EE 121: Introduction to Digital Communication Systems

Solution to problem 2 for Discussion Section 3

Mon 2/11/2007 and Wed 2/13/2007

2. (a) The stationary distribution is easily calculated. The steady-state probabilities are found by solving

$$\pi(0)\alpha_0 + (1-\alpha_1)(1-\pi(0)) = \pi(0)$$
$$\Rightarrow \pi(0) = \frac{1-\alpha_1}{2-\alpha_0-\alpha_1}$$

By symmetry

$$\pi(1) = \frac{1-\alpha_0}{2-\alpha_0-\alpha_1}.$$

Thus

$$\begin{aligned} H(X_2|X_1) &= \sum_x \Pr(X_n = x) H(X_n|X_{n-1} = x) \\ &= \frac{1-\alpha_1}{2-\alpha_0-\alpha_1}\left(-\alpha_0 \log_2 \alpha_0 - (1-\alpha_0)\log_2(1-\alpha_0)\right) \\ &\quad + \frac{1-\alpha_0}{2-\alpha_1-\alpha_0}\left(-\alpha_1 \log_2 \alpha_1 - (1-\alpha_1)\log_2(1-\alpha_1)\right) \end{aligned}$$

b) The entropy rate is at most one bit as the process has only two states. This rate can be achieved if (and only if) $\alpha_0 = \alpha_1 = 1/2$, in which case the process is actually i.i.d. with $\Pr(X_i = 0) = \Pr(X_i = 1) = 1/2$.

c) As a special case of the general two-state Markov-chain, the entropy rate is

$$H(X_2|X_1) = \frac{1}{2-p}\left(-p\log_2 p - (1-p)\log_2(1-p)\right).$$

d) By straightforward calculus, we find the maximum value of $H(X)$ of part (c) occurs for $p = (\sqrt{(5)} - 1)/2$ (note this value is the reciprocal of the Golden Ratio). The maximum value is then about 0.694 bits.

e) The Markov chain of part (c) forbids consecutive ones. Consider any allowable sequence of symbols of length n. If the first symbol is 1, then the next symbol must be zero; the remaining $N(n-2)$ symbols can form any allowable sequence. If the first symbol is 0, then the remaining $N(n-1)$ symbols can be any allowable sequence. So the number of allowable sequences of length n satisfies the recurrence relation

$$N(n) = N(n-1) + N(n-2)$$

with initial conditions $N(1) = 2, N(2) = 3$. This is the well known Fibonacci sequence. The above difference equation can be solved by considering the characteristic equation $z^2 - z - 1 = 0$ which has solutions $z = \phi, \phi - 1$ where $\phi = (\sqrt{5} + 1)/2$ is the Golden Ratio. Using the initial conditions we find

$$N(n) = \frac{\phi^n - (1 - \phi)^n}{\sqrt{5}}.$$

Therefore

$$\begin{aligned} H_0 &= \lim_{n \to \infty} \frac{\log_2 N(n)}{n} \\ &= \lim_{n \to \infty} \frac{\log_2(\phi^n - (1 - \phi)^n)/\sqrt{5}}{n} \\ &= \lim_{n \to \infty} \frac{n \log_2 \phi - \log_2 \sqrt{5}}{n} \\ &= \log_2 \phi \\ &= 0.694 \end{aligned}$$

Since there are only $N(n)$ possible outcomes for $X_1, \ldots, X_n$, an upper bound on $H(X_1, \ldots, X_n)$ is $\log_2 N(n)$, and so the entropy rate of the Markov chain of part (c) is at most H_0. In fact, we saw in part (d) that this upper bound can be achieved.