Today

Review for Midterm.

First there was logic...

A statement is a true or false.

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Statements?

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Statements?

$3 = 4 - 1$?

First there was logic...

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Statements?

$3 = 4 - 1$? Statement!

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Statements?

- $3 = 4 - 1$? Statement!
- $3 = 5$?

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- 3 ?

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- $n = 3$?

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- $n = 3$? Not a statement...but a predicate.

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Predicate: Statement with free variable(s).

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Example: $x = 3$

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Example: $x = 3$ Given a value for x, becomes a statement.

Predicate?

$n > 3$?

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Predicate: Statement with free variable(s).

Example: $x = 3$ Given a value for x, becomes a statement.

Predicate?

$n > 3$? Predicate: $P(n)$!

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Example: $x = 3$ Given a value for x, becomes a statement.

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$x = y$?

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- $n > 3$? Predicate: $P(n)$!
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Quantifiers:

$(\forall x)\ P(x)$. For every x, $P(x)$ is true.

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$(\forall x \in R)(\exists y \in R) y > x$.

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Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

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You got this!

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Propositional Expressions and Logical Equivalence

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Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$

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Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

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Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
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Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
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Proofs: truth table or manipulation of known formulas.

Connecting Statements

$A \wedge B$, $A \vee B$, $\neg A$.

You got this!

Propositional Expressions and Logical Equivalence

$(A \implies B) \equiv (\neg A \vee B)$
$\neg(A \vee B) \equiv (\neg A \wedge \neg B)$

Proofs: truth table or manipulation of known formulas.

$(\forall x)(P(x) \wedge Q(x)) \equiv (\forall x)P(x) \wedge (\forall x)Q(x)$

..and then proofs...

Direct: $P \implies Q$

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Example: a is even $\implies$ a^2 is even.

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Approach: What is even?

..and then proofs...

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Approach: What is even? $a = 2k$

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What is even?

$a^2 = 2(2k^2)$

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Integers closed under multiplication!

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Contrapositive: $P \implies Q$

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Example: a^2 is odd $\implies$ a is odd.

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Example: a^2 is odd $\implies$ a is odd.

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$\neg P \implies$ **false**

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$\neg P \implies R \wedge \neg R$

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Useful for prove something does not exist:

..and then proofs...

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$

..and then proofs...

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

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Direct: $P \implies Q$

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

Example: finite set of primes

..and then proofs...

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Useful for prove something does not exist:

Example: rational representation of $\sqrt{2}$ does not exist.

Example: finite set of primes does not exist.

Example: rogue couple does not exist.

...jumping forward..

Contradiction in induction:

...jumping forward..

Contradiction in induction:

contradict place where induction step doesn't hold.

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Well Ordering Principle.

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Stable Marriage:

...jumping forward..

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first day where women does not improve.

...jumping forward..

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first day where women does not improve.
first day where any man rejected by optimal women.

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...and then induction...

$$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\, P(n).$$

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$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\, P(n).$

Thm: For all $n \geq 1$, $8 | 3^{2n} - 1$.

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\, P(n).$

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Induction on n.

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Induction on n.

Base: $8|3^2 - 1$.

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Base: $8|3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\, P(n)$.

Thm: For all $n \geq 1$, $8|3^{2n} - 1$.

Induction on n.

Base: $8|3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.

Induction Step: Prove $P(n+1)$

...and then induction...

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$3^{2n+2} - 1 =$

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Induction Hypothesis: Assume $P(n)$: True for some n.

Induction Step: Prove $P(n+1)$

$3^{2n+2} - 1 = 9(3^{2n}) - 1$

...and then induction...

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Thm: For all $n \geq 1$, $8|3^{2n} - 1$.

Induction on n.

Base: $8|3^2 - 1$.

Induction Hypothesis: Assume $P(n)$: True for some n.
$(3^{2n} - 1 = 8d)$

Induction Step: Prove $P(n+1)$

$3^{2n+2} - 1 = 9(3^{2n}) - 1$ (by induction hypothesis)

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\, P(n)$.

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$(3^{2n} - 1 = 8d)$

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$$\begin{aligned} 3^{2n+2} - 1 &= 9(3^{2n}) - 1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1) - 1 \end{aligned}$$

...and then induction...

$P(0) \wedge ((\forall n)(P(n) \implies P(n+1) \equiv (\forall n \in N)\, P(n)$.

Thm: For all $n \geq 1$, $8|3^{2n}-1$.

Induction on n.

Base: $8|3^2-1$.

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($3^{2n}-1 = 8d$)

Induction Step: Prove $P(n+1)$

$$\begin{aligned} 3^{2n+2}-1 &= 9(3^{2n})-1 \quad \text{(by induction hypothesis)} \\ &= 9(8d+1)-1 \\ &= 72d+8 \end{aligned}$$

...and then induction...

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Divisible by 8.

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Divisible by 8.

□

Stable Marriage: a study in definitions and WOP.

n-men, *n*-women.

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n-men, *n*-women.

Each person has completely ordered preference list

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Each person has completely ordered preference list
contains every person of opposite gender.

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Pairing.
Set of pairs (m_i, w_j) containing all people *exactly* once.

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How many pairs?

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How many pairs? n.

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People in pair are **partners** in pairing.

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Rogue Couple in a pairing.
A m_j and w_k who like each other more than their partners

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Stable Pairing.
Pairing with no rogue couples.

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Pairing.
Set of pairs (m_i, w_j) containing all people *exactly* once.
How many pairs? n.
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Rogue Couple in a pairing.
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Stable Pairing.
Pairing with no rogue couples.

Does stable pairing exist?

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Pairing with no rogue couples.

Does stable pairing exist?

No, for roommates problem.

TMA.

Traditional Marriage Algorithm:

TMA.

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Each Day:

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All men propose to favorite woman who has not yet rejected him.

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Useful Algorithmic Definitions:

TMA.

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Woman's current proposer is "**on string.**"

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"Propose and Reject."

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Key Property: Improvement Lemma:

TMA.

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Every day, if man on string for woman,

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Key Property: Improvement Lemma:
Every day, if man on string for woman,
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rogue couple (M,W)

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rogue couple (M,W)
$\Longrightarrow$ M proposed to W
$\Longrightarrow$ W ended up with someone she liked better than *M*.

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$\Longrightarrow$ M proposed to W
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Not rogue couple!

Optimality/Pessimal

Optimal partner if best partner in any stable pairing.

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Not necessarily first in list.

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Thm: TMA produces male optimal pairing, *S*.

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Thm: TMA produces male optimal pairing, *S*.
  First man *M* to lose optimal partner.

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Thm: TMA produces male optimal pairing, S.
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TMA: M asked W first!

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TMA: M asked W first!
There is M' who bumps M in TMA.

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W prefers M'.

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M' likes W at least as much as optimal partner.

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    Not first bump.

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  TMA: M asked W first!
  There is M' who bumps M in TMA.
  W prefers M'.
  M' likes W at least as much as optimal partner.
    Not first bump.
  M' and W is rogue couple in T.

Thm: woman pessimal.

Optimality/Pessimal

Optimal partner if best partner in any stable pairing.
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Man-optimal pairing is pairing where every man gets optimal partner.

Thm: TMA produces male optimal pairing, S.
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Different stable pairing T.
TMA: M asked W first!
There is M' who bumps M in TMA.
W prefers M'.
M' likes W at least as much as optimal partner.
Not first bump.
M' and W is rogue couple in T.

Thm: woman pessimal.

Man optimal $\Longrightarrow$ Woman pessimal.

Optimality/Pessimal

Optimal partner if best partner in any stable pairing.
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TMA: M asked W first!
There is M' who bumps M in TMA.
W prefers M'.
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Thm: woman pessimal.

Man optimal $\Longrightarrow$ Woman pessimal.
Woman optimal $\Longrightarrow$ Man pessimal.

...Graphs...

$G = (V, E)$

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Directed: ordered pair of vertices.

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Adjacent, Incident, Degree.

...Graphs...

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Adjacent, Incident, Degree.
In-degree, Out-degree.

Thm: Sum of degrees is $2|E|$.

...Graphs...

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Directed: ordered pair of vertices.

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Edge is incident to 2 vertices.

...Graphs...

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  Degree of vertices is total incidences.

Pair of Vertices are Connected:

...Graphs...

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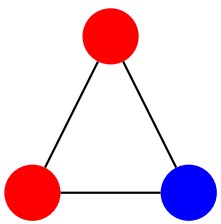

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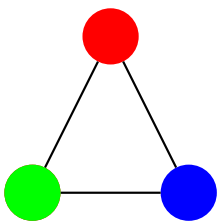

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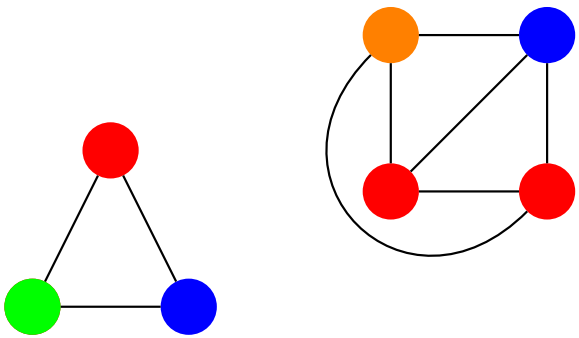

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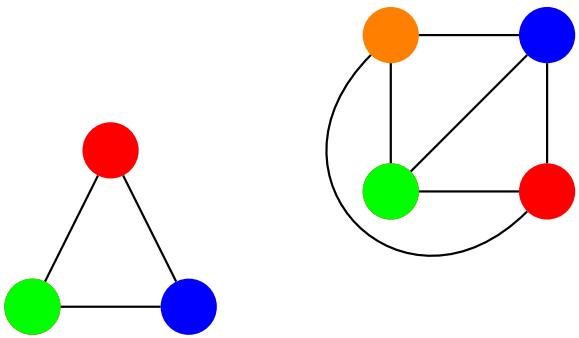

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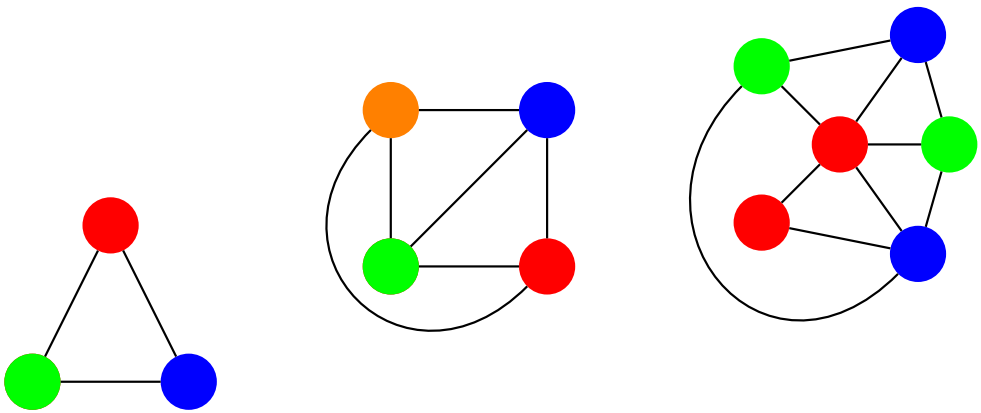

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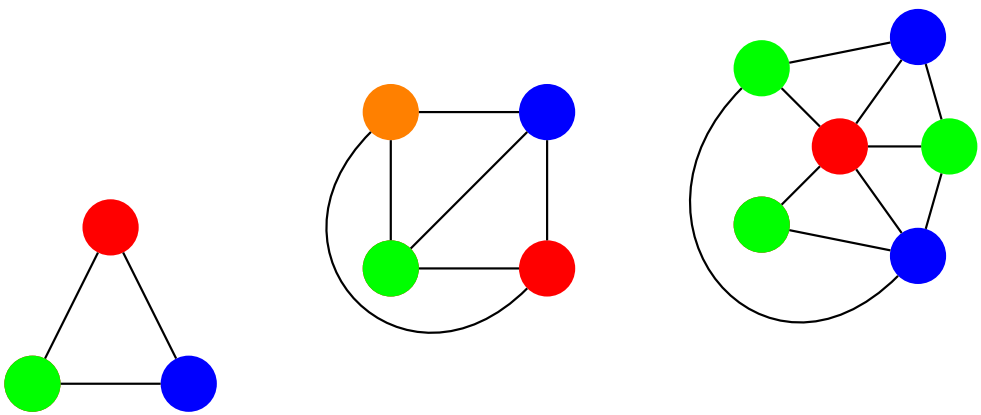

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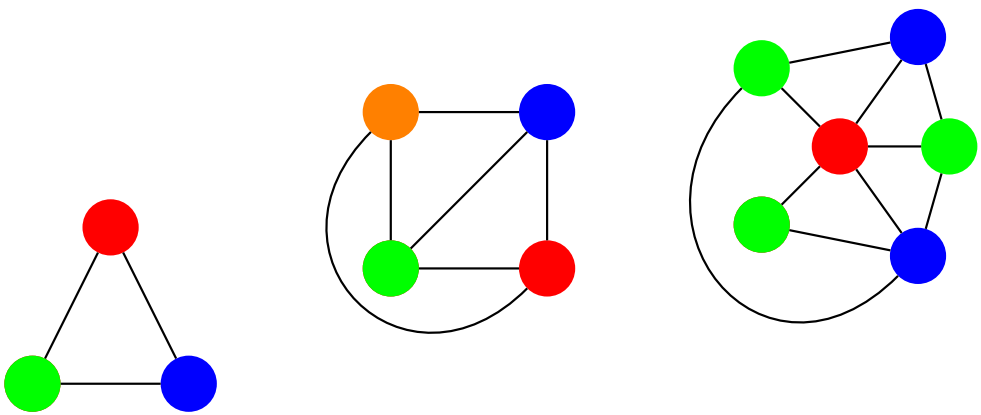

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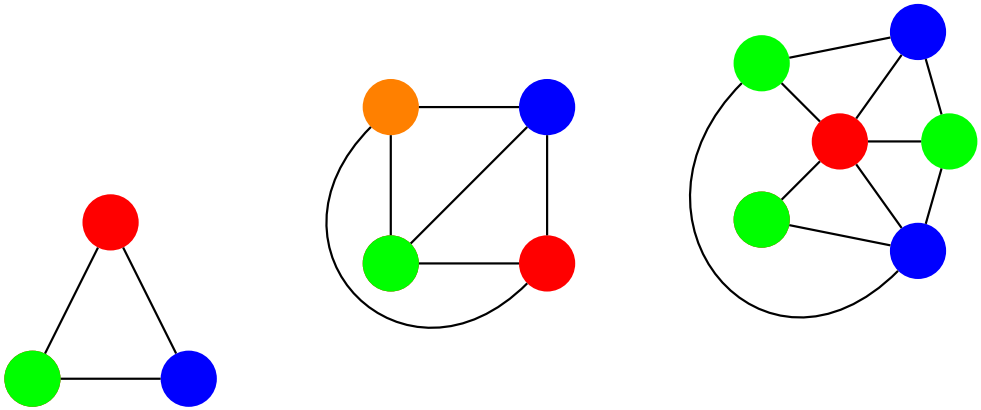

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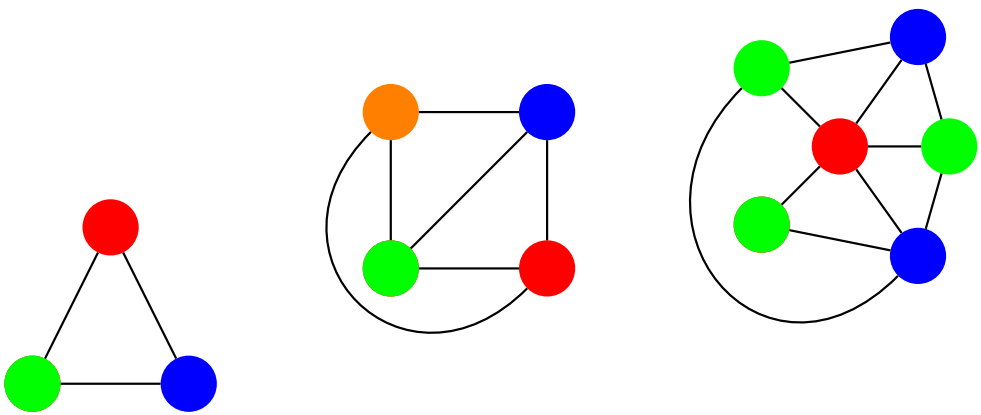

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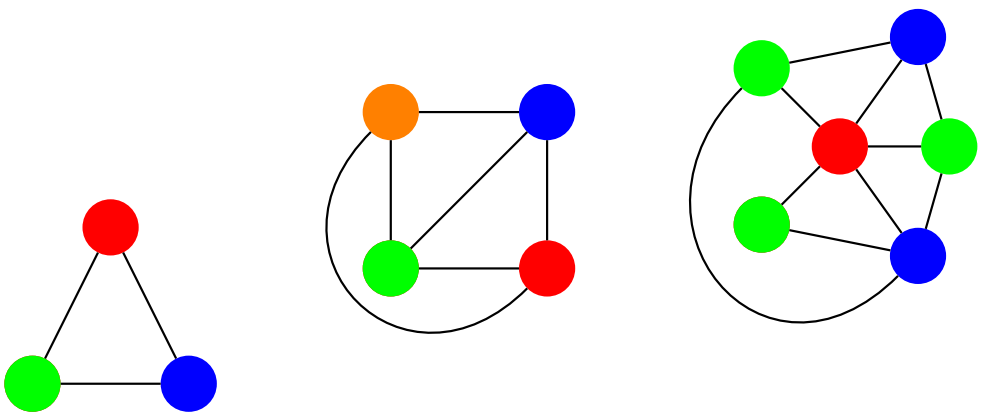

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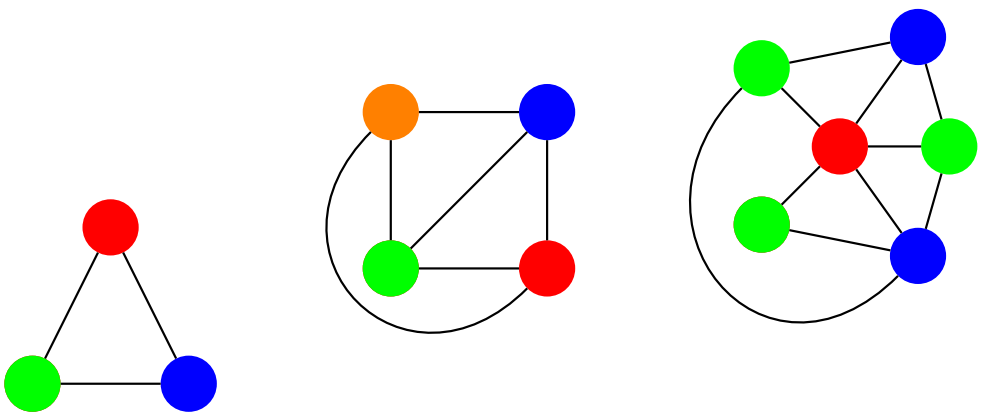

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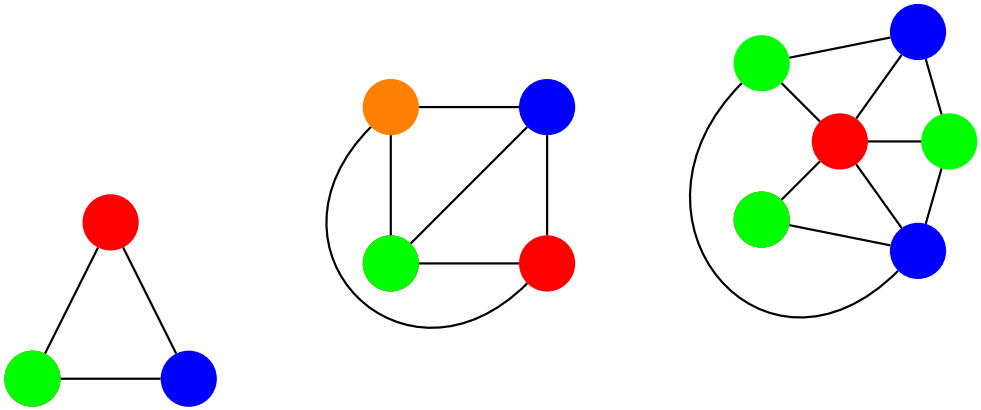

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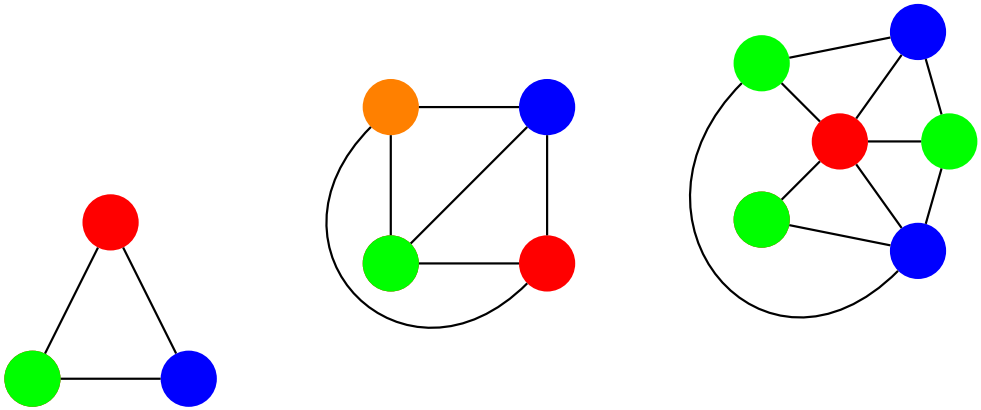

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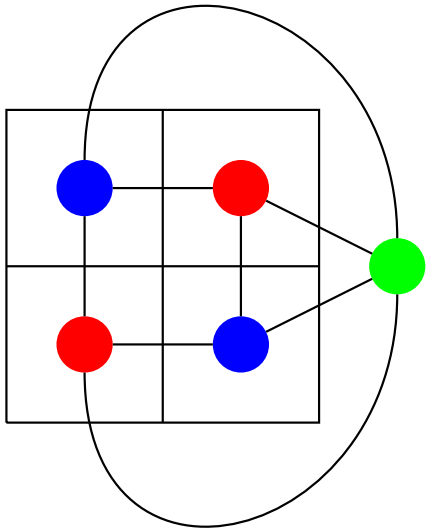
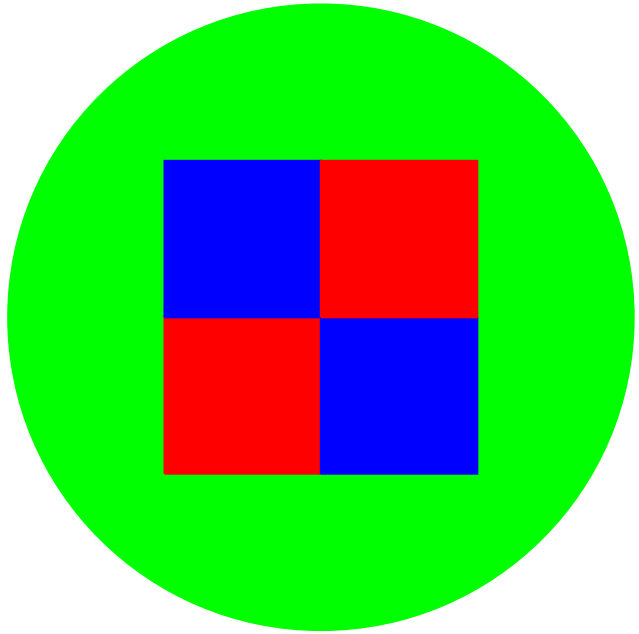
Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





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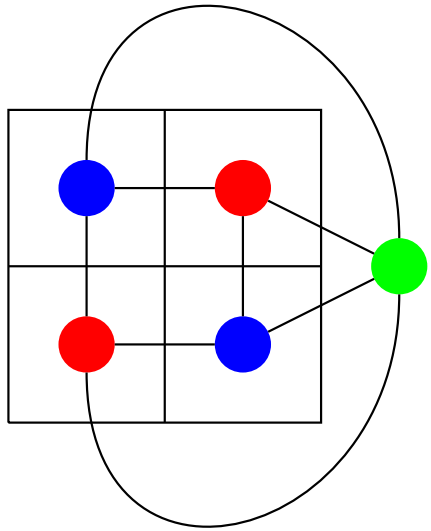

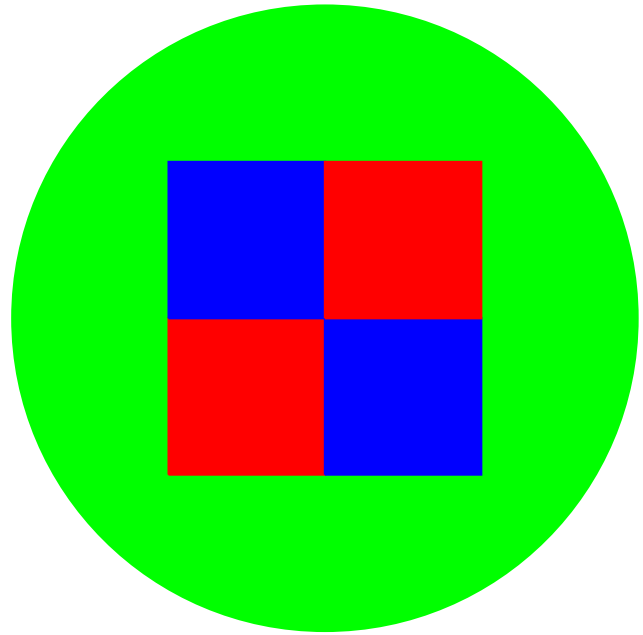

Four color theorem is about planar graphs!

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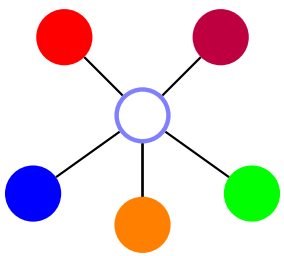

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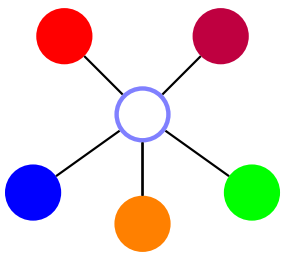

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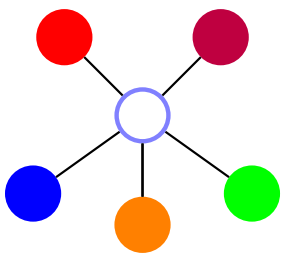

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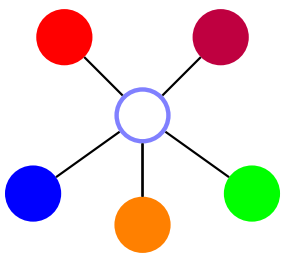

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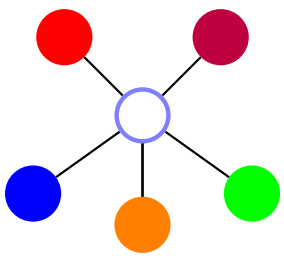

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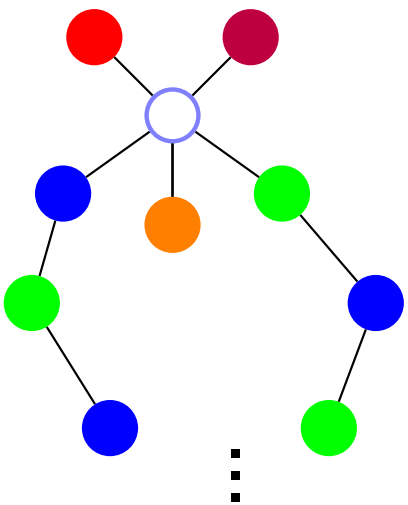

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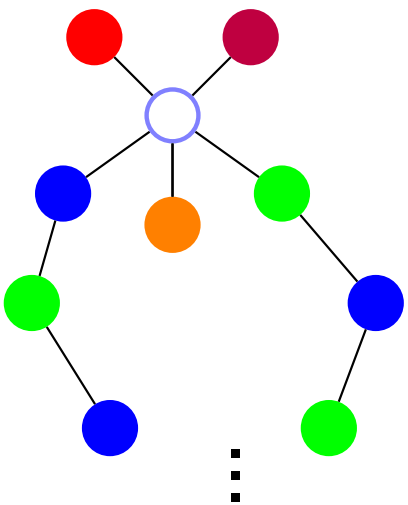

Assume neighbors are colored all differently.
Otherwise done.
Switch green to blue in component.
Done. Unless blue-green path to blue.
Switch red to orange in its component.

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

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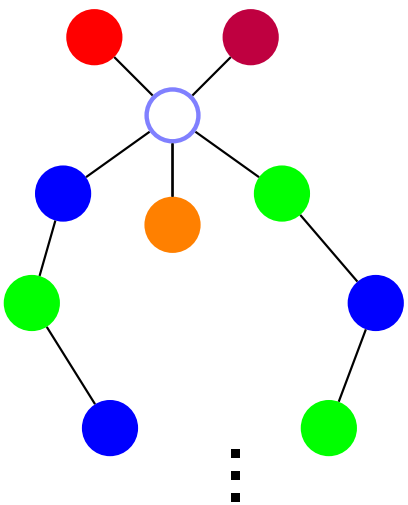

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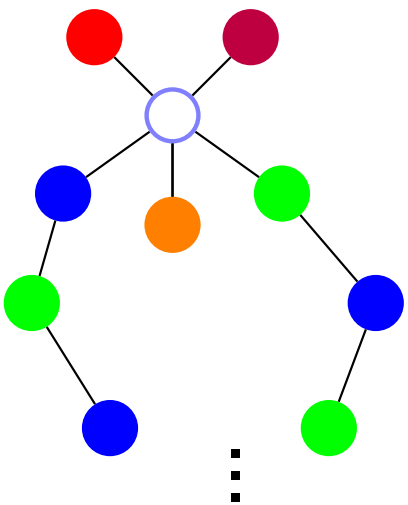

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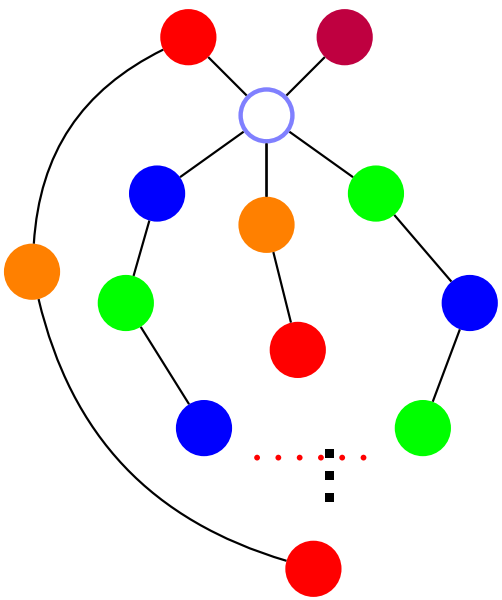

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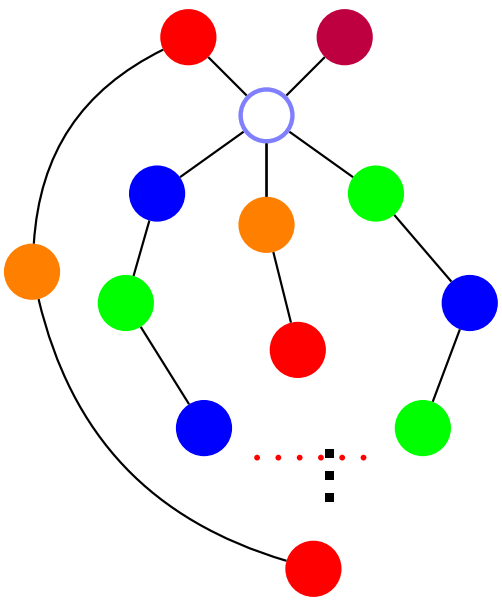

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Planar.

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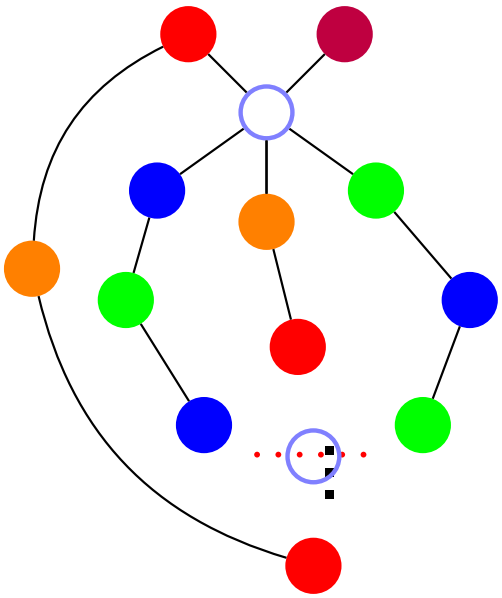

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Planar. $\Longrightarrow$ paths intersect at a vertex!

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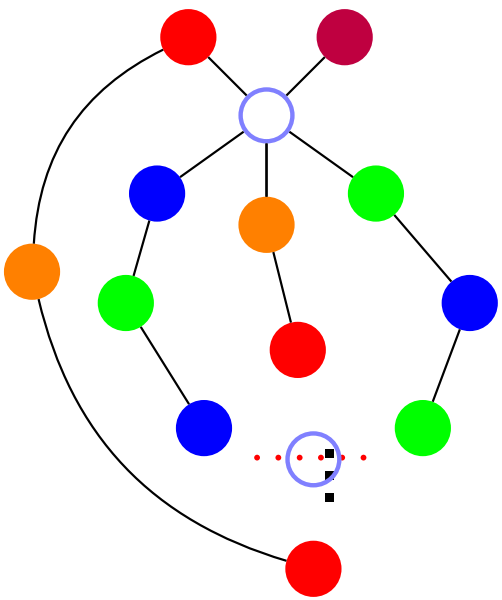

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What color is it?

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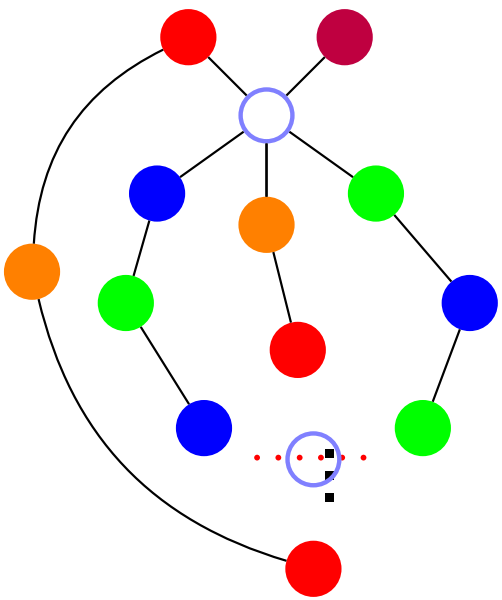

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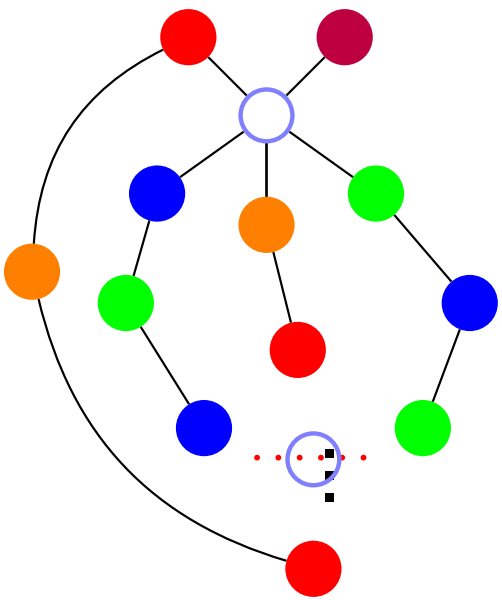

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Must be blue or green to be on that path.

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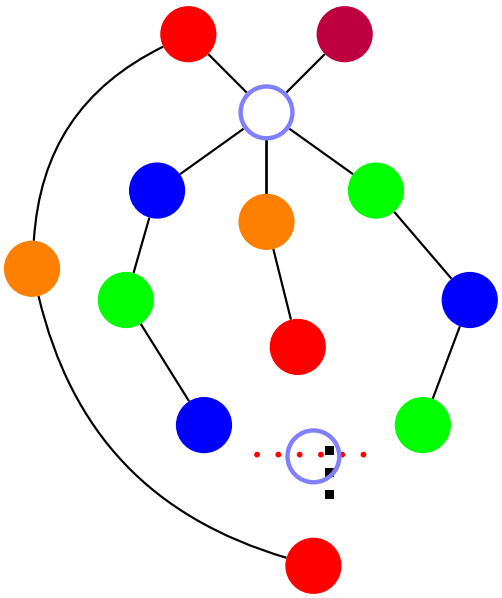

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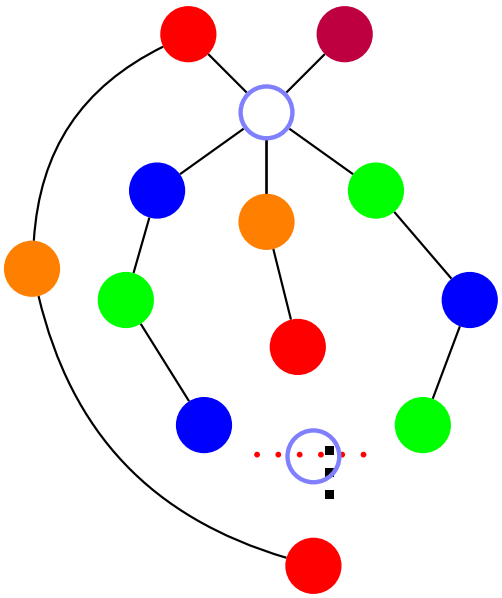

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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

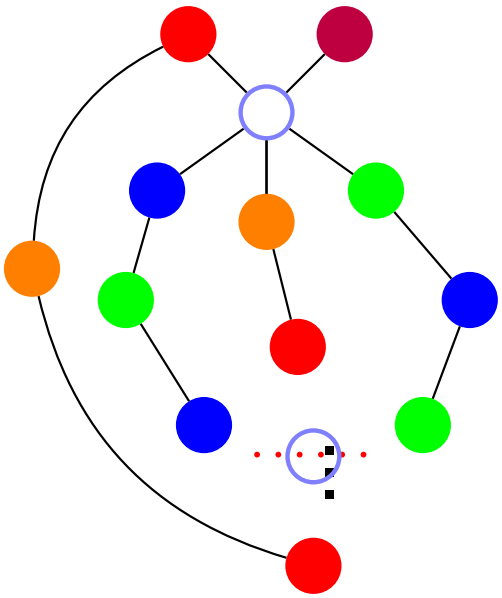
Contradiction.

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof:
Again with the degree 5 vertex. Again recurse.



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Switch red to orange in its component.
Done. Unless red-orange path to red.

Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

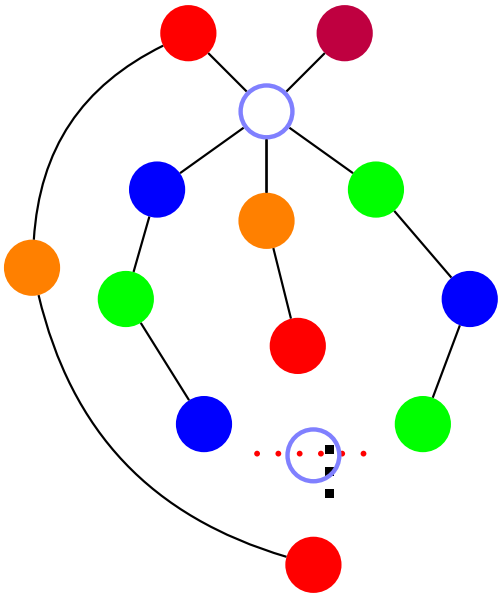
Contradiction. Can recolor one of the neighbors.
And recolor “center” vertex.

Five color theorem

Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof:
Again with the degree 5 vertex. Again recurse.



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Planar. $\Longrightarrow$ paths intersect at a vertex!

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Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.
And recolor "center" vertex. □

Four Color Theorem

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof:

Four Color Theorem

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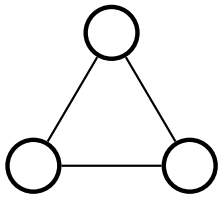
Proof: Not Today!

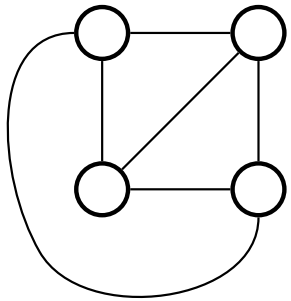
Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

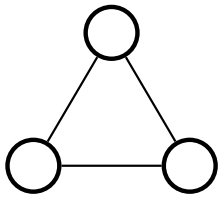
Proof: Not Today!

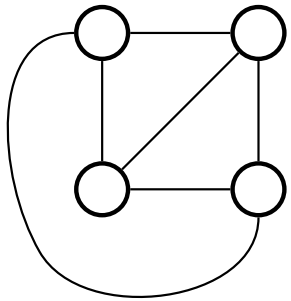
Graph Types: Complete Graph.





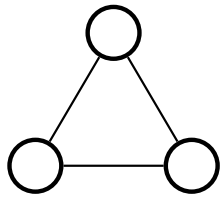
Graph Types: Complete Graph.

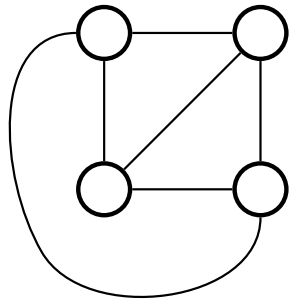




K_n, $|V| = n$

Graph Types: Complete Graph.

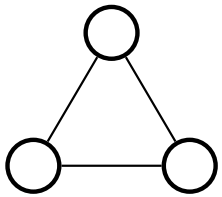


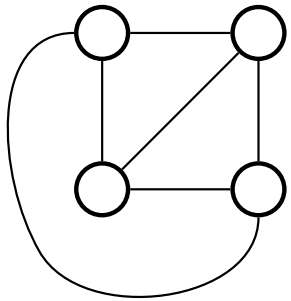


K_n, $|V| = n$

every edge present.

Graph Types: Complete Graph.





K_n, $|V| = n$

every edge present.
degree of vertex?

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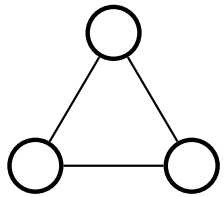
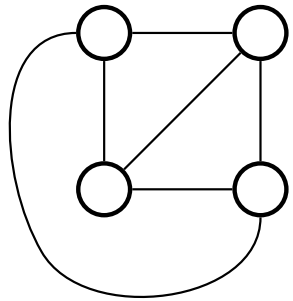
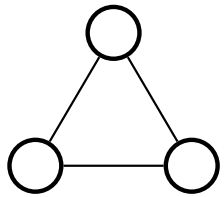


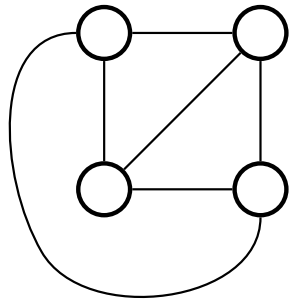


K_n, $|V| = n$

every edge present.
degree of vertex? $|V| - 1$.

Graph Types: Complete Graph.



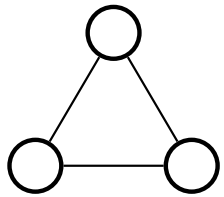


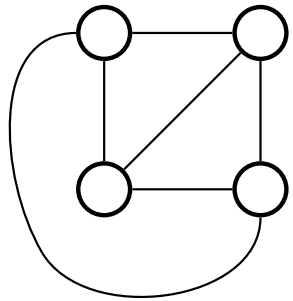
K_n, $|V| = n$

every edge present.
degree of vertex? $|V| - 1$.

Very connected.

Graph Types: Complete Graph.



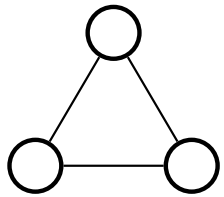


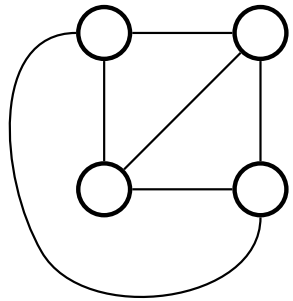
K_n, $|V| = n$

every edge present.
degree of vertex? $|V| - 1$.

Very connected.
Lots of edges:

Graph Types: Complete Graph.



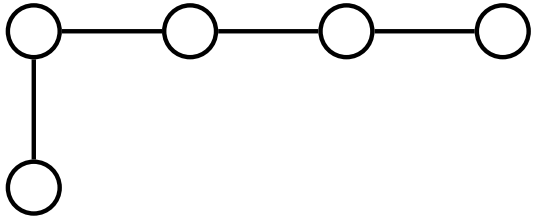


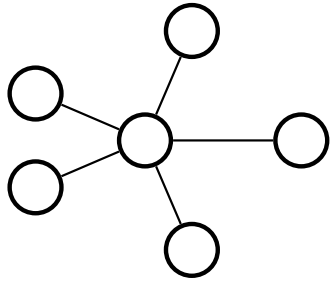
K_n, $|V| = n$

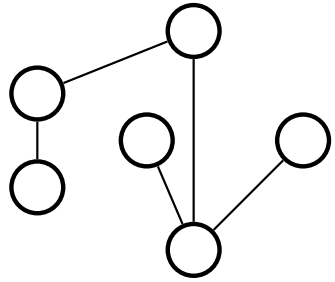
every edge present.
degree of vertex? $|V| - 1$.

Very connected.
Lots of edges: $n(n-1)/2$.

Trees.

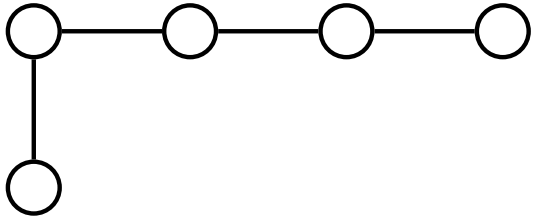


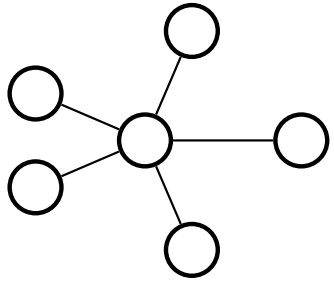


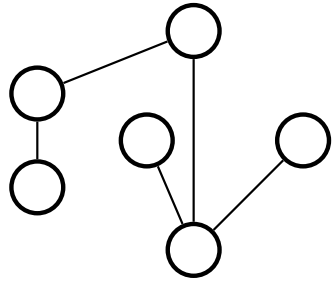


Definitions:

Trees.



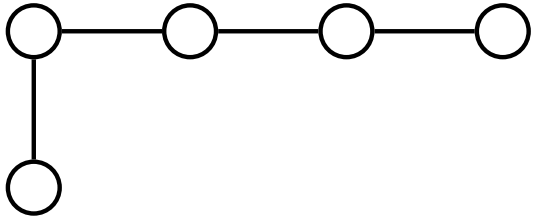


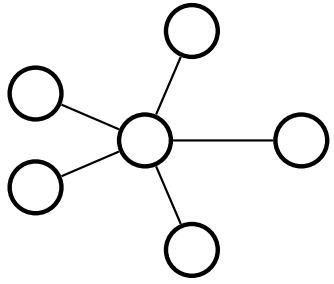


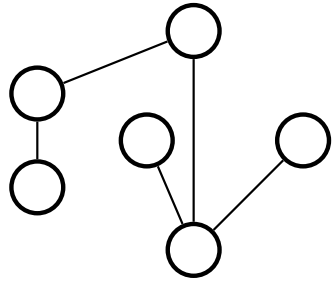
Definitions:

A connected graph without a cycle.

Trees.







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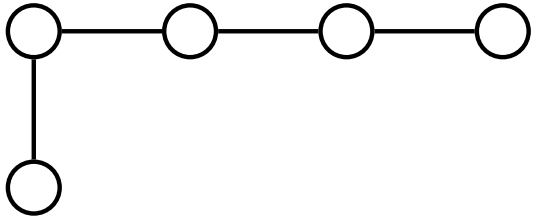

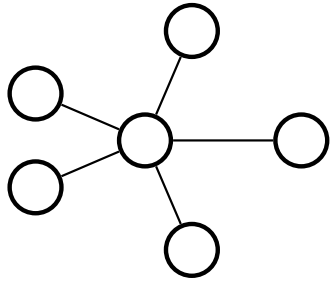

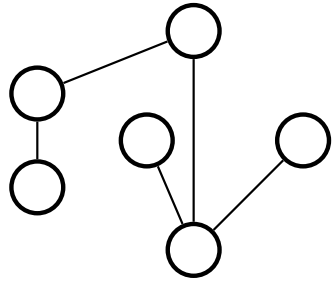

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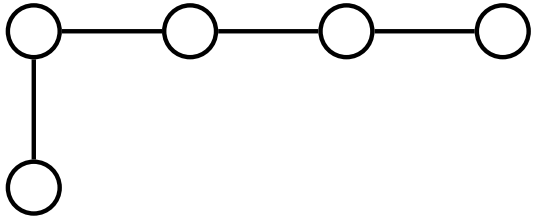

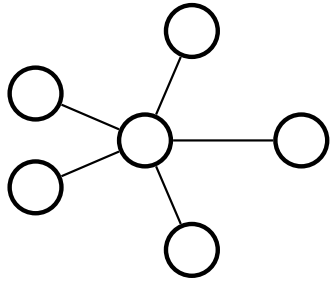

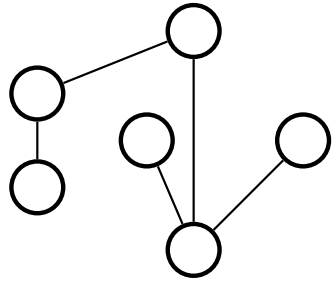

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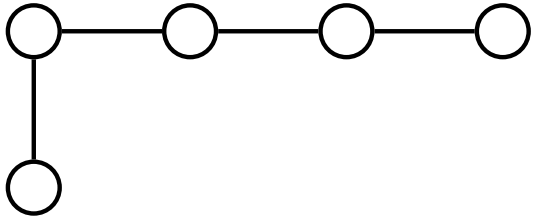

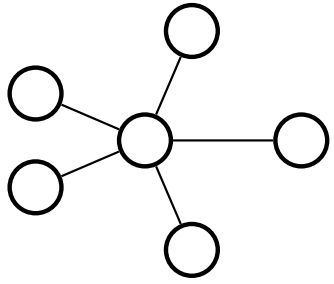

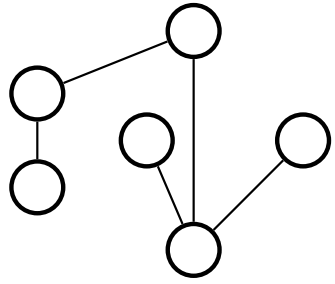

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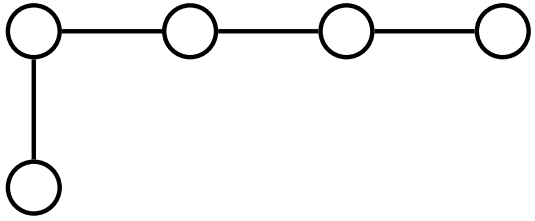

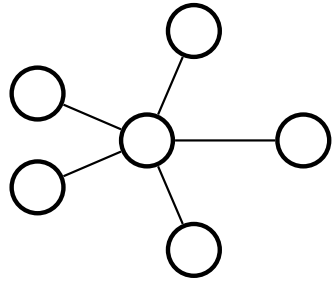

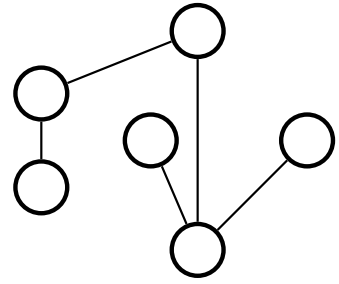

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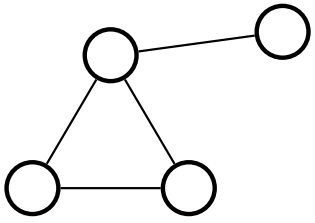

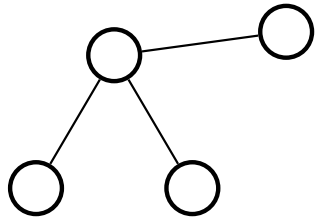

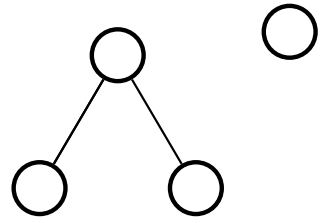

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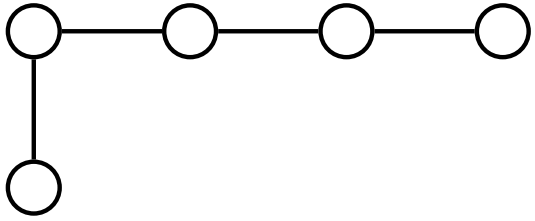

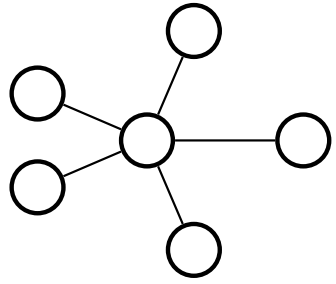

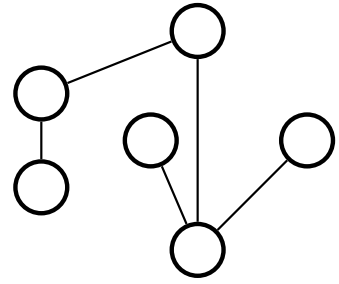

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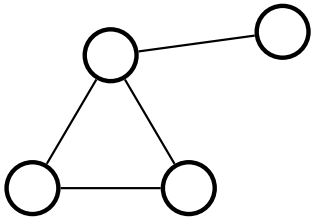

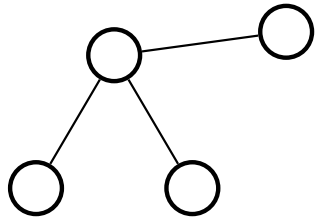

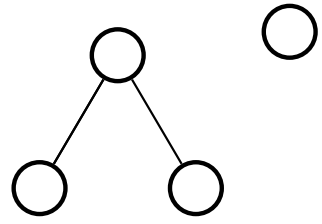

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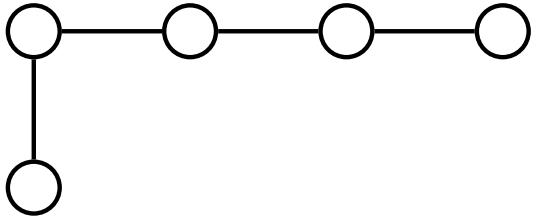

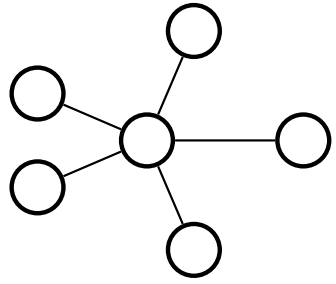

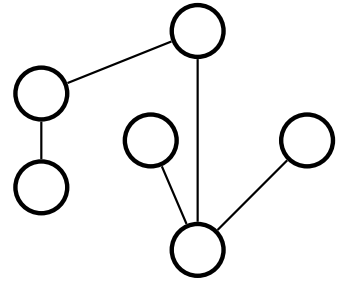

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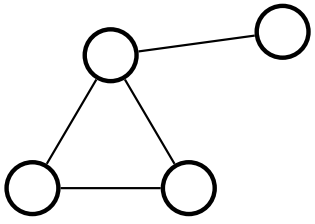

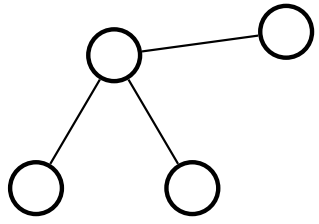

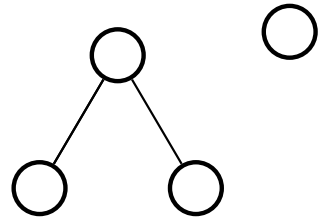

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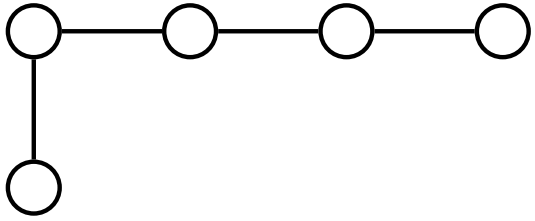

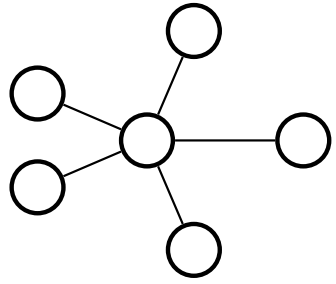

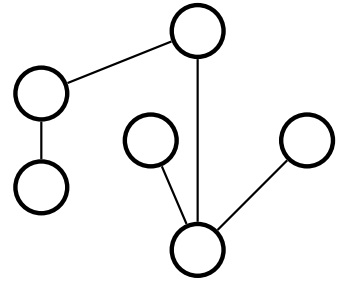

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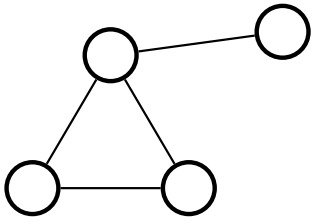

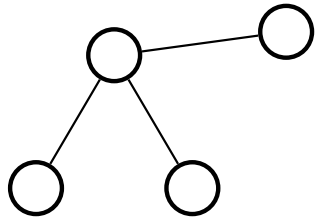

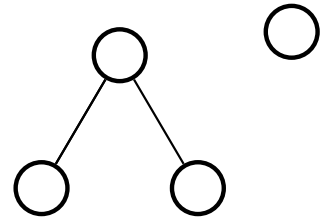

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Property:
Can remove a single node and break into components of size at most $|V|/2$.

Hypercube

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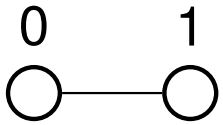

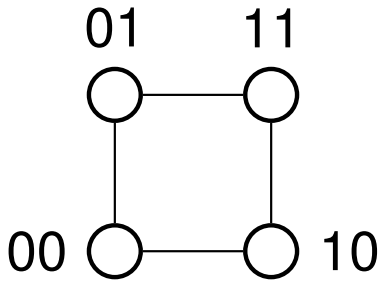

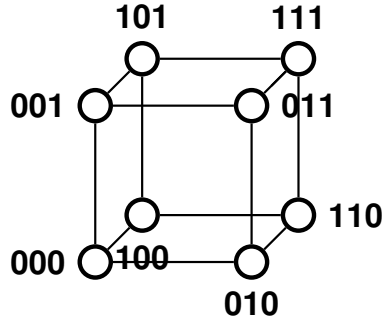

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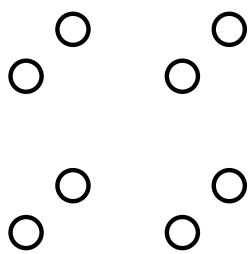

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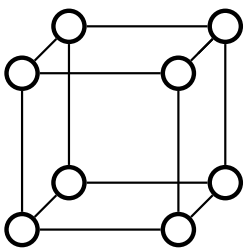

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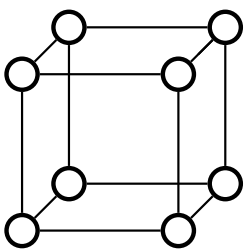

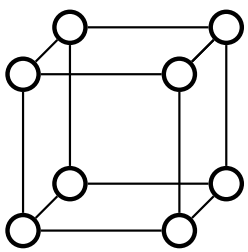

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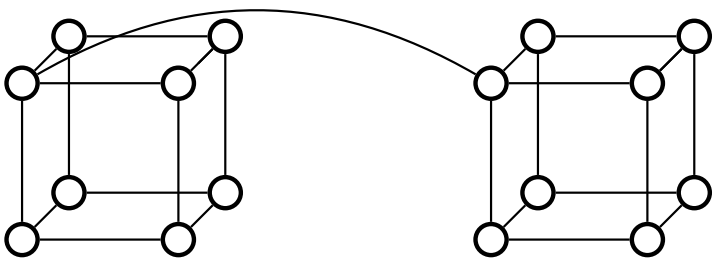
An n-dimensional hypercube consists of a 0-subcube (1-subcube) which is a $n-1$-dimensional hypercube with nodes labelled $0x$ ($1x$) with the additional edges $(0x, 1x)$.



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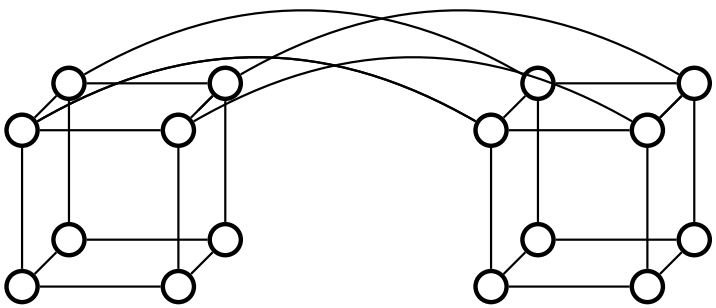

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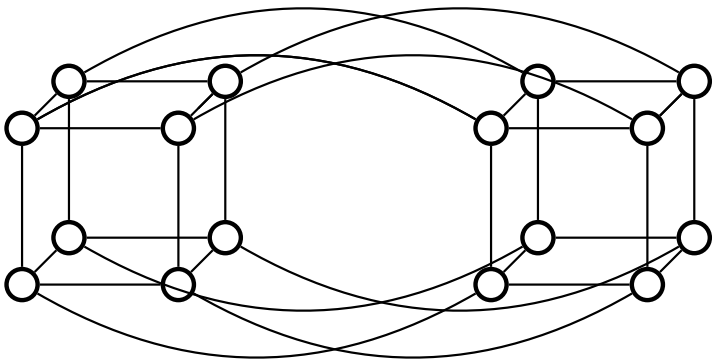

Hypercube:properties

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Good communication network!

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Elements of equivalence classes of integers.

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Additive inverses are intuitively negative numbers.

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$d = e^{-1} \pmod{(p-1)(q-1)}$.

Public Key Cryptography:

$D(E(m, K), k) = (m^e)^d \mod N = m$.

Signature scheme:

$S(C) = D(C)$.

Announce $(C, S(C))$

RSA, Public Key, and Signatures.

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$E(D(C, k), K) = (C^d)^e = C \pmod{N}$

Midterm format

Time: 120 minutes.

Midterm format

Time: 120 minutes.

Some short answers.

Midterm format

Time: 120 minutes.

Some short answers.
  Get at ideas that you learned.

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.
Know material well:

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.
Know material well: fast,

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.
Know material well: fast, correct.

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.
Know material well: fast, correct.
Know material medium:

Midterm format

Time: 120 minutes.

Some short answers.
Get at ideas that you learned.
Know material well: fast, correct.
Know material medium: slower,

Midterm format

Time: 120 minutes.

Some short answers.

Get at ideas that you learned.

Know material well: fast, correct.

Know material medium: slower, less correct.

Midterm format

Time: 120 minutes.

Some short answers.

Get at ideas that you learned.

Know material well: fast, correct.

Know material medium: slower, less correct.

Know material not so well:

Midterm format

Time: 120 minutes.

Some short answers.

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Know material well:	fast, correct.
Know material medium:	slower, less correct.
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Midterm format

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Some longer questions.

Midterm format

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Some longer questions.
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Will post midterm from 4 years ago to get an idea.

Midterm format

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- Back when I was young

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Fermat/RSA

$3^6 \pmod 7$?

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$2^{12} \pmod{21}$?

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Wrapup.

Wrapup.

If you sent me email about Midterm conflicts

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Other arrangements.

Wrapup.

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- Other arrangements.
- Should have recieved an email from me.

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