In vector notation, Eq. (1.31) becomes simply

Nonlinear branch equation

$$\mathbf{i} = \mathbf{g}(\mathbf{v}) \tag{1.32}$$

Since the independent current sources do not form cut sets (by assumption), Eq. (1.14) remains valid. Substituting Eq. (1.32) for **i** in Eq. (1.14), we obtain

$$\mathbf{A}\mathbf{g}(\mathbf{v}) = \mathbf{i}_s(t) \tag{1.33}$$

Substituting next Eq. (1.16) for **v** in Eq. (1.33), we obtain:

Nonlinear node equation

$$\mathbf{A}\mathbf{g}(\mathbf{A}^T\mathbf{e}) = \mathbf{i}_s(t) \tag{1.34}$$

For each solution **e** of Eq. (1.34), we can calculate the corresponding *branch voltage vector* **v** by direct substitution into Eq. (1.16), namely, $\mathbf{v} = \mathbf{A}^T\mathbf{e}$. This in turn can be used to calculate the *branch current vector* **i** by direct substitution into Eq. (1.32), namely, $\mathbf{i} = \mathbf{g}(\mathbf{v})$. Hence, the basic problem is to solve the nonlinear node equation (1.34). The rest is trivial.

In general, *nonlinear* equations do not have closed form solutions. Consequently, they must be solved by numerical techniques. In Sec. 3, we will study the most widely used method called the *Newton-Raphson algorithm*. Before we do this, however, let us first study how to formulate circuit equations in the most general case.

2 TABLEAU ANALYSIS FOR RESISTIVE CIRCUITS

The only, albeit major, shortcoming of *node analysis* is that it disallows many standard circuit elements from the class of allowable circuits, e.g., the voltage source, ideal transformer, ideal op amp, CCCS, CCVS, VCVS, current-controlled nonlinear resistor (such as neon bulb), etc. In this section, we overcome this objection by presenting a *completely general* analysis method—one that works for *all* resistive circuits. Conceptually, this method is simpler than node analysis: It consists of writing out the complete list of linearly independent KCL equations, linearly independent KVL equations, and the branch equations. For obvious reasons, this list of equations is called *tableau equations*.

Since no variables are eliminated (recall both **v** and **i** must be eliminated in node analysis, leaving **e** as the only variable) in listing the tableau equations, all three vectors **e**, **v**, and **i** are present as variables. Since we must have as many tableau equations as there are variables, it is clear that the price we pay for the increased generality is that the *tableau analysis* involves many more equations than *node analysis* does. In our era of computer-aided circuit analysis, however, this objection turns out to be a blessing in disguise because the matrix associated with tableau analysis is often extremely sparse, thereby allowing highly efficient numerical algorithms to be brought to bear.

The significance of tableau analysis actually transcends the above more mundane numerical considerations. As we will see over and over again in this book, tableau analysis is a powerful *analytic tool* which allows us to derive many profound results with almost no pain at all—at least compared to other approaches.

2.1 Tableau Equation Formulation: Linear Resistive Circuits

To write the tableau equation for any *linear* resistive circuit, we simply use the following algorithm:[7]

Step 1. Draw the digraph of the circuit and hinge it if necessary so that the resulting digraph is connected. Pick an arbitrary datum node and formulate the reduced-incidence matrix **A**.

Step 2. Write a complete set of linearly independent KCL equations:[8]

$$\mathbf{A}\mathbf{i}(t) = \mathbf{0} \tag{2.1}$$

Step 3. Write a complete set of linearly independent KVL equations:

$$\mathbf{v}(t) - \mathbf{A}^T\mathbf{e}(t) = \mathbf{0} \tag{2.2}$$

Step 4. Write the branch equations. Since the circuit is *linear*, these equations can always be recast into the form

$$\mathbf{M}(t)\mathbf{v}(t) + \mathbf{N}(t)\mathbf{i}(t) = \mathbf{u}_s(t) \tag{2.3}$$

Together, Eqs. (2.1), (2.2), and (2.3) constitute the tableau equations. If the digraph has n nodes and b branches, Eqs. (2.1), (2.2), and (2.3) will contain $n-1$, b, and b equations, respectively. Since the vectors **e**, **v**, and **i** also contain $n-1$, b, and b variables, respectively, the tableau equation for a linear resistive circuit always consists of $(n-1)+2b$ linear equations in $(n-1)+2b$ variables.

Example Consider the linear resistive circuit shown in Fig. 2.1*a*. It contains only three elements: a voltage source, an ideal transformer described by $v_1 = (n_1/n_2)v_2$ and $i_2 = -(n_1/n_2)i_1$, and a time-varying linear resistor described by $v_3(t) = R(t)i_3(t)$. Note that the first two elements are not allowed in node analysis because they are not voltage-controlled. The third

[7] The reader may wish to scan the following illustrative example after *each* step in order to get familiarized first with the notations used in writing the tableau equation.

[8] Note that unlike Eq. (1.14), tableau analysis deals with the *original* digraph where each independent current source is represented by a branch.

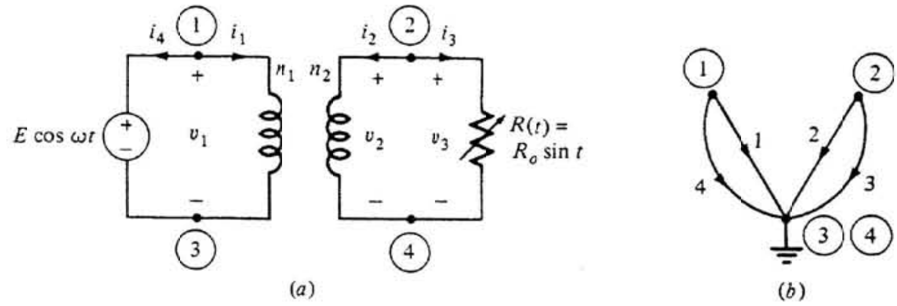

Figure 2.1 All three elements in this circuit are disallowed in node analysis.

element, which would normally be acceptable, is also disallowed here because its conductance $G(t) = 1/(R_o \sin t) = \infty$, at $t = 0, 2\pi, 4\pi, \ldots$, and is therefore not defined for all t.

Applying the preceding recipe, we hinge nodes ③ and ④ and draw the *connected* digraph shown in Fig. 2.1*b*. Choosing the hinged node as datum, the tableau equation is formulated as follows:

KCL: $$\mathbf{AI} = \mathbf{0} \Leftrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}}_{\mathbf{i}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\mathbf{0}} \tag{2.1$'$}$$

KVL: $$\mathbf{v} - \mathbf{A}^T\mathbf{e} = \mathbf{0} \Leftrightarrow \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}} - \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}}_{\mathbf{A}^T} \underbrace{\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}}_{\mathbf{e}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{0}} \tag{2.2$'$}$$

Branch equations:

$$\left.\begin{aligned} n_2 v_1 - n_1 v_2 &= 0 \\ n_1 i_1 + n_2 i_2 &= 0 \\ v_3 - R(t) i_3 &= 0 \\ v_4 &= E \cos \omega t \end{aligned}\right\} \Leftrightarrow \underbrace{\begin{bmatrix} n_2 & -n_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{M}(t)} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}}_{\mathbf{v}}$$

$$+ \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ n_1 & n_2 & 0 & 0 \\ 0 & 0 & -R(t) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{N}(t)} \underbrace{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \end{bmatrix}}_{\mathbf{i}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ E \cos \omega t \end{bmatrix}}_{\mathbf{u}_s(t)} \tag{2.3$'$}$$

Note that $n = 3$ and $b = 4$ for the digraph in Fig. 2.1*b*. Consequently, we expect the tableau equation to contain $(n-1)+2b=10$ equations involving 10 variables, namely, $e_1, e_2, v_1, v_2, v_3, v_4, i_1, i_2, i_3$, and i_4. An inspection of Eqs. (2.1)′, (2.2)′, and (2.3)′ shows that indeed we have 10 equations involving only these 10 variables. Note that had it been possible to write the node equation for this circuit, we would have to solve only two equations in two variables.

Note also that the branch equation (2.3)′ is of the form stipulated in Eq. (2.3), where $\mathbf{M}(t) = \mathbf{M}$ is a *constant* matrix. The second matrix $\mathbf{N}(t)$ is a function of t in view of the entry $-R(t)$. Clearly, if the circuit is time-invariant, then both $\mathbf{M}(t)$ and $\mathbf{N}(t)$ in Eq. (2.3) will be constant matrices.

The vector $\mathbf{u}_s(t)$ on the right-hand side of Eq. (2.3) does *not* depend on any *variable* e_j, v_j, or i_j, and is therefore due only to *independent* voltage and current sources in the circuit. Consequently, element k of $\mathbf{u}_s(t)$ will be *zero* whenever branch k is *not* an *independent* source. Note that *controlled* source coefficients always appear in the matrices $\mathbf{M}(t)$ and/or $\mathbf{N}(t)$, never in $\mathbf{u}_s(t)$.

An inspection of Eq. (2.3)′ reveals that each *row* k of $\mathbf{M}(t)$ and $\mathbf{N}(t)$ contains coefficients or time functions which define uniquely the *linear relation between* v_k and i_k of *branch* k in the digraph, assuming branch k corresponds to a resistor. If branch k happens to be an *independent* source, then the kth diagonal element is equal to *one* in $\mathbf{M}(t)$ (for a *voltage* source) or $\mathbf{N}(t)$ (for a current source), while *all other* elements in row k are zeros. In this case, the kth element of $\mathbf{u}_s(t)$ will contain either a constant (for a dc source) or a time function which specifies uniquely this independent source. On the other hand, if branch k is *not* an *independent* source, then the kth element of $\mathbf{u}_s(t)$ is always zero. It follows from the above interpretation that both $\mathbf{M}(t)$ and $\mathbf{N}(t)$ are $b \times b$ matrices and $\mathbf{u}_s(t)$ is a $b \times 1$ vector, where b is the number of branches in the digraph.

Finally, note that we can state that a *resistive circuit* is *linear* iff its branch equations can be written in the form stipulated in Eq. (2.3), and that it is *time-invariant* iff both $\mathbf{M}(t)$ and $\mathbf{N}(t)$ are constant real matrices.

The tableau matrix Since Eqs. (2.1)′, (2.2)′, and (2.3)′ which constitute the tableau equation consist of a system of *linear* equations, it is convenient and more illuminating to rewrite them as a single matrix equation; Eq. (2.4)′, page 229.

In the general case, Eqs. (2.1), (2.2), and (2.3) can be recast into the following compact matrix form, where **0** and **1** denote a *zero* and a *unit* matrix of appropriate dimension, respectively as shown in Eq. (2.4), page 229.

It is natural to call $\mathbf{T}(t)$ the *tableau matrix* associated with the linear resistive circuit. If the circuit is time-invariant, the tableau matrix $\mathbf{T}(t) = \mathbf{T}$ is a *constant* real matrix.

Every linear resistive circuit is associated with a *unique* $[(n-1)+2b] \times$

$$\underbrace{\left[\begin{array}{cc:cccc:cccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
\hdashline
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & n_2 & -n_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & n_1 & n_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -R(t) & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{array}\right]}_{\mathbf{T}(t)}
\underbrace{\left[\begin{array}{c}
e_1 \\ e_2 \\ \hdashline v_1 \\ v_2 \\ v_3 \\ v_4 \\ \hdashline i_1 \\ i_2 \\ i_3 \\ i_4
\end{array}\right]}_{\mathbf{w}(t)}
=
\underbrace{\left[\begin{array}{c}
0 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \\ E\cos\omega t
\end{array}\right]}_{\mathbf{u}(t)}
\begin{array}{l} \\ \\ \\ \\ \\ \\ \left.\begin{array}{c} \\ \\ \\ \\ \end{array}\right\} \mathbf{u}_s(t) \end{array}
\qquad (2.4)'$$

Linear tableau equation

$$\underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{A} \\ -\mathbf{A}^T & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}(t) & \mathbf{N}(t) \end{bmatrix}}_{\mathbf{T}(t)} \underbrace{\begin{bmatrix} \mathbf{e}(t) \\ \mathbf{v}(t) \\ \mathbf{i}(t) \end{bmatrix}}_{\mathbf{w}(t)} = \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{u}_s(t) \end{bmatrix}}_{\mathbf{u}(t)} \qquad (2.4)$$

$[(n-1)+2b]$ square tableau matrix $\mathbf{T}(t)$, and a *unique* $[(n-1)+2b]\times 1$ vector $\mathbf{u}(t)$.[9] The following result testifies to the significance of the tableau matrix:

Existence and uniqueness theorem: Linear resistive circuits A linear resistive circuit has a *unique* solution at any time t_0 if and only if

$$\det[\mathbf{T}(t_0)] \neq 0 \qquad (2.5)$$

PROOF The *inverse* matrix $\mathbf{T}^{-1}(t_0)$ exists at time t_0 if and only if $\det[\mathbf{T}(t_0)] \neq 0$. Hence, the solution at $t = t_0$ is given *uniquely* by

$$\mathbf{w}(t_0) = \mathbf{T}^{-1}(t_0)\mathbf{u}(t_0) \qquad (2.6)$$

∎

REMARK The linear tableau equation (2.4) also holds for circuits containing $(n+1)$-terminal or n-port resistors described by (see *Exercise 6*):

$$\mathbf{Av} + \mathbf{Bi} + \mathbf{c} = \mathbf{0}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices and $\mathbf{c}$ is an n-vector. Such an element is called an *affine resistor*.[10]

Note that when $\mathbf{c} = \mathbf{0}$, an *affine* resistor reduces to a *linear* resistor. Hence, a two-terminal affine (but *not* linear) resistor is characterized by a *straight line* in the v-i plane which does *not* include the origin.

Affine resistors arise naturally in the analysis of *nonlinear* resistive circuits in Sec. 3.

[9] The "uniqueness" is of course relative to a particular choice of element and node numbers.

[10] In mathematics, $\mathbf{f}(\mathbf{v}, \mathbf{i}) \triangleq \mathbf{Av} + \mathbf{Bi} + \mathbf{c}$ is called an *affine function* if $\mathbf{c} \neq \mathbf{0}$, and a *linear function* if $\mathbf{c} = \mathbf{0}$.

Exercise 1 Write the tableau equation for the circuit in Fig. 1.2, and identify the tableau matrix $\mathbf{T}(t)$.

Exercise 2 Specify the dimension of each submatrix in the tableau matrix $\mathbf{T}(t)$ in Eq. (2.4) for a connected digraph with n nodes and b branches.

Exercise 3 Write the tableau equation for the subclass of linear resistive circuits studied in Sec. 1.1; i.e., specify $\mathbf{M}(t)$, $\mathbf{N}(t)$, and $\mathbf{u}_s(t)$ in terms of $\mathbf{Y}_b(t)$ and $\mathbf{i}_s(t)$ in Eq. (1.17).

Exercise 4 (*a*) Show that the circuit in Fig. 2.1*a* does *not* have a unique solution at $t_0 = 0, 2\pi, 4\pi, \ldots, m2\pi$ for any integer m, by showing $\det[\mathbf{T}(t_0)] = 0$ for the tableau matrix in Eq. (2.4)′.

(*b*) Give a circuit interpretation which explains why the circuit in Fig. 2.1*a* does *not* have a unique solution at the above time instants. Do this for the two cases $E \neq 0$ and $E = 0$, respectively.

(*c*) Show that the circuit has a unique solution if $R(t) \neq 0$ *by inspection*. *Hint*: Start with the determinant expansion on the last column.

Exercise 5 (*a*) Replace the ideal transformer in Fig. 2.1*a* by a two-port resistor described by

$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 6 \end{bmatrix}$$

Write the tableau equation in the form of Eq. (2.4). Explain why the first two elements of $\mathbf{u}_s(t)$ are no longer zero in this case.

(*b*) Show that the above two-port resistor can be realized by a *linear* two-port resistor with port 1 in parallel with a 4-A current source and port 2 in series with a 6-V voltage source.

Exercise 6 Show that the tableau equation (2.4) remains unchanged if the circuit considered in this section also includes *affine* resistors. Explain why an entry k of the vector $\mathbf{u}_s(t)$ need *not* be zero in this case even if the corresponding branch k is not an independent source.

2.2 Tableau Equation Formulation: Nonlinear Resistive Circuits

Exactly the same principle is used to formulate the tableau equation for *nonlinear* resistive circuits: Simply list the linearly independent KCL and KVL equations, and the branch equations, which are now nonlinear. Hence, the first three steps of the algorithm at the beginning of Sec. 2.1 remain unchanged. Only Step 4 needs to be modified because Eq. (2.3) is valid only for *linear* resistive circuits. The following example will illustrate and suggest the modified form of Eq. (2.3).