EE 121 - Introduction to Digital Communications
Final Exam practice problems

May 16, 2008

Note: some of these problems are of a suitable length for the exam and some are too long. All should be useful in preparing.

1. Consider a Markov chain $\{X_i\}$ where $X_i = 0$ or 1 and the transition probability from state 0 to 1 is the same as that from state 1 to 0, say α.

(a) What should be the initial distribution on X_1 such that the Markov chain is stationary? We will assume this distribution for the rest of the question.

(b) For what choice of α would $\{X_i\}$ be i.i.d.? State the weak law of large numbers for $\{X_i\}$ and prove it.

(c) Show now that for all values of α not equal to 0 or 1, the weak law of large numbers still holds for $\{X_i\}$. (Hint: mimic the proof in the i.i.d. case). What happens when $\alpha = 0$ and $\alpha = 1$?

2. Consider sending a single bit using repetition coding over an AGN channel over two time instants. There is an energy constraint of E Joules in each time instant. The Gaussian noise is independent from time to time but the variance is different: say σ_1^2 at time 1 and σ_2^2 at time 2.

(a) Derive the ML rule based on the two received voltages y_1 and y_2. Does the rule make intuitive sense?

(b) Compute the average error probability of the ML rule.

3. Consider transmitting 3 bits over two time instants on an additive Gaussian noise channel with independent noises at the two time instants. The 8 messages (3 bit sequences) correspond to points on a circle centered around the origin with radius $\sqrt{E}$ and angles

$\theta_1, \ldots, \theta_8$ with

$$\theta_i = \frac{(i-1)\pi}{4} \quad \text{radians}, \quad i = 1, \ldots, 8.$$

This modulation scheme is called *phase shift keying*. Sketch the constellation diagram and the ML decision regions.

4. Consider the detection problem of x given y_1 and y_2:

$$y_1 = x + z_1$$
$$y_2 = z_1 + z_2.$$

Here, x is equally likely to be $\sqrt{E}$ or $-\sqrt{E}$, and z_1, z_2 are i.i.d. $\mathcal{N}(0, \sigma^2)$ noise independent of x.

(a) Is y_1 a sufficient statistic for the detection problem? Give a rigorous as well as an intuitive explanation.

(b) Find the optimal detector of x given y_1 and y_2 and compute the probability of error.

(c) Consider now the best detector of x using y_1 only. Find the additional energy, if any, needed to achieve the same error probability as the optimal detector in (b).

5. Consider the L-tap ISI channel model:

$$y[n] = \sum_{l=1}^{L} h_l x[n-l] + w[n], \quad n \geq 1,$$

and suppose plain bit-by-bit signaling over this channel. So, $x[n]$ is i.i.d. in time n and equally likely to be $\pm\sqrt{E}$. As usual, $w[n]$ is additive white Gaussian noise (with energy per symbol equal to 2) and is independent of the input $x[n]$. So, the average energy of the input signal per symbol is E. Calculate the average energy, $\mathbb{E}[y^2[n]]$, of the output signal per symbol. Conclude from your calculation that ISI increases the average SNR of the channel, i.e., the more the ISI, the better the SNR is.

6. Consider the 2-tap ISI channel model:

$$y[n] = x[n] + \frac{1}{2}x[n-1] + w[n].$$

Suppose an infinite stream of symbols $\ldots, x[-1], x[0], x[1], \ldots$ is transmitted.

(a) Express the DTFT (Discrete-Time Fourier Transform) of $y[n]$ in terms of the DTFT of $x[n]$.

(b) Using part (a) or otherwise, derive an expression for the zero-forcing (ZF) equalizer $G_{\text{ZF}}[n]$. (Hint: the ZF equalizer satisfies

$$(G_{\text{ZF}} * y)[n] = x[n]$$

when the noise is zero, i.e. when $w[n] = 0$ for all n).

7. Consider the following 8-QAM signal set:

$$s_m(t) = g_T(t) \cos\left(2\pi f_c t + \frac{2\pi m}{8}\right), \qquad m = 1, 2, 3, 4,$$

and

$$s_m(t) = 2g_T(t) \cos\left(2\pi f_c t + \frac{2\pi m}{8}\right), \qquad m = 5, 6, 7, 8,$$

where f_c is the carrier frequency and $g_T(t)$ is given by

$$g_T(t) = \frac{1}{4T}\left(1 - \cos\left(\frac{\pi t}{T}\right)\right), \qquad t \in [0, T],$$

and is equal to zero outside the time interval $[0, T]$. What is the complex baseband form of $s(t)$?

8. Suppose that the sampled output of a receive filter is given by

$$r_m = a_m + 2a_{m-1} + \nu_m,$$

where $\{a_m\}$ is the data taking values +1 or -1 and $\{v_m\}$ is an i.i.d. Gaussian sequence with mean zero and variance 1. Suppose $a_0 = +1$ and supposed that the observed outputs at the first three samples time instants are $r_1 = 1, r_2 = -1$, and $r_3 = 0$.

(a) Use the Viterbi algorithm to find the optimal ML sequence decision for a_1 based on r_1, r_2 and r_3. Clearly draw the trellis, identify the cost metric for each branch of the trellis and explain how you have used the Viterbi algorithm, i.e. it should be clear from your solution that you have indeed used the Viterbi algorithm and that you have not obtained the shortest path by exhaustive enumeration.

(b) Compute the probability that the Viterbi algorithm decides that the path corresponding to $a_1 = +1$, $a_2 = -1$, $a_3 = +1$, has a smaller cost than the path corresponding to the actual transmitted bits $a_1 = -1$, $a_2 = +1$, and $a_3 = +1$.

9. Consider the transmission of two equally likely signals $+A$ or $-A$ over a channel where the noise n has the pdf

$$p(n) = \frac{1}{\sqrt{2\sigma^2}} e^{-\frac{|n|\sqrt{2}}{\sigma}}, \quad -\infty < n < \infty.$$

In other words, the output of the channel is $r = s + n$, where s can be either $+A$ or $-A$, and the noise pdf is given as above.

(a) What is the optimal decision rule? Identify the decision regions for the two signals $+A$ and $-A$.

(b) Under the decision rule in part (a), compute the probability of error.

10. Each of the following are *short* questions, and require only brief answers.

(a) Compare the spectral efficiencies of QPSK and BPSK.

(b) Compare M-ary PAM with M-ary PPM in terms of bandwidth and energy used.

(c) Draw the 8-PSK signal constellation and assign symbols to the signals such that adjacent symbols differ in at most one bit.