CHAPTER

THREE

MULTITERMINAL RESISTORS

While the conventional resistor is probably the most familiar circuit element, the transistor is certainly the most useful electronic device. A transistor is a three-terminal device which behaves like a two-terminal *resistor* when viewed from any pair of terminals at dc. This is why its inventors (Nobel laureates Bardeen, Bratten, and Shockley) christened it a "*transfer resistor*," or transistor in brief. Numerous n-terminal electronic devices available commercially also behave like n-terminal *resistors* at dc and low-frequency operations. The purpose of this chapter is to generalize our study from two-terminal resistors to three-terminal and multiterminal resistors.

A resistive two-port is a two-port which is made of resistive elements only. We will start this chapter with the analysis of a simple linear resistive two-port. Recall that a three-terminal element can be viewed as a grounded two-port. Thus by studying resistive two-ports we automatically bring three-terminal resistors into consideration. Both are specified in terms of a set of two voltages and a set of two currents. Therefore, we need to generalize the v-i characteristic of a two-terminal resistor (or a resistive one-port) to a vector relation among the two voltages and two currents of a three-terminal resistor (or a resistive two-port).

We will introduce some ideal two-port elements which are especially useful in modeling. We will present the dc characteristics of the bipolar and MOS transistors and illustrate them with examples. We conclude the chapter with brief discussions of two useful resistive three-ports and a four-terminal element. A more general discussion of two-ports and n-ports will be given in Chap. 13.

1 RESISTIVE TWO-PORTS

Figure 1.1 shows a three-terminal element with node ③ chosen as the datum node. There are two (terminal) voltages v_1 and v_2 and two (terminal) currents i_1 and i_2, and these are the circuit variables describing the three-terminal element. Figure 1.2 shows a two-port with port voltages v_1 and v_2 and port currents i_1 and i_2 as its circuit variables. Whereas three-terminal resistors often pertain to an intrinsic device in practice, two-port resistors are usually made up of an interconnection of *resistive* elements (e.g., two-terminal resistors, three-terminal resistors, etc.). Therefore, for convenience, we often speak of resistive two-ports in this and succeeding sections instead of three-terminal resistors.

The generalization from a two-terminal resistor to a resistive two-port amounts to extending from scalar voltage and current variables to two-dimensional vector voltage and current variables. In other words, the v-i characteristic of a resistive one-port is generalized to a relation between two vectors: the *port voltage vector* $\mathbf{v}$ and the *port current vector* $\mathbf{i}$, where

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad \text{and} \quad \mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

and we need *two* equations in general to express the relation.

A *three-terminal* element or a *two-port* will be called a (time-invariant) resistor if its port voltages and port currents satisfy the following relation:

$$\mathcal{R}_R = \{(v_1, v_2, i_1, i_2): f_1(v_1, v_2, i_1, i_2) = 0 \text{ and } f_2(v_1, v_2, i_1, i_2) = 0\} \qquad (1.1)$$

This relation,[1] similar to the two-terminal resistor given by Eq. (1.2) in Chap. 2, will be called the *v-i characteristic of a three-terminal resistor or a resistive two-port*. The difference is that we need *two* scalar functions $f_1(\cdot)$ and $f_2(\cdot)$ to characterize a two-port and there are four scalar variables v_1, v_2, i_1, and i_2; the characteristic is in general a two-dimensional surface in a four-dimensional space.

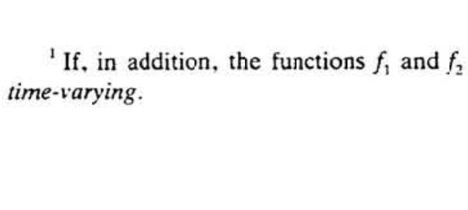


Figure 1.1 Three-terminal element with node ③ chosen as the datum node.

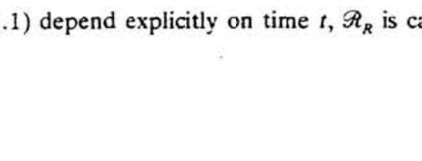


Figure 1.2 A two-port with its port voltage v_1, v_2 and port currents i_1, i_2.

[1] If, in addition, the functions f_1 and f_2 in Eq. (1.1) depend explicitly on time t, $\mathcal{R}_R$ is called *time-varying*.

When we deal with two-ports, we often need to distinguish the ports, so one of them is marked port 1 and the other is marked port 2 as shown in Fig. 1.2. As a tradition port 1 is often referred to as the *input port* and port 2 is often referred to as the *output port*.

In the following we will first consider linear resistors and use them to bring out pertinent concepts in the generalization from a one-port to a two-port.

1.1 A Linear Resistive Two-Port Example

Consider a resistive two-port made up of three linear resistors as shown in Fig. 1.3. Let us apply two independent current sources to the two-port as shown. KCL applied to nodes ①, ②, and ③ yields

$$i_{s1} = i_1 \qquad i_{s2} = i_2 \qquad i_3 = i_1 + i_2$$

Using Ohm's law and KVL for node sequences ①-③-④-① and ②-③-④-②, we obtain the two equations characterizing the resistive two-port:

$$v_1 = R_1 i_1 + R_3(i_1 + i_2) = (R_1 + R_3)i_1 + R_3 i_2 \tag{1.2a}$$

$$v_2 = R_2 i_2 + R_3(i_1 + i_2) = R_3 i_1 + (R_2 + R_3)i_2 \tag{1.2b}$$

In terms of the port voltage vector and the port current vector, we can rewrite the above equations in matrix form as

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \mathbf{Ri} = \begin{bmatrix} R_1 + R_3 & R_3 \\ R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{1.3}$$

where

$$\mathbf{R} \triangleq \begin{bmatrix} R_1 + R_3 & R_3 \\ R_3 & R_2 + R_3 \end{bmatrix} \tag{1.4}$$

is called the *resistance matrix* of the *linear* resistive two-port. It is linear because $\mathbf{v} = \mathbf{Ri}$ expresses $\mathbf{v}$ as a linear function of $\mathbf{i}$.[2] Equation (1.3) gives the

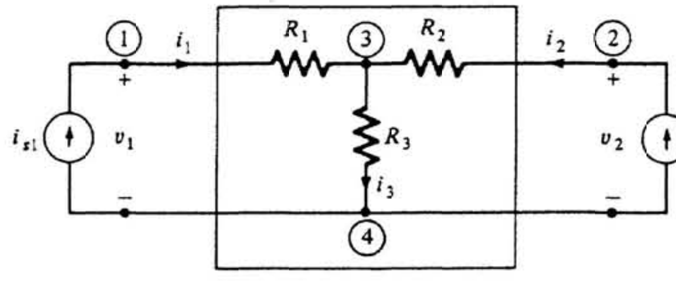


Figure 1.3 A linear resistive two-port.

[2] As will be seen later, a two-port containing independent sources and linear circuit elements is defined as a linear two-port. This is similar to the definition of a linear circuit. If a linear two-port contains independent sources, it will have an *affine* representation. This is discussed in Chap. 5, Sec. 4.

current-controlled representation of the linear resistive two-port because the voltages are expressed as functions of currents. The vector equation $\mathbf{v} = \mathbf{Ri}$ represents two linear constraints imposed by the two-port on the four variables v_1, v_2, i_1, and i_2. [See Eq. (1.1).]

In the circuit in Fig. 1.3, the two currents $i_1 = i_{s1}$ and $i_2 = i_{s2}$ are the *sources* and the two voltages v_1 and v_2 are the *responses*. Thus i_1 and i_2 in Eqs. (1.2*a* and *b*) are the independent variables and v_1 and v_2 are the dependent variables. Of course, we can also solve for i_1 and i_2 in terms of v_1 and v_2 from the two equations (1.2*a* and *b*), or directly from the vector equation (1.3) to obtain

$$\mathbf{i} = \mathbf{Gv} \tag{1.5}$$

where
$$\mathbf{G} \triangleq \mathbf{R}^{-1} = \frac{1}{R_1R_2 + R_2R_3 + R_3R_1}\begin{bmatrix} R_2 + R_3 & -R_3 \\ -R_3 & R_1 + R_3 \end{bmatrix} \tag{1.6}$$

is called the *conductance matrix* of the *linear* resistive two-port. In scalar form, Eqs. (1.5) and (1.6) can be written as

$$i_1 = \frac{R_2 + R_3}{R_1R_2 + R_2R_3 + R_3R_1}\, v_1 - \frac{R_3}{R_1R_2 + R_2R_3 + R_3R_1}\, v_2 \tag{1.7}$$

$$i_2 = \frac{-R_3}{R_1R_2 + R_2R_3 + R_3R_1}\, v_1 + \frac{R_1 + R_3}{R_1R_2 + R_2R_3 + R_3R_1}\, v_2 \tag{1.8}$$

The two equations above give an alternative representation of the same two-port. It is called the *voltage-controlled representation*. If we view the resistance matrix $\mathbf{R}$ of a two-port as the generalization of the resistance R of a linear two-terminal resistor, then the conductance matrix $\mathbf{G} = \mathbf{R}^{-1}$ of the same two-port is the generalization of the conductance $G = 1/R$ of the same two-terminal resistor.

1.2 Six Representations

With four scalar variables v_1, v_2, i_1, and i_2 and two equations to characterize a resistive two-port, there exist other representations besides the two just introduced. Since in most instances we may choose any two of the four variables as independent variables, the remaining two are then the dependent variables. Thus, altogether there are $C_2^4 = (4 \times 3)/2 = 6$ possibilities. Table 3.1 gives the classification of the six representations according to independent and dependent variables. Table 3.2 gives the equations of the six possible representations of a linear resistive two-port. As pointed out in Sec. 1.1, $\mathbf{G}$ is the inverse matrix of $\mathbf{R}$. Similarly, from Table 3.2, we also have $\mathbf{H}' = \mathbf{H}^{-1}$ and $\mathbf{T}' = \mathbf{T}^{-1}$. We call $\mathbf{H}$ and $\mathbf{H}'$ the *hybrid matrices* because both the dependent and independent variables are mixtures of a voltage and a current. We call $\mathbf{T}$ and $\mathbf{T}'$ the *transmission matrices* because they relate the variables pertaining to one port to that pertaining to the other and the two-port serves as a

Table 3.1 Six representations of a two-port

Representations	Independent variables	Dependent variables
Current-controlled	i_1, i_2	v_1, v_2
Voltage-controlled	v_1, v_2	i_1, i_2
Hybrid 1	i_1, v_2	v_1, i_2
Hybrid 2	v_1, i_2	i_1, v_2
Transmission 1	v_2, i_2	v_1, i_1
Transmission 2	v_1, i_1	v_2, i_2

Table 3.2 Equations for the six representations of a linear resistive two-port

Representations	Scalar equations	Vector equations
Current-controlled	$v_1 = r_{11}i_1 + r_{12}i_2$ $v_2 = r_{21}i_1 + r_{22}i_2$	$\mathbf{v} = \mathbf{R}\mathbf{i}$
Voltage-controlled	$i_1 = g_{11}v_1 + g_{12}v_2$ $i_2 = g_{21}v_1 + g_{22}v_2$	$\mathbf{i} = \mathbf{G}\mathbf{v}$
Hybrid 1	$v_1 = h_{11}i_1 + h_{12}v_2$ $i_2 = h_{21}i_1 + h_{22}v_2$	$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \mathbf{H}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$
Hybrid 2	$i_1 = h'_{11}v_1 + h'_{12}i_2$ $v_2 = h'_{21}v_1 + h'_{22}i_2$	$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \mathbf{H}'\begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$
Transmission 1†	$v_1 = t_{11}v_2 - t_{12}i_2$ $i_1 = t_{21}v_2 - t_{22}i_2$	$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = \mathbf{T}\begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$
Transmission 2†	$v_2 = t'_{11}v_1 + t'_{12}i_1$ $-i_2 = t'_{21}v_1 + t'_{22}i_1$	$\begin{bmatrix} v_2 \\ -i_2 \end{bmatrix} = \mathbf{T}'\begin{bmatrix} v_1 \\ i_1 \end{bmatrix}$

† For historical reasons, a minus sign is used in conjunction with i_2. Because of the reference direction chosen for i_2, $-i_2$ gives the current leaving the output port.

transmission media. Transmission matrices are important in the study of communication networks and will be treated in Chap. 13.

Example Consider the two-port in Fig. 1.3. Let $R_1 = 1\,\Omega$, $R_2 = 2\,\Omega$, and $R_3 = 3\,\Omega$. Equations (1.2*a* and *b*) give the following current-controlled representation:

$$v_1 = 4i_1 + 3i_2 \tag{1.9}$$

$$v_2 = 3i_1 + 5i_2 \tag{1.10}$$

The voltage-controlled representation is given by Eqs. (1.7) and (1.8):

$$i_1 = \tfrac{5}{11}v_1 - \tfrac{3}{11}v_2 \tag{1.11}$$

$$i_2 = -\tfrac{3}{11}v_1 + \tfrac{4}{11}v_2 \tag{1.12}$$

It is straightforward to derive the other four representations from the equations above. The general treatment will be given in Chap. 13; however, it is easy to obtain, for example, the hybrid representations: Using Eqs. (1.10) and (1.11) and solving for i_1 in terms of v_1 and i_2, we have

$$i_1 = \tfrac{1}{4}v_1 - \tfrac{3}{4}i_2 \tag{1.13}$$

which is the first equation of the hybrid 2 representation. Similarly, we obtain the second equation

$$v_2 = \tfrac{3}{4}v_1 + \tfrac{11}{4}i_2 \tag{1.14}$$

Equations (1.13) and (1.14) define the hybrid 2 matrix

$$\mathbf{H}' = \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} \end{bmatrix}$$

The hybrid 1 matrix can be obtained by simply taking the inverse of $\mathbf{H}'$, thus

$$\mathbf{H} = \mathbf{H}'^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{3}{4} & \frac{11}{4} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{11}{5} & \frac{3}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{bmatrix}$$

Exercise Determine the transmission 1 and the transmission 2 reptions of the resistive two-port shown in Fig. 1.3.

1.3 Physical Interpretations

In the example in Sec. 1.1, we derived the current-controlled representation by using two current sources at the two ports and determining the two port voltages. This is shown in Fig. 1.4*a*. We can similarly interpret the voltage-controlled and the two hybrid representations by using appropriate sources as shown in Fig. 1.4*b*, *c*, and *d*. It is seen that in the two hybrid representations, we use a current source and a voltage source as inputs, thus the responses are a voltage and a current.

Current-controlled representation In Chap. 2 we defined a linear two-terminal resistor as one having a straight line characteristic passing through the origin in the v-i plane. For two-ports we have four variables and two equations, e.g., the current-controlled representation is

$$\begin{aligned} v_1 &= r_{11}i_1 + r_{12}i_2 \\ v_2 &= r_{21}i_1 + r_{22}i_2 \end{aligned} \tag{1.15}$$

These two equations impose two linear constraints on the port voltages and the port currents and hence the point representing the four variables; namely,

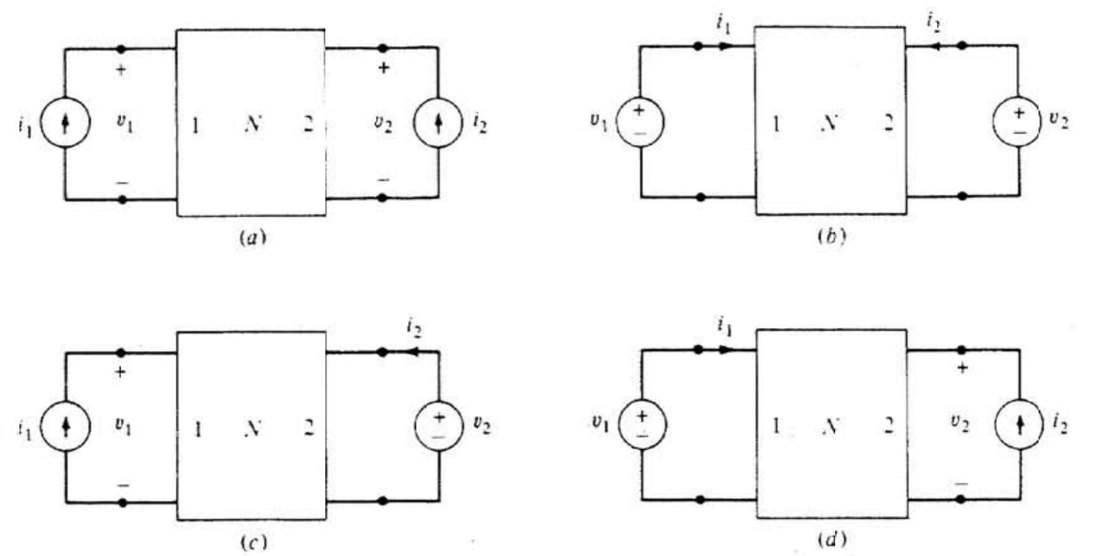


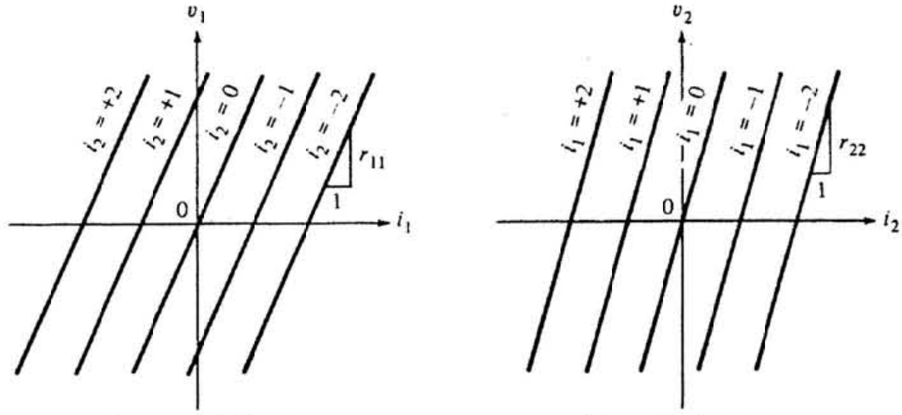
Figure 1.4 Sources and responses to two-ports for (*a*) current-controlled representation, (*b*) voltage-controlled representation, (*c*) hybrid 1 representation, and (*d*) hybrid 2 representation.

(v_1, v_2, i_1, i_2), is constrained to a two-dimensional subspace in the four-dimensional space spanned by v_1, v_2, i_1, i_2. Of course this is difficult to visualize. However, if we take one equation at a time, we can represent it by a family of curves in the appropriate *i-v* planes. Consider plotting, in the i_1-v_1 *plane*, the straight lines

$$v_1 = r_{11}i_1 + r_{12}i_2$$

where i_2 is considered as a *fixed parameter* which is given successively several values. The result is a family of parallel straight lines with a slope equal to r_{11}. These straight lines are as shown in Fig. 1.5*a*. Similarly, in Fig. 1.5*b* we plot, in the i_2-v_2 plane, the straight lines

$$v_2 = r_{21}i_1 + r_{22}i_2$$



(*a*) $v_1 = r_{11}i_1 + r_{12}i_2$ (*b*) $v_2 = r_{21}i_1 + r_{22}i_2$

Figure 1.5 Two-port characteristics plotted on the i_1-v_1 plane with i_2 as a parameter and on the i_2-v_2 plane with i_1 as a parameter.

and we use i_1 as a parameter. These two families of parallel straight lines in the i_1-v_1 plane and i_2-v_2 plane depict the current-controlled representation of the linear resistive two-port described by Eq. (1.15).

From the first equation in (1.15) we can give the following interpretations to r_{11} and r_{12}:

$$r_{11} = \left.\frac{v_1}{i_1}\right|_{i_2=0} \tag{1.16}$$

Thus r_{11} is called the *driving-point resistance at port* 1 when $i_2 = 0$, i.e., port 2 is kept open-circuited. Also, from the first equation of (1.15), if i_1 is set to zero, we obtain $v_1 = r_{12}i_2$. Thus the v_1-axis intercept of the $i_2 = 1$ characteristic in the i_1-v_1 plane is equal to r_{12}. Like Eq. (1.16), r_{12} can be interpreted by

$$r_{12} = \left.\frac{v_1}{i_2}\right|_{i_1=0} \tag{1.17}$$

which is called the *transfer resistance* when $i_1 = 0$, i.e., port 1 is kept open-circuited.

Similarly, from the second equation in (1.15), we can give the following interpretations:

$$r_{21} = \left.\frac{v_2}{i_1}\right|_{i_2=0} \tag{1.18}$$

$$r_{22} = \left.\frac{v_2}{i_2}\right|_{i_1=0} \tag{1.19}$$

where r_{22} is the *driving-point resistance at port* 1 when $i_1 = 0$, i.e., port 1 is kept open-circuited; and r_{21} is the *transfer resistance* when $i_2 = 0$, i.e., port 2 is kept open-circuited.

In Fig. 1.6 we give the physical interpretations of r_{11}, r_{12}, r_{21}, and r_{22} according to Eqs. (1.16) to (1.19). Note that in each case the input is a current source and the response is a voltage across a port which is open-circuited. Recalling, from Chap. 2, that an independent current source has infinite internal resistance, we call the resistance matrix **R** the *open-circuit resistance matrix* and the four parameters r_{11}, r_{12}, r_{21} and r_{22} *open-circuit resistance parameters* of the linear resistive two-port. More specifically, r_{11} and r_{22} are the *open-circuit driving-point resistances* at port 1 and port 2, respectively; r_{21} and r_{12} are the *open-circuit forward* and *reverse transfer resistances*, respectively.

It is easy to go through a dual treatment to give the corresponding interpretations for the voltage-controlled representation. We shall leave that as an exercise.

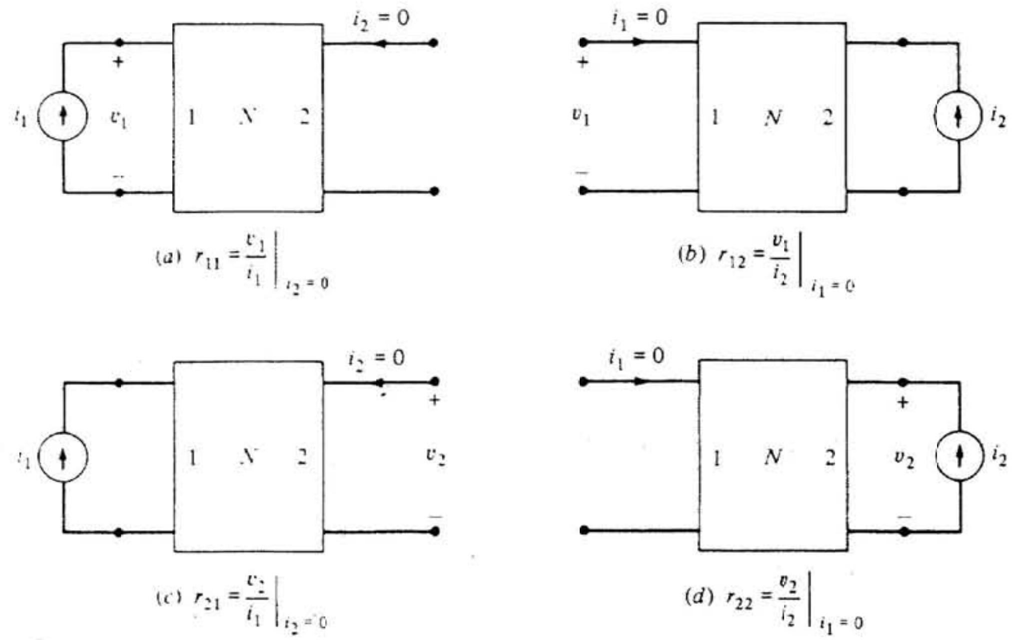


Figure 1.6 Interpretations of (*a*) r_{11}, (*b*) r_{12}, (*c*) r_{21}, and (*d*) r_{22}.

Exercise Give the physical interpretation of the voltage-controlled representation of a linear resistive two-port. Use Fig. 1.7 to interpret the *short-circuit conductances* g_{11}, g_{12}, g_{21}, and g_{22} in terms of port voltages and port currents.

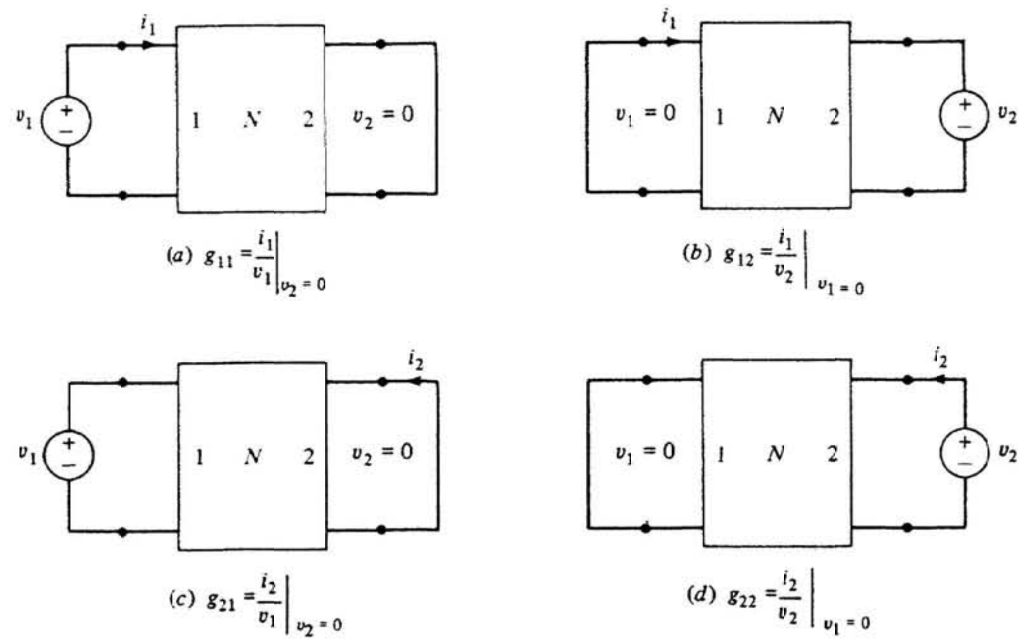


Figure 1.7 Interpretation of (*a*) g_{11}, (*b*) g_{12}, (*c*) g_{21}, and (*d*) g_{22}.

Hybrid representation The two equations for the hybrid 1 representation read:

$$v_1 = h_{11}i_1 + h_{12}v_2 \tag{1.20}$$

$$i_2 = h_{21}i_1 + h_{22}v_2 \tag{1.21}$$

Following the same treatment as the current-controlled representation, we write

$$h_{11} = \left.\frac{v_1}{i_1}\right|_{v_2=0} \tag{1.22}$$

$$h_{12} = \left.\frac{v_1}{v_2}\right|_{i_1=0} \tag{1.23}$$

$$h_{21} = \left.\frac{i_2}{i_1}\right|_{v_2=0} \tag{1.24}$$

$$h_{22} = \left.\frac{i_2}{v_2}\right|_{i_1=0} \tag{1.25}$$

The physical interpretation in terms of sources, responses, and external connections for the four hybrid parameters is given in Fig. 1.8.

Note that the four hybrid parameters h_{11}, h_{12}, h_{21}, and h_{22} represent a driving-point resistance, a reverse *voltage transfer ratio*, a forward *current transfer ratio*, and a driving-point conductance, respectively. Furthermore, in each case, the external connection for port 1 is either a current source or an open circuit, and the external connection for port 2 is either a voltage source

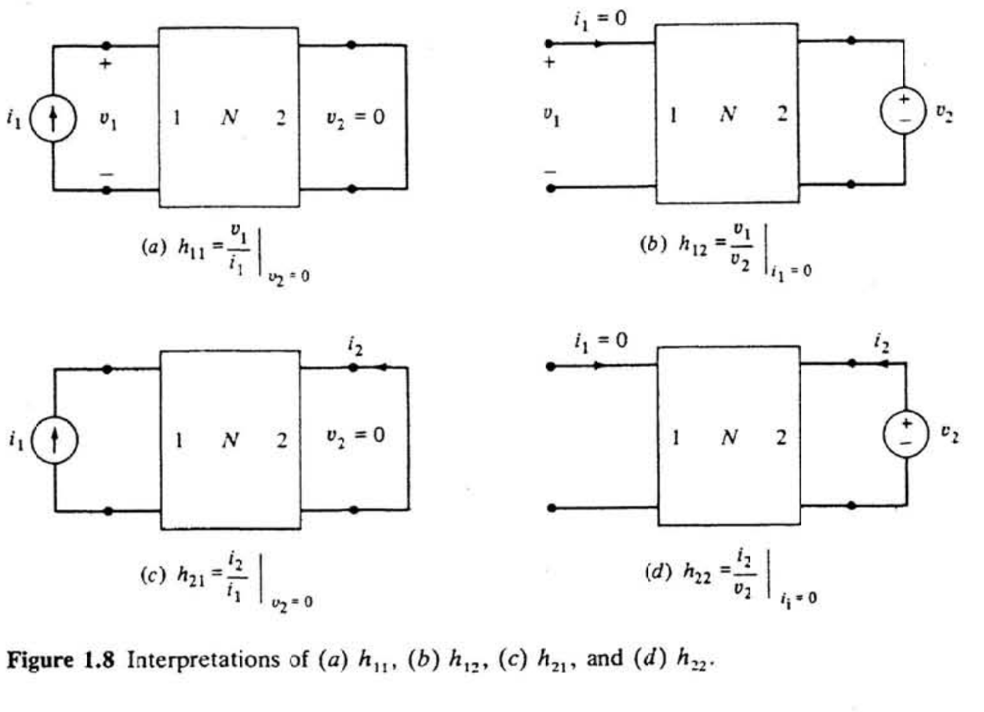


Figure 1.8 Interpretations of (*a*) h_{11}, (*b*) h_{12}, (*c*) h_{21}, and (*d*) h_{22}.

or a short circuit. As will be seen later in the chapter, hybrid representation is commonly used in dealing with the common-emitter configuration of the bipolar transistor.

2 USEFUL RESISTIVE TWO-PORTS

There are a number of (ideal) two-port circuit elements which are extremely useful in modeling and in exhibiting specific properties of devices. We will describe the most important ones in this section, namely, the linear controlled sources, the ideal transformer, and the gyrator. All of them are *linear* circuit elements and are characterized in terms of the four port variables v_1, v_2, i_1, and i_2. Thus they are resistive *two-ports* according to our definition.

2.1 Linear Controlled Sources

Up to this point we have encountered independent voltage sources and current sources. Independent sources are used as inputs to a circuit. In this section we will introduce another type of sources, called *controlled sources* or *dependent sources*. A *controlled source* is a resistive two-port element consisting of two branches: a primary branch which is either an open circuit or a short circuit and a secondary branch which is either a voltage source or a current source. The voltage or current waveform in the secondary branch is *controlled* by (or dependent upon) the voltage or current of the primary branch. Therefore, there exist four types of controlled sources depending on whether the primary branch is an open circuit or a short circuit and whether the secondary branch is a voltage source or a current source. The four types of controlled sources are shown in Fig. 2.1. They are the *current-controlled voltage source* (CCVS), the *voltage-controlled current source* (VCCS), the *current-controlled current source* (CCCS), and the *voltage-controlled voltage source* (VCVS). Note that we use a

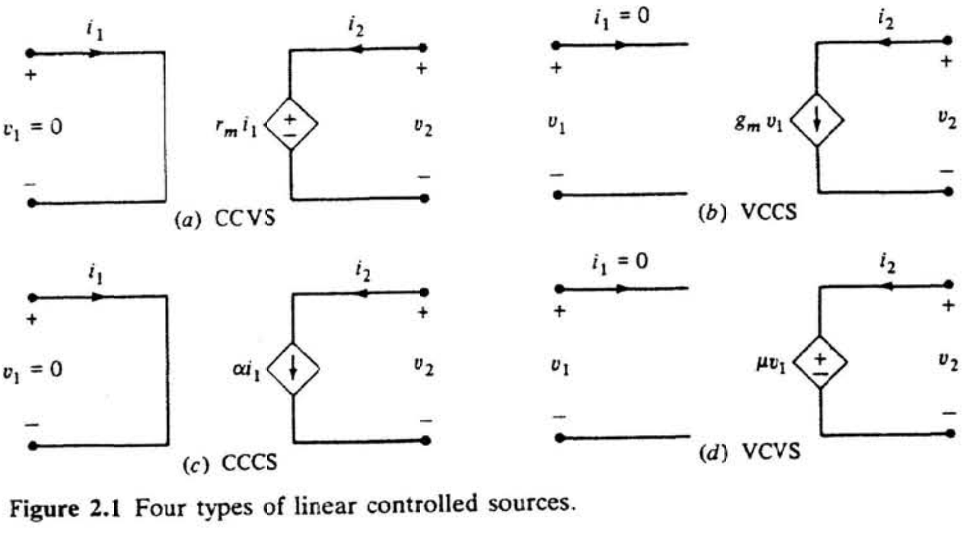


Figure 2.1 Four types of linear controlled sources.

diamond-shaped symbol to denote controlled sources. This is to differentiate them from the independent sources which, as we see repeatedly in the sequel, have very different properties. Each linear controlled source is characterized by two linear equations.

CCVS: $\quad v_1 = 0 \quad v_2 = r_m i_1$ (2.1)

VCCS: $\quad i_1 = 0 \quad i_2 = g_m v_1$ (2.2)

CCCS: $\quad v_1 = 0 \quad i_2 = \alpha i_1$ (2.3)

VCVS: $\quad i_1 = 0 \quad v_2 = \mu v_1$ (2.4)

where r_m is called the *transresistance*, g_m is called the *transconductance*, α is called the *current transfer ratio*, and μ is called the *voltage transfer ratio*. They are all constants, thus the four controlled sources are linear time-invariant two-port resistors. More generally, if a CCVS is characterized by the two equations: $v_1 = 0$ and $v_2 = f(i_1)$, where $f(\cdot)$ is a given *nonlinear* function, then that CCVS is a *nonlinear controlled source*. Similarly, if a CCCS is characterized by the two equations: $v_1 = 0$, $i_2 = \alpha(t) i_1$, where $\alpha(\cdot)$ is a given function of time, then this CCCS is a *linear time-varying controlled source*.

In the previous section we demonstrated with an example that a linear resistive two-port has six representations. In the case of linear controlled sources, Eqs. (2.1) to (2.4) can be put in matrix form each corresponding to one representation:

CCVS: $$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ r_m & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \tag{2.5}$$

VCCS: $$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ g_m & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{2.6}$$

CCCS: $$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix} \tag{2.7}$$

VCVS: $$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix} \tag{2.8}$$

In Eq. (2.5) we have the current-controlled representation for the CCVS. Since the resistance matrix is singular, its inverse does not exist; therefore, there is no voltage-controlled representation for a CCVS. As a matter of fact, it is easy to see that neither of the hybrid representations exist as well. We can make similar statements for the other three controlled sources, i.e., only one of the representations in the first four rows of Table 3.2 exists.

Linear controlled sources are ideal *coupling* circuit elements, yet they are extremely useful in modeling electronic devices and circuits. This will be illustrated later in Sec. 4 of the chapter. In Chap. 4 we will see that all four controlled sources can be realized physically (to a good approximation) by using operational amplifiers.

Equivalent circuits of linear resistive two-ports Controlled sources are useful in modeling resistive two-ports. Consider the current-controlled representation of a linear resistive two-port:

$$v_1 = r_{11}i_1 + r_{12}i_2 \tag{2.9}$$

$$v_2 = r_{21}i_1 + r_{22}i_2 \tag{2.10}$$

Equation (2.9) can be interpreted as a series connection of a linear resistor with resistance r_{11} and a CCVS whose voltage is dependent on the current i_2 at port 2. Similarly, Eq. (2.10) can be interpreted as a series connection of a linear resistor with resistance r_{22} and a CCVS whose voltage is dependent on the current i_1 at port 1. Therefore, we may use the equivalent circuit shown in Fig. 2.2 to represent the two-port. This equivalent circuit puts in evidence the meanings of the four parameters r_{11}, r_{12}, r_{21}, and r_{22} introduced earlier. The reader should review Eqs. (1.16) to (1.19) and use Fig. 2.2 to give appropriate interpretations of the four resistance parameters.

Exercises

1. Show that the circuit shown in Fig. 2.3 is the equivalent circuit of a voltage-controlled linear resistive two-port.
2. Show that the circuit shown in Fig. 2.4 is the equivalent circuit corresponding to the hybrid 1 representation of a linear resistive two-port.

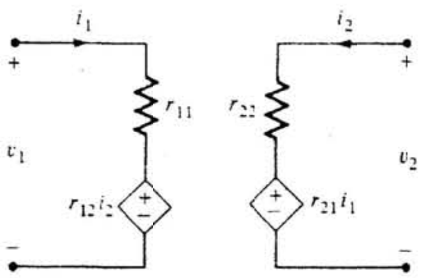


Figure 2.2 Equivalent circuit of a current-controlled linear resistive two-port.

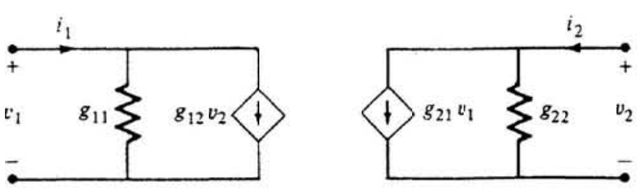


Figure 2.3 Equivalent circuit of a voltage-controlled linear resistive two-port.

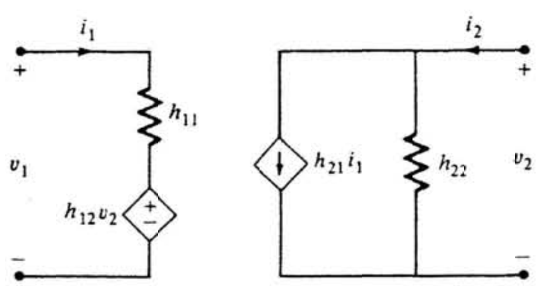


Figure 2.4 Equivalent circuit based on the hybrid 1 representation of a linear resistive two-port: h_{11} is a resistance (in ohms) and h_{22} is a conductance (in siemens).

In the following we will give three examples to illustrate additional properties of controlled sources.

Example 1 Consider the circuit shown in Fig. 2.5 where a battery with internal resistance R_b is connected to the primary branch of a VCVS. Since $i_1 = 0$ and $v_2 = \mu v_1 = \mu E_b$, the output behaves like an independent dc voltage source with zero internal resistance irrespective of the battery resistance R_b.

Example 2 In Fig. 2.6 the primary branch and the secondary branch of a CCVS are connected in such a fashion that the resulting one-port behaves like a linear resistor. Here $i_2 = i_1$ and $v_2 = r_m i_1 = r_m i_2$. Thus looking backwards from port 2, we have a linear resistor with resistance equal to r_m.

Exercise Modify the connection of the circuit in Fig. 2.6 so that the resistance seen at port 2 is $-r_m$.

Example 3 Consider the circuit in Fig. 2.7 where the CCCS has its two branches connected at a common node ① to a linear resistor with resistance R_1. An independent voltage source v_i serves as the input, and

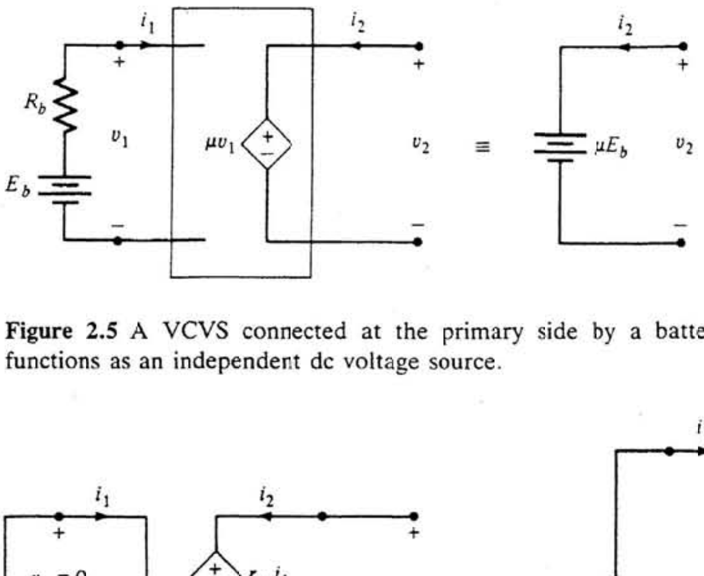


Figure 2.5 A VCVS connected at the primary side by a battery with internal resistance R_b functions as an independent dc voltage source.

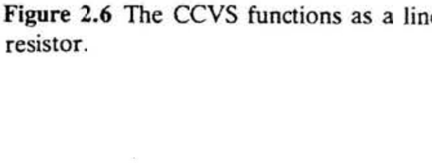


Figure 2.6 The CCVS functions as a linear resistor.

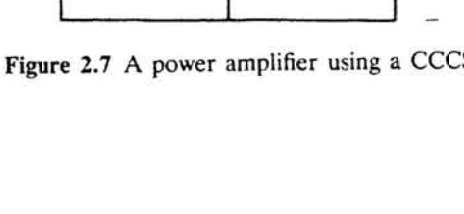


Figure 2.7 A power amplifier using a CCCS.

we wish to determine the output voltage v_o across the second linear resistor with resistance R_2 acting as a load. Using KCL at node ①, we have $i_3 = i_1 + i_2 = (1+\alpha)i_1$. Thus

$$v_i = R_1(1+\alpha)i_1$$

$$v_o = R_2(-i_2) = -R_2\alpha i_1$$

The power delivered by the source v_i to the circuit is

$$p_i = v_i i_1 = R_1(1+\alpha)i_1^2$$

The power delivered by the circuit to the load resistor R_2 is

$$p_o = v_o(-i_2) = R_2\alpha^2 i_1^2$$

Therefore the power gain is

$$\frac{p_o}{p_i} = \frac{\alpha^2}{1+\alpha}\frac{R_2}{R_1}$$

Clearly, by choosing R_1 and R_2 we can obtain an arbitrarily large gain for any given value of α. Thus controlled sources can be used in the design of a power amplifier.

Exercise Demonstrate that a nonlinear controlled source in Fig. 2.8*a* can be realized with the circuit shown in Fig. 2.8*b*, which contains a two-terminal nonlinear resistor, a CCCS, and a CCVS.

2.2 Ideal Transformer

The *ideal transformer* is an ideal two-port resistive circuit element which is characterized by the following two equations:

$$v_1 = nv_2 \tag{2.11}$$

and

$$i_2 = -ni_1 \tag{2.12}$$

where n is a real number called the *turns ratio*. The symbol for the ideal

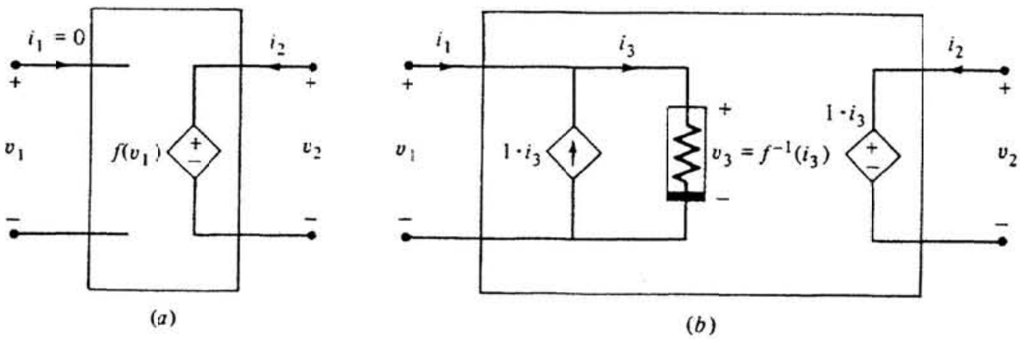


Figure 2.8 (*a*) A nonlinear VCVS and (*b*) its equivalent realization using linear controlled sources.

transformer is shown in Fig. 2.9. The ideal transformer is a *linear* resistive two-port since its equations impose *linear* constraints on its port voltages and port currents. Note that neither the current-controlled representation nor the voltage-controlled representation exists for the ideal transformer. Equations (2.11) and (2.12) can be written in matrix form in terms of the hybrid matrix representation:

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \mathbf{H}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix}\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} \tag{2.13}$$

The ideal transformer is an idealization of a physical transformer that is used in many applications. The properties of the physical transformer will be discussed in Chap. 8.

We wish to stress that because the ideal transformer is an ideal element defined by Eqs. (2.11) and (2.12), the relation between port voltages and between port currents holds for *all* waveforms and for *all* frequencies, including dc.

Two fundamental properties of the ideal transformer

1. The ideal transformer neither dissipates nor stores energy. Indeed, the power entering the two-port at time t from Eq. (2.13) is

$$p(t) = v_1(t)i_1(t) + v_2(t)i_2(t) = 0 \tag{2.14}$$

Thus, like the ideal diode, the ideal transformer is a *non-energic element.*

2. When an ideal transformer is terminated at the output port with an R-Ω linear resistor, the input port behaves as a linear resistor with resistance $n^2R\ \Omega$. This situation is shown in Fig. 2.10 where

$$v_2 = -Ri_2$$

Therefore

$$\frac{v_1}{i_1} = \frac{nv_2}{-i_2/n} = n^2R \tag{2.15}$$

is the resistance of the equivalent linear resistor at the input port.

Mechanical analog The ideal transformer is the electrical analog of an ideal pair of mechanical gears shown in Fig. 2.11. Since the velocity at A, the point

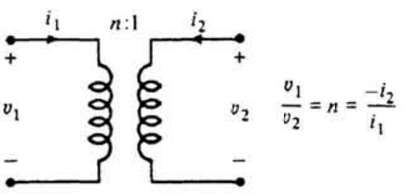


Figure 2.9 An ideal transformer defined by the single parameter n, the turns ratio.

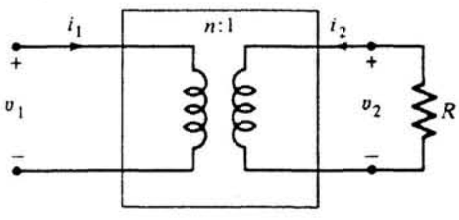


Figure 2.10 An ideal transformer terminated at the output port with a resistor.

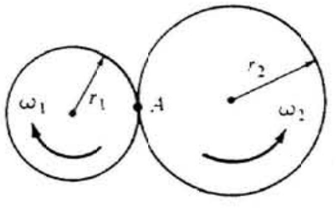


Figure 2.11 A pair of gears.

of contact of the gears, must be the same for both gears, we have

$$\omega_1 r_1 = \omega_2 r_2 = \text{velocity of } A$$

where ω_1 and ω_2 are the angular velocities of the two gears [radians per second (rad/s)], and r_1 and r_2 are the radii. On the other hand the forces applied by one gear to the other at point A must be equal in magnitude and opposite in direction, thus

$$\frac{\tau_1}{r_1} = -\frac{\tau_2}{r_2} = \text{force}$$

where τ_1 and τ_2 are the torques applied to the two gears.

Comparing the two defining equations for the ideal transformer with the two equations above for the pair of gears, we note the following analogs:

Electric circuits	Mechanical systems
Voltage v	Angular velocity ω
Current i	Torque τ
Turns ratio n	Radius ratio r_2/r_1

Mechanical analogs are helpful in understanding the property of electric circuits. In Chap. 7 we will encounter other analogous mechanical systems when we introduce other circuit elements: capacitors and inductors.

2.3 Gyrator

A *gyrator* is an ideal two-port element defined by the following equations:

$$i_1 = Gv_2 \tag{2.16}$$

and

$$i_2 = -Gv_1 \tag{2.17}$$

where the constant G is called the *gyration conductance*. In vector form we have the voltage-controlled representation:

$$\mathbf{i} = \begin{bmatrix} 0 & G \\ -G & 0 \end{bmatrix} \mathbf{v} \tag{2.18}$$

The symbol for a gyrator is shown in Fig. 2.12. It is easy to check that the ideal

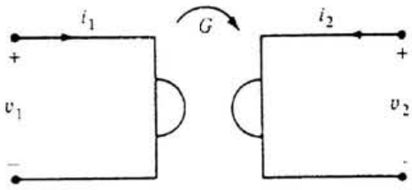


Figure 2.12 An ideal gyrator.

gyrator is also a non-energic element, i.e., at all times the power delivered to the two-port is identically zero.

The fundamental property of an ideal gyrator is given by the following equation:

$$\frac{v_1}{i_1} = \frac{-i_2/G}{Gv_2} = \frac{1}{G^2}\,\frac{-i_2}{v_2} \tag{2.19}$$

That is, when a gyrator is terminated at the output port with an R_L-Ω linear resistor as shown in Fig. 2.13, the input port behaves as a linear resistor with resistance G_L/G^2 Ω, where $G_L \triangleq 1/R_L$. As will be clear in Chap. 6, if the output port of an ideal gyrator is terminated with a capacitor, the input port behaves like an inductor. Thus a gyrator is a useful element in the design of inductorless filters.

Another interesting observation is the following: If the output port of a gyrator is connected to a current-controlled two-terminal resistor, i.e., $v_2 = f(-i_2)$, then the input port becomes a voltage-controlled resistor. For example, setting $G = 1$ in Eqs. (2.16) and (2.17), we easily obtain

$$i_1 = v_2 = f(-i_2) = f(v_1)$$

The resulting current-controlled resistor is then the dual of the original voltage-controlled resistor.

Physical gyrators which approximate the property of an ideal gyrator over low operating frequencies (below 10 kHz) are available commercially in the form of integrated circuit modules.

3 NONLINEAR RESISTIVE TWO-PORTS

In the previous two sections we discussed linear resistive two-ports and their various characterizations and properties. In the real world we need to deal with

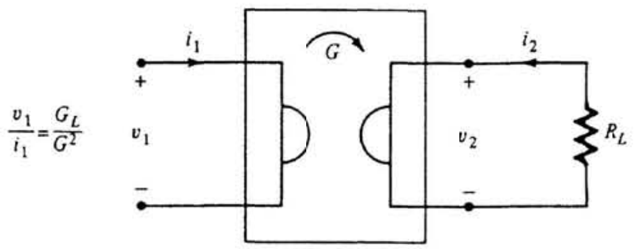


Figure 2.13 A gyrator terminated at the output port with a resistor.

nonlinear resistive two-ports and three-terminal devices such as transistors. Much of the material given in the previous two sections can be extended and generalized to the *nonlinear* case. This section illustrates the basic idea of nonlinear two-port representation with an example. Sections 4 and 5 will treat the modeling and characterizations of bipolar and MOS transistors together with dc analysis and the small-signal analysis.

Recall that the definition of a resistive two-port is expressed by Eq. (1.1) in terms of two functions and four variables. Thus, for example, the following two equations characterize a nonlinear two-port resistor:

$$\begin{aligned} f_1(v_1, v_2, i_1, i_2) &= i_1 + 2i_2 - (v_1 + v_2)^3 - 2(v_2 - v_1)^{1/3} = 0 \\ f_2(v_1, v_2, i_1, i_2) &= 2i_1 - i_2 - 2(v_1 + v_2)^3 + (v_2 - v_1)^{1/3} = 0 \end{aligned} \tag{3.1}$$

Similar to linear resistive two-ports, there are six possible explicit representations for nonlinear resistive two-ports, which express two variables in terms of the two others. This is a contrived example which has the remarkable property that we can find the analytic forms of all six representations. They are given as follows:

$$\begin{array}{ll} 1.\ v_1 = \frac{1}{2}(i_1^{1/3} - i_2^3) & 2.\ i_1 = (v_1 + v_2)^3 \\ \quad v_2 = \frac{1}{2}(i_2^3 + i_1^{1/3}) & \quad i_2 = (v_2 - v_1)^{1/3} \\ 3.\ v_1 = (i_1^{1/3} - v_2) & 4.\ i_1 = (2v_1 + i_2^3)^3 \\ \quad i_2 = (2v_2 - i_1^{1/3})^{1/3} & \quad v_2 = (i_2^3 + v_1) \\ 5.\ v_1 = (v_2 - i_2^3) & 6.\ v_2 = -v_1 + i_1^{1/3} \\ \quad i_1 = (-i_2^3 + 2v_2)^3 & \quad i_2 = (i_1^{1/3} - 2v_1)^{1/3} \end{array} \tag{3.2}$$

These are the six representations of the same nonlinear resistive two-port.

As in the case of the linear two-port (shown in Table 3.2), we summarize the six representations in Table 3.3.

Each of the six representations can be plotted as two families of curves, each parameterized by a third variable as shown in Fig. 3.1*a* through *f*, respectively.

Table 3.3 Equation for the six representations of a nonlinear resistive two-port

Current-controlled representation	Voltage-controlled representation
$v_1 = \hat{v}_1(i_1, i_2)$	$i_1 = \hat{i}_1(v_1, v_2)$
$v_2 = \hat{v}_2(i_1, i_2)$	$i_2 = \hat{i}_2(v_1, v_2)$
Hybrid 1 representation	Hybrid 2 representation
$v_1 = \hat{v}_1(i_1, v_2)$	$i_1 = \hat{i}_1(v_1, i_2)$
$i_2 = \hat{i}_2(i_1, v_2)$	$v_2 = \hat{v}_2(v_1, i_2)$
Transmission 1 representation	Transmission 2 representation
$v_1 = \hat{v}_1(v_2, -i_2)$	$v_2 = \hat{v}_2(v_1, i_1)$
$i_1 = \hat{i}_1(v_2, -i_2)$	$-i_2 = \hat{i}_2(v_1, i_1)$

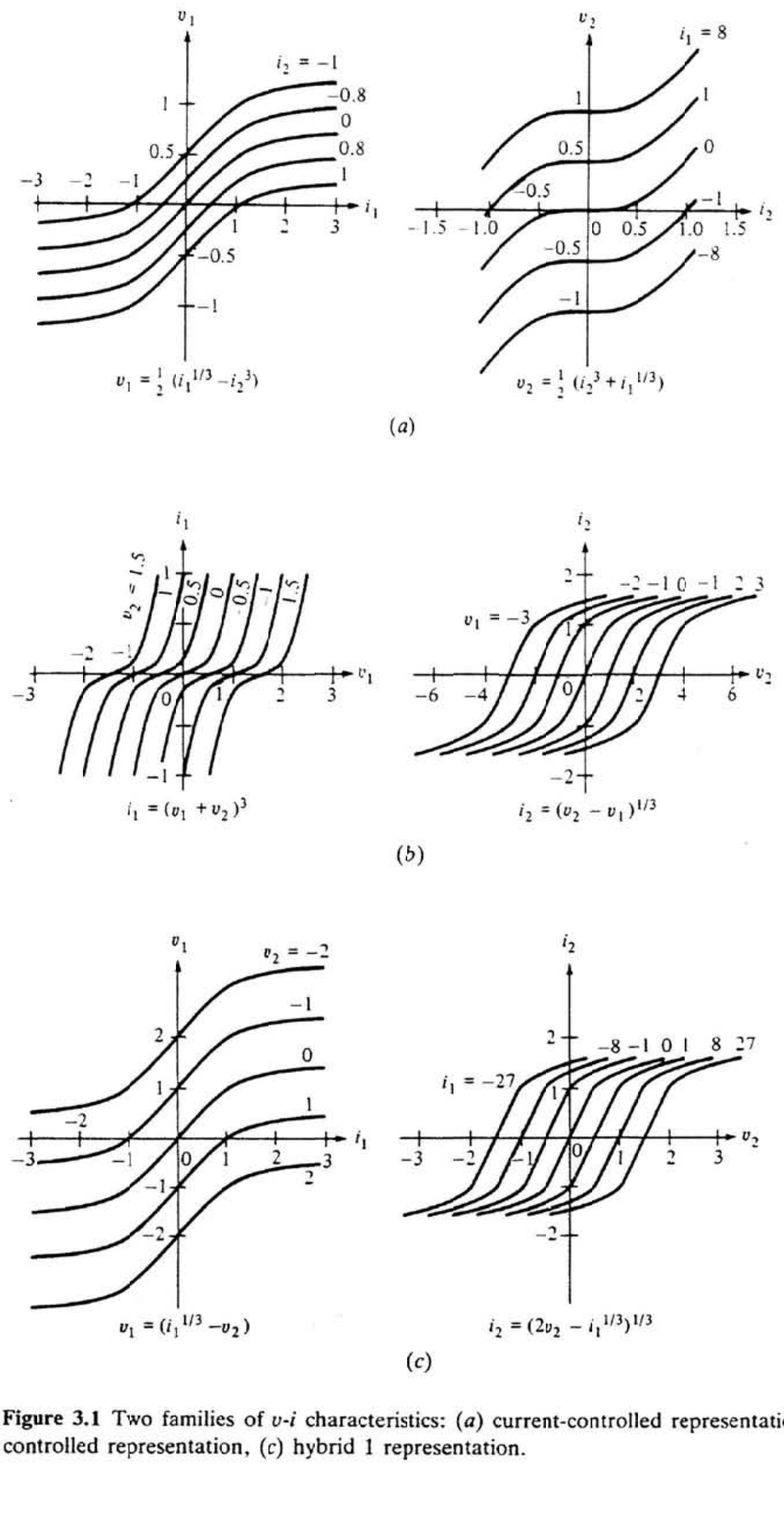
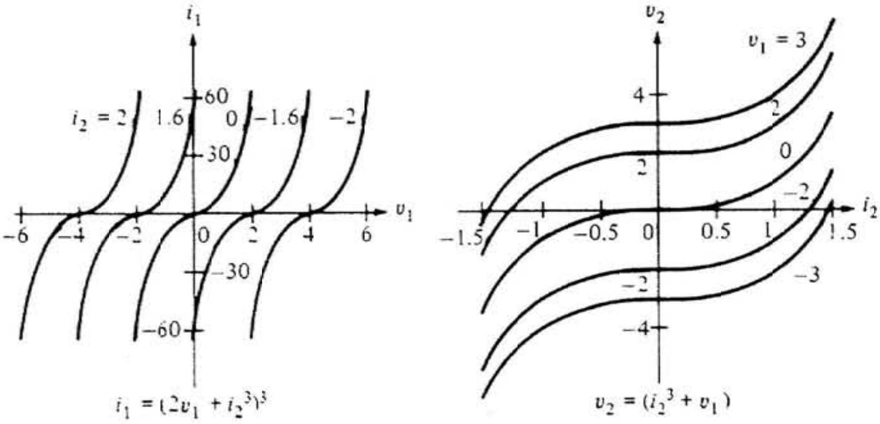
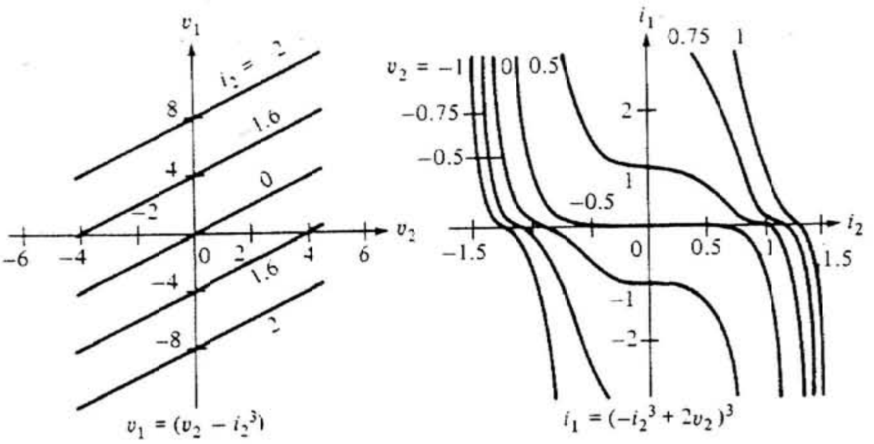


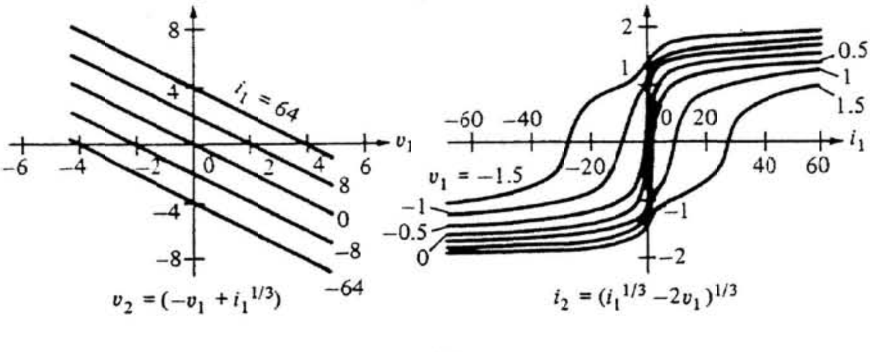
Figure 3.1 Two families of v-i characteristics: (*a*) current-controlled representation, (*b*) voltage-controlled representation, (*c*) hybrid 1 representation.



(d)



(e)



(f)

Figure 3.1 (*Continued*) Two families of v-i characteristics: (d) hydrid 2 representation, (e) transmission 1 representation, and (f) transmission 2 representation.