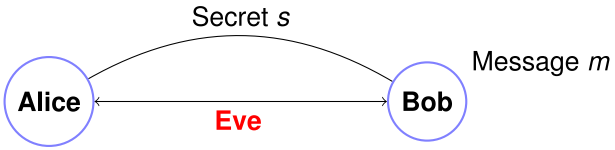

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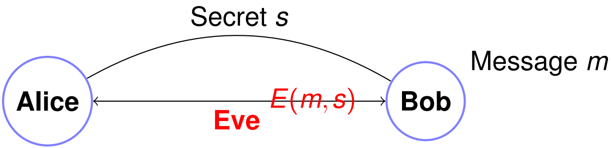

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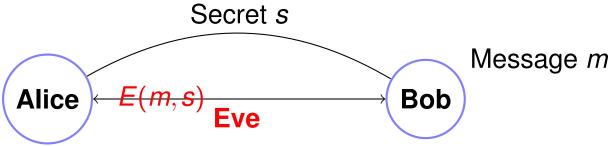

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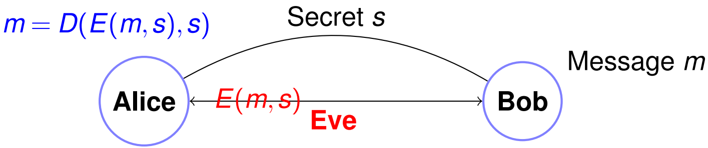

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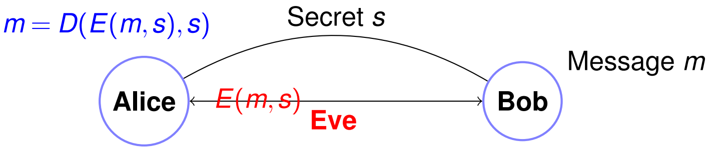
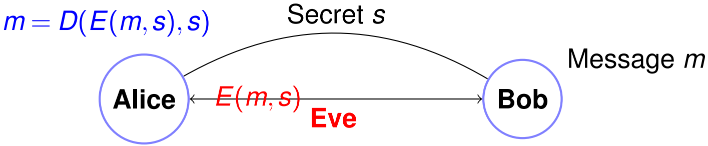


Example:

Cryptography ...



Example:
One-time Pad: secret s is string of length $|m|$.

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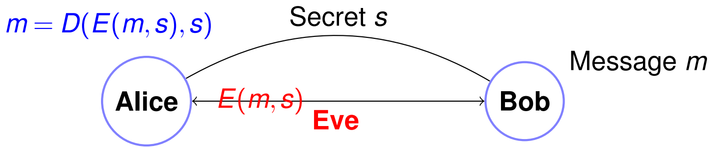

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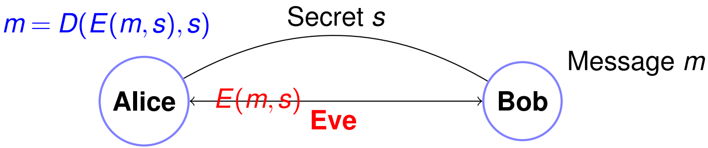

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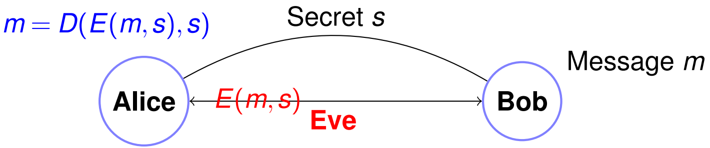

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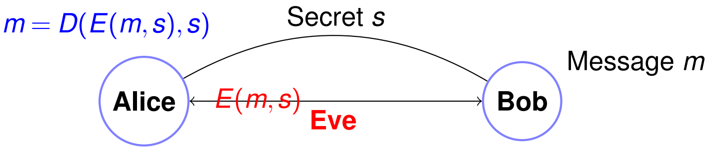

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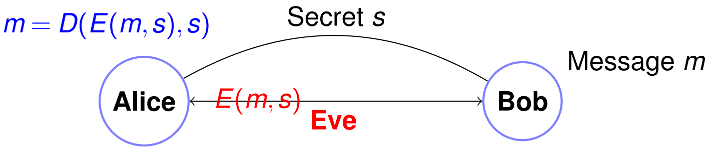

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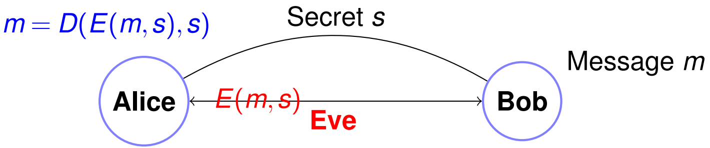

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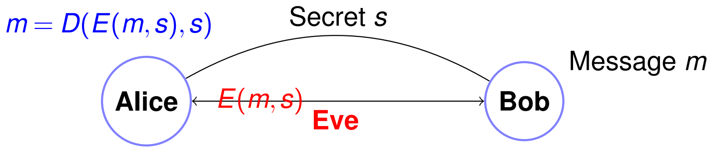

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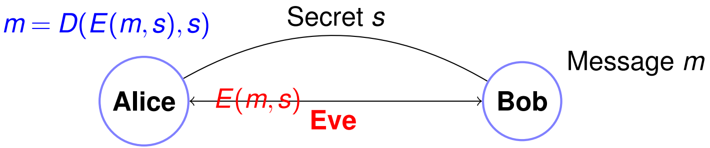

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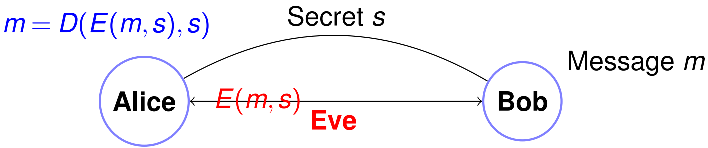

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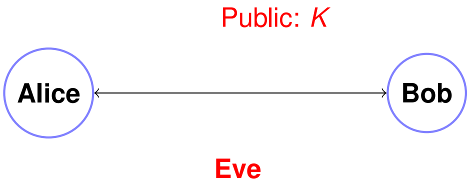

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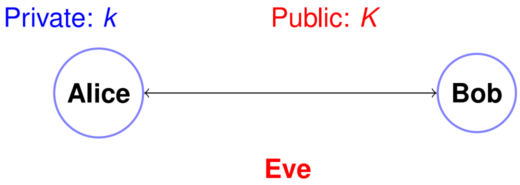

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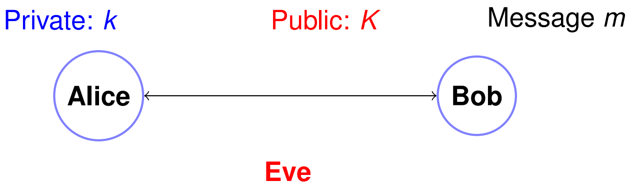
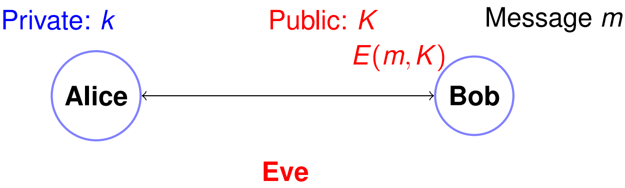


Public key crypography.



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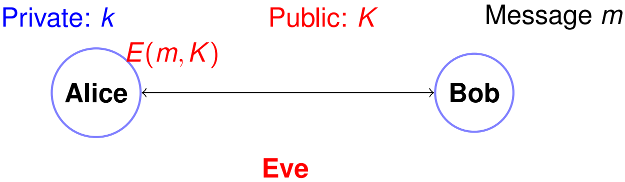
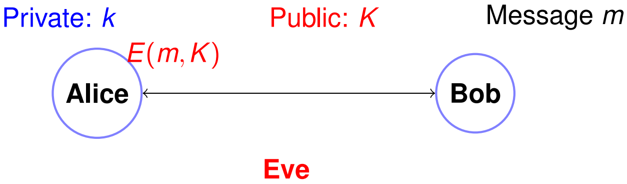


Public key crypography.

$m = D(E(m, K), k)$



Public key crypography.

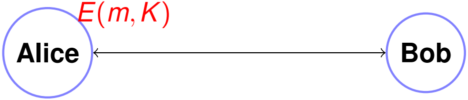
$m = D(E(m,K),k)$

Private: k

Public: K

Message m

$E(m,K)$



Eve

Everyone knows key K!

Public key crypography.

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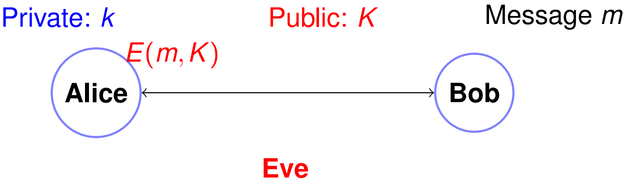

Everyone knows key K!
Bob (and Eve

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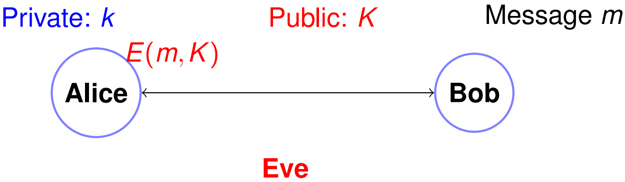

Everyone knows key K!
Bob (and Eve and me

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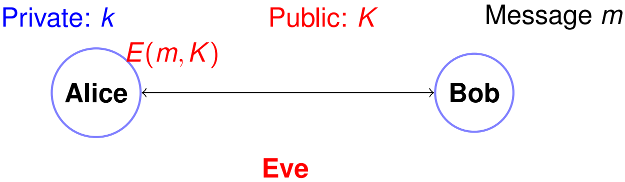

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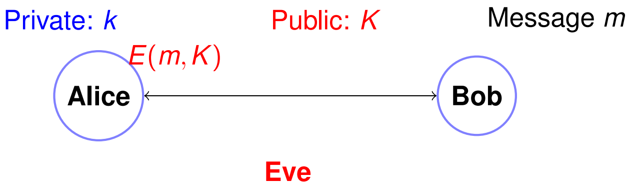

Everyone knows key K!

Bob (and Eve and me and you and you ...) can encode.

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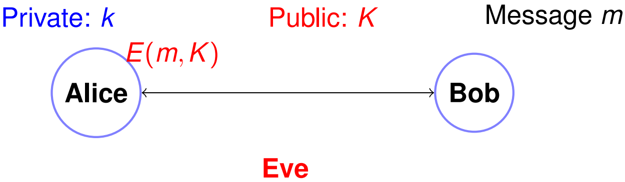

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Bob (and Eve and me and you and you ...) can encode.
Only Alice knows the secret key k for public key K.

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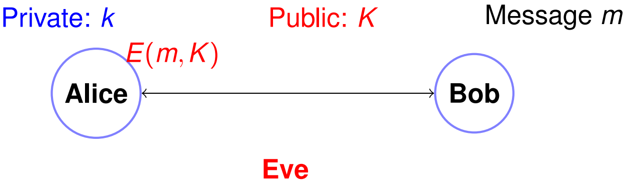

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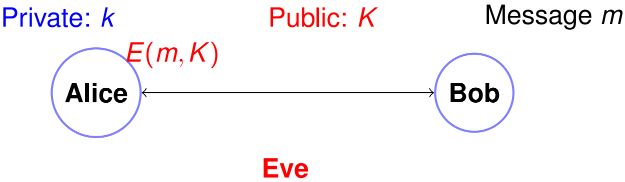

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Is this even possible?

Is public key crypto possible?

[2] Typically small, say $e = 3$.

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We don't really know.

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Iterative Extended GCD.

Example: $p = 7$, $q = 11$.

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$$7(0) + 60(1) = 60$$

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$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \end{aligned}$$

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$$\begin{aligned} 7(0) + 60(1) &= 60 \\ 7(1) + 60(0) &= 7 \\ 7(-8) + 60(1) &= 4 \end{aligned}$$

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Confirm:

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Confirm: $-119 + 120 = 1$

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Confirm: $-119 + 120 = 1$

$d = e^{-1} = -17 = 43 = \pmod{60}$

Encryption/Decryption Techniques.

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Public Key: $(77, 7)$

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Message Choices: $\{0, \ldots, 76\}$.

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$E(2)$

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$E(2) = 2^e$

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$E(2) = 2^e = 2^7$

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$E(2) = 2^e = 2^7 \equiv 128 \pmod{77}$

Encryption/Decryption Techniques.

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Message: 2!

$E(2) = 2^e = 2^7 \equiv 128 \pmod{77} = 51 \pmod{77}$

Encryption/Decryption Techniques.

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$E(2) = 2^e = 2^7 \equiv 128 \pmod{77} = 51 \pmod{77}$
$D(51) = 51^{43} \pmod{77}$

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uh oh!

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Obvious way: 43 multiplcations. Ouch.

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In general, $O(N)$ multiplications!

Repeated squaring.

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Notice: $43 = 32 + 8 + 2 + 1$.

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4 multiplications sort of...

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Need to compute $51^{32} \ldots 51^{1}$.?

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$51^1 \equiv 51 \pmod{77}$
$51^2 =$

Repeated squaring.

Notice: $43 = 32+8+2+1$. $51^{43} = 51^{32+8+2+1} = 51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} \pmod{77}$.
4 multiplications sort of...
Need to compute $51^{32} \ldots 51^{1}$.?
$51^{1} \equiv 51 \pmod{77}$
$51^{2} = (51) * (51) = 2601 \equiv 60 \pmod{77}$

Repeated squaring.

Notice: $43 = 32 + 8 + 2 + 1$. $51^{43} = 51^{32+8+2+1} = 51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} \pmod{77}$.

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Repeated squaring.

Notice: $43 = 32 + 8 + 2 + 1$. $51^{43} = 51^{32+8+2+1} = 51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} \pmod{77}$.
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Need to compute $51^{32} \ldots 51^{1}$.?
$51^1 \equiv 51 \pmod{77}$
$51^2 = (51) * (51) = 2601 \equiv 60 \pmod{77}$
$51^4 = (51^2) * (51^2) = 60 * 60 = 3600 \equiv 58 \pmod{77}$

Repeated squaring.

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Need to compute $51^{32} \ldots 51^{1}$.?

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$51^8 =$

Repeated squaring.

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$51^8 = (51^4) * (51^4) = 58 * 58 = 3364 \equiv 53 \pmod{77}$

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$51^{8} = (51^{4}) * (51^{4}) = 58 * 58 = 3364 \equiv 53 \pmod{77}$

$51^{16} = (51^{8}) * (51^{8}) = 53 * 53 = 2809 \equiv 37 \pmod{77}$

Repeated squaring.

Notice: $43 = 32 + 8 + 2 + 1$. $51^{43} = 51^{32+8+2+1} = 51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} \pmod{77}$.

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$51^{16} = (51^8) * (51^8) = 53 * 53 = 2809 \equiv 37 \pmod{77}$

$51^{32} = (51^{16}) * (51^{16}) = 37 * 37 = 1369 \equiv 60 \pmod{77}$

Repeated squaring.

Notice: $43 = 32+8+2+1$. $51^{43} = 51^{32+8+2+1} = 51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} \pmod{77}$.
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5 more multiplications.

Repeated squaring.

Notice: $43 = 32+8+2+1$. $51^{43} = 51^{32+8+2+1} = 51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} \pmod{77}$.

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Need to compute $51^{32} \ldots 51^{1}$.?

$51^1 \equiv 51 \pmod{77}$

$51^2 = (51)*(51) = 2601 \equiv 60 \pmod{77}$

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$51^{32} = (51^{16})*(51^{16}) = 37*37 = 1369 \equiv 60 \pmod{77}$

5 more multiplications.

$51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} = (60)*(53)*(60)*(51) \equiv 2 \pmod{77}$.

Repeated squaring.

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Need to compute $51^{32} \ldots 51^{1}$.?

$51^{1} \equiv 51 \pmod{77}$

$51^{2} = (51) * (51) = 2601 \equiv 60 \pmod{77}$

$51^{4} = (51^{2}) * (51^{2}) = 60 * 60 = 3600 \equiv 58 \pmod{77}$

$51^{8} = (51^{4}) * (51^{4}) = 58 * 58 = 3364 \equiv 53 \pmod{77}$

$51^{16} = (51^{8}) * (51^{8}) = 53 * 53 = 2809 \equiv 37 \pmod{77}$

$51^{32} = (51^{16}) * (51^{16}) = 37 * 37 = 1369 \equiv 60 \pmod{77}$

5 more multiplications.

$51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} = (60) * (53) * (60) * (51) \equiv 2 \pmod{77}$.

Decoding got the message back!

Repeated squaring.

Notice: $43 = 32 + 8 + 2 + 1$. $51^{43} = 51^{32+8+2+1} = 51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} \pmod{77}$.

4 multiplications sort of...

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Repeated Squaring took 9 multiplications

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$51^{32} \cdot 51^{8} \cdot 51^{2} \cdot 51^{1} = (60)*(53)*(60)*(51) \equiv 2 \pmod{77}$.

Decoding got the message back!

Repeated Squaring took 9 multiplications versus 43.

Repeated Squaring: x^y

Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute x^1,

Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute $x^1, x^2,$

Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

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Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute $x^1, x^2, x^4, \ldots, x^{2^{\lfloor \log y \rfloor}}$.

Repeated Squaring: x^y

Repeated squaring $O(\log y)$ multiplications versus y!!!

1. x^y: Compute $x^1, x^2, x^4, \ldots, x^{2^{\lfloor \log y \rfloor}}$.
2. Multiply together x^i where the $(\log(i))$th bit of y (in binary) is 1.

Repeated Squaring: x^y

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   Example:

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Modular Exponentiation: $x^y \mod N$.

Repeated Squaring: x^y

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Modular Exponentiation: $x^y \mod N$. All n-bit numbers. Repeated Squaring:

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Conclusion: $x^y \mod N$

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Conclusion: $x^y \mod N$ takes $O(n^3)$ time.

RSA is pretty fast.

Modular Exponentiation: $x^y \mod N$.

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versus $a^{k(p-1)(q-1)+1} = a \pmod{pq}$.

Similar, not same, but useful.

Correct decoding...

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□

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$a^{1+b(p-1)} \equiv a \pmod{p}$
Proof:

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$a^{1+b(p-1)} \equiv a^1 * (a^{p-1})^b \equiv a * (1)^b \equiv a \pmod{p}$

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All steps are polynomial in $O(\log N)$, the number of bits.

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Breaking in general sense $\implies$ factoring algorithm.

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CS161...

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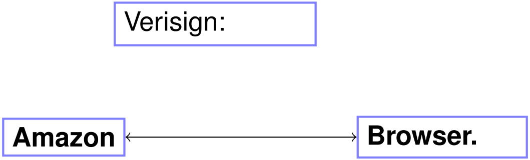

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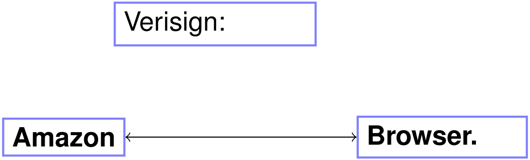

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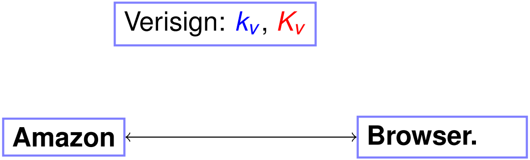

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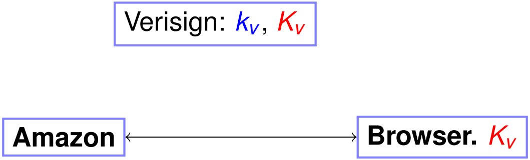

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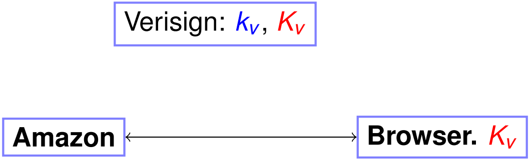

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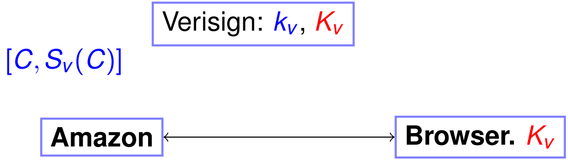

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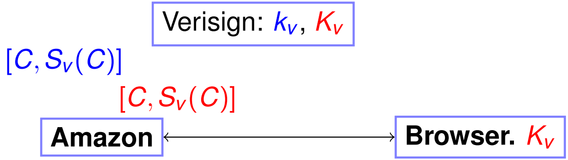

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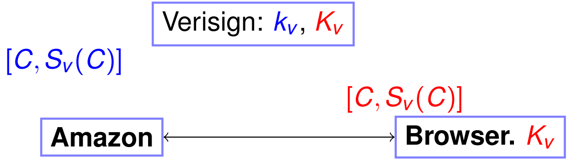

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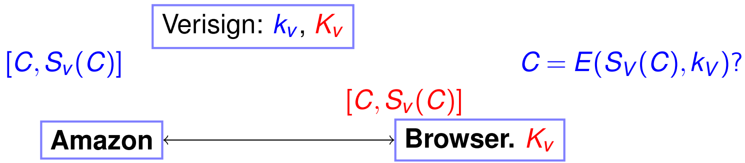

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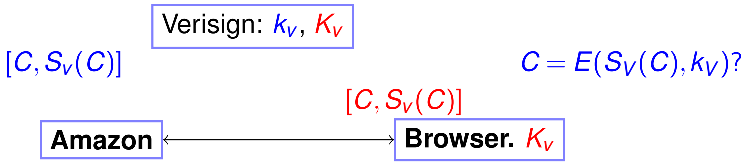

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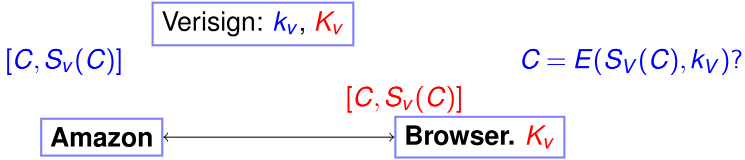

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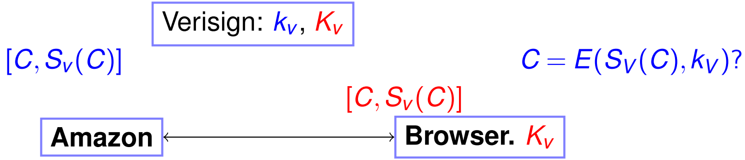

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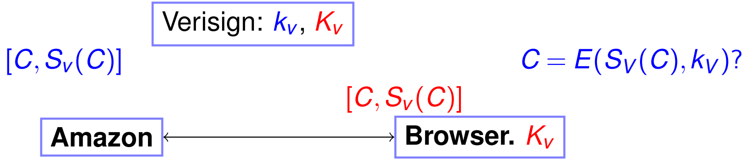

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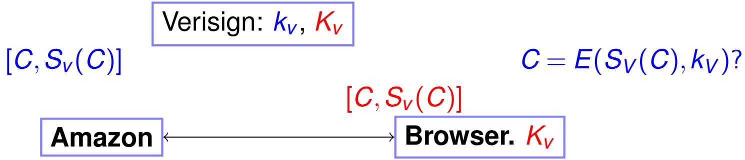

Certificate Authority: Verisign, GoDaddy, DigiNotar,...

Verisign's key: $K_V = (N, e)$ and $k_V = d$ $(N = pq.)$

Browser "knows" Verisign's public key: K_V.

Amazon Certificate: C = "I am Amazon. My public Key is K_A."

Versign signature of C: $S_v(C)$: $D(C, k_V) = C^d \mod N$.

Browser receives: $[C, y]$

Checks $E(y, K_V) = C$?

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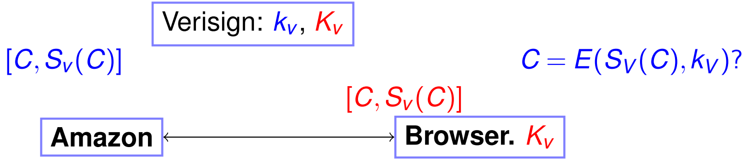

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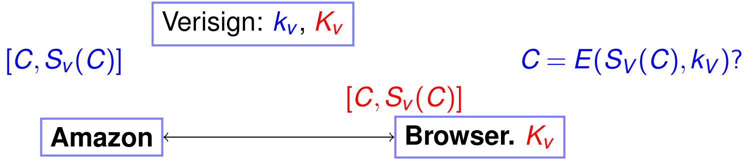

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Security: Eve can't forge unless she "breaks" RSA scheme.

RSA

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RSA

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Public-Key Encryption.

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RSA Scheme:

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