EE 121 - Introduction to Digital Communications
Discussion Section 7, Partial Solutions

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1. (a) We use a constellation with 2^n symbols $\mathbf{x}_i = \alpha(2i-1, 2i-1)$ for $i = -2^{n-1}+1, \ldots, 2^{n-1}$. The constant α needs to be chosen such that the constellation satisfies the average power constraint. As the string contains n data symbols and also has entropy n bits, the symbols are distributed independently and uniformly. Thus $\Pr(\mathbf{X} = \mathbf{x}_i) = 2^{-n}$ for all i and

$$\begin{aligned}
\mathbb{E}X(1)^2 + \mathbb{E}X(2)^2 &= \alpha^2 2^{-n} \sum_{i=-2^{n-1}+1}^{2^{n-1}} (2i-1)^2 + (2i-1)^2 \\
&= \alpha^2 2^{2-n} \sum_{i=1}^{2^{n-1}} (2i-1)^2 \\
&= \alpha^2 2^{2-n} \sum_{i=1}^{2^{n-1}} 4i^2 - 4i + 1 \\
&= \alpha^2 2^{2-n} \left(4.2^{n-1}(2^{n-1}+1)(2^n+1)/6 - 4.2^{n-1}(2^{n-1}+1)/2 + 2^{n-1}\right) \\
&= \alpha^2 2^{2-n} \left(2^n(2^{n-1}+1)(2^n+1)/3 - 2^n(2^{n-1}+1) + 2^{n-1}\right) \\
&= \frac{2}{3}\alpha^2(2^{2n}-1)
\end{aligned}$$

Setting the right hand side equal to $2E$ we have

$$\alpha = \sqrt{\frac{3E}{2^{2n}-1}}$$

The distance between constellation points is

$$d = \sqrt{8}\alpha.$$

Therefore the probability of error is then roughly (i.e. assuming n is large so that we can ignore the boundary constellation points)

$$P_e(1) = \mathrm{Q}\left(\sqrt{\frac{3}{2^{2n+1}} \cdot \frac{E}{\sigma^2}}\right)$$

(b) Now we use a constellation with 2^n symbols $\mathbf{x}_{i,j} = \beta(2i-1, 2j-1)$ for $i = -2^{n/2-1}+1, \ldots, 2^{n/2-1}$ and $j = -2^{n/2-1}+1, \ldots, 2^{n/2-1}$, where we assume n is even. We choose β to satisfy the power constraint.

$$\begin{aligned}
\mathbb{E}X(1)^2 + \mathbb{E}X(2)^2 &= \beta^2 2^{-n} \sum_{i=-2^{n/2-1}+1}^{2^{n/2-1}} \sum_{j=-2^{n/2-1}+1}^{2^{n/2-1}} (2i-1)^2 + (2j-1)^2 \\
&= \beta^2 2^{-n} \cdot 2 \cdot 2^{n/2} \sum_{i=-2^{n/2-1}+1}^{2^{n/2-1}} (2i-1)^2 \\
&= \frac{2}{3}\beta^2(2^n - 1)
\end{aligned}$$

Thus

$$\beta = \sqrt{\frac{3E}{2^n - 1}}.$$

The probability of error is then roughly

$$P_e(2) = \mathrm{Q}\left(\sqrt{\frac{3}{2^{n+1}} \cdot \frac{E}{\sigma^2}}\right)$$

(c) When n is large the probability using the repetition code

$$P_e(1) \approx e^{-3 \cdot 2^{-2(n+1)} \cdot E/\sigma^2}$$

versus

$$P_e(2) \approx e^{-3 \cdot 2^{-(n+1)} \cdot E/\sigma^2}$$

for the code of part (b).

The trouble with the repetition scheme of part (a) is that it makes inefficient use of the degrees of freedom available. In part (b) we use a code that fills the entire 2D signalling space, hence the distance between codewords will shrink much more slowly —like $2^{-n/2}$ instead of 2^{-n}. The error probability reflects this.